Conformal Field Theory

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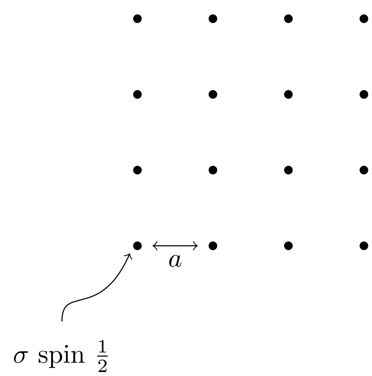
Contents

Introduction

Aim: to understand CFT's in 2d.

Why?

(A) Second order phase transitions in 2d systems (so-called critical phenomena). For example, the 2d Ising model. Take a 2d lattice of critical sites (in principal, this is assumed to be infinite):



These spins are equipped with a nearest neighbor interaction. The energy for the system is given by

$$E(\{\sigma\}) = -\epsilon \sum_{\substack{\text{adjacent} \\ \text{lattice sites}}} \sigma_i \sigma_j$$

where $\sigma_i \in \pm\frac{1}{2}$. There are ground states for the system: all states having the same spin (ie either all $|\uparrow\rangle$ or all $|\downarrow\rangle$). The partition function is

$$Z = \sum_{\{\sigma\}} \exp\left(-E(\{\sigma\})\beta\right)$$

where $\beta = \frac{1}{T}$ is the inverse temperature.

We can then compute the correlation functions:

$$\langle \sigma_i, \sigma_j \rangle - \frac{1}{Z} \sum_{\{\sigma\}} \exp\left(-E(\{\sigma\})\right) \sigma_i \sigma_j \sim \exp\left(\frac{-|i-j|}{\zeta(T)}\right)$$

where $|i-j|$ is the distance from the site i to the site j.

If $T \to \infty$, then $\zeta(T) \to 0$, so that $\langle \sigma_i, \sigma_j \rangle \to 0$. If $T \to T_c$, $\zeta(T) \to \infty$, and the system reaches criticality.

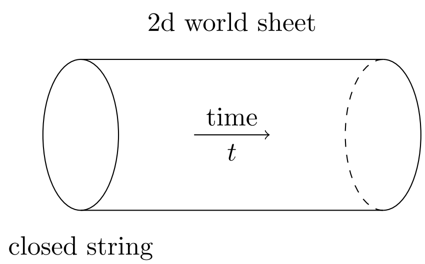
At criticality, $T = T_c$, we have that

$$\langle \sigma_i, \sigma_j \rangle \sim \frac{1}{|i-j|^{\frac{1}{4}}}$$

This theory is scale-invariant (for scales $\gg a$), and can be solved by use of a 2-dimensional CFT[1]

[1] Onsager won the '68 Nobel prize in Chemistry for his solution of the 2d Ising model at criticality.

(B) String theory provides other examples where CFT's can be given meaning. It concerns itself with the time evolution of 1d objects ("strings"):



This time evolution is described by the *Polyakov action*, which is reparametrization invariant and Weyl invariant. Together these give us conformal invariance, which leads to a 2d CFT description of 2d worldsheets.

1 Conformal Field Theories in d dimensions

1.1 Conformal group in dimension d ($d > 2$)

Definition (Local Conformal Transformations)**.** Let M, N be smooth d-dimensional manifolds with metrics g and h respectively[2]. A local diffeomorphism of open sets

$$\phi : U \subset M \to \phi(U) \subset N$$

is called a *local conformal transformation* if

$$\phi^* h = \Lambda \cdot g$$

for a smooth scale function

$$\Lambda : U \to \mathbb{R}_{>0}$$

[2] Very generally, we can take (M, g) and (N, h) to be semi-Riemannian. That is, the 'metrics' can be taken to give symmetric, nondegenerate, bilinear pairings on the tangent spaces (not necessarily positive definite).

Remark. (local) conformal transformations preserve angles. Given two vectors $X, Y \in T_pM$, for $p \in U$, we then have

$$\begin{aligned}\cos\angle(\phi_* X, \phi_* Y) &= \frac{h(\phi_* X, \phi_* Y)}{|\phi_* X|_h |\phi_* Y|_h} \\ &= \frac{\phi^* h(X, Y)}{|X|_{\phi^* H} |Y|_{\phi^* h}} \\ &= \frac{\Lambda g(X, Y)}{\sqrt{\Lambda}|X|_g \sqrt{\Lambda}|Y|_g} \\ &= \cos\angle(X, Y)\end{aligned}$$

Remark. • Conformal transformations define 'new' metrics on a space for $M = N$

$$\tilde{g} = \phi^* g = \Lambda \cdot g$$

for $\phi : M \to M$. This implies, with a little computation, that the Weyl Tensor remains invariant under conformal transformations, but the Riemann tensor does not.

We will, for now, work with *conformally flat space* $M = N = \mathbb{R}^d$, with the metric

$$\eta = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{n}, \underbrace{1, \ldots, 1}_{m})$$

where $n + m = d$. So we can write a conformal transformation $\phi : \mathbb{R}^d \to \mathbb{R}^d$ $X^\mu \mapsto \phi^\alpha(X^\mu)$, by requiring the relation

$$\boxed{\eta_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial X^\mu} \frac{\partial \phi^\beta}{\partial X^\nu} = \Lambda \eta_{\mu,\nu}}$$

- The 2d/3d Weyl tensor always vanishes on any 2d/3d manifold that is locally conformally flat.
- In 2d, we can see from cartography that we can map a globe minus a point to a sheet of paper by preserving angles.

We will now consider an infinitesmal transformation$X^\mu \mapsto X^\mu + \epsilon^\mu(X)$. Which gives us

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dX^\mu dX^\nu$$

as the transformation of line elements[3]

This transformation is conformal if

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \eta_{\mu\nu} C$$

By taking the trace of this relation, we can find that

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \mathrm{div}(\epsilon) \eta_{\mu\nu}} \qquad (*)$$

Furthermore

$$\partial_\mu \partial_\nu (\mathrm{div}\epsilon) = \partial_\mu \partial^\rho (\partial_\nu \epsilon_\rho + \partial_\rho \epsilon_\nu - \partial_\rho \epsilon_\nu) \overset{(*)}{=} \frac{2}{d} \partial_\mu \partial_\nu (\mathrm{div}\epsilon) - \Box \partial_\mu \epsilon_\nu$$

Since this expression is symmetric under the interchange of μ and ν, we can simplify to

$$\frac{2}{d} \partial_\mu \partial_\nu (\mathrm{div}\epsilon) - \frac{1}{2} \Box (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$$

and, applying $(*)$ again, we get

$$(\eta_{\mu\nu} \Box + (d-2) \partial_\mu \partial_\nu)(\mathrm{div}\epsilon) = 0$$

[3] The line element is, as usual, defined by
$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu$$

Taking the trace, we then get

$$\boxed{\Box(\mathrm{div}\epsilon) = 0}$$

Ansatz: For the infinitesmal transformations:

$$\partial_\mu \partial_\nu (\mathrm{div}\epsilon) = 0 \quad \Rightarrow \quad \mathrm{div}\epsilon = A + B_\mu X^\mu$$

Which implies

$$\epsilon_\mu = a_\mu + b_{\mu\nu} X^\nu + C_{\mu\nu\rho} X^\nu X^\rho$$

where $C_{\mu\nu\rho} = C_{\mu\rho\nu}$

From this, we can classify infinitesmal conformal transformations:

(i) $\epsilon_\mu = a_\mu$, called *translations*

(ii) $\epsilon_\mu = b_{\mu\nu} X^\nu$. In $(*)$, we get:

$$b_{\mu\nu} + b_{\mu\nu} = \frac{2}{d}\eta_{\mu\nu}(\eta^{\tau\sigma} b_{\tau\sigma})$$

  a) $\epsilon_\mu = w_{\mu\nu} X^\nu$, where $w_{\mu\nu} = -w_{\nu\mu}$, which we call *rotations.*

  b) $\epsilon_\mu = \lambda \cdot \eta_{\mu\nu} X^\nu = \lambda X_\nu$ which we call *dilations.*

(iii) $\epsilon_\mu = C_{\mu\nu\rho} X^\nu X^\rho$. In $(*)$, we get[4]:

$$C_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$$

so that

$$\epsilon_\mu = 2(b \cdot X) X_\mu - b_\mu |X|^2$$

which we call *special conformal transformations (SCTs).*

[4] Where $b_\mu = \frac{1}{d} C_{\rho\sigma\mu}\eta^{\rho\sigma}$

By integrating these infinitesmal versions, we can obtain expressions for finite conformal transformations:

(i) Translations:

$$\tilde{X}^\mu = X^\mu + a^\mu$$

(ii) Rotations:

$$\tilde{X}^\mu = M^\mu_\nu X^\nu$$

for $M^\mu_\nu \in SO(n,m)$.

Poincaré Subgroup

(iii) Dilations:

$$\tilde{X}^\mu = \Lambda X^\mu$$

(iv) SCTs:

$$\tilde{X}^\mu = \frac{X^\mu - b^\mu |X|^2}{1 - 2 b_\mu X^\mu + |b|^2 |X|^2}$$

As it happens, the conformal group acting on X^μ is precisely
$SO(n+1, m+1)$
INTERLUDE: Projective space $\mathbb{RP}^{d+1} = \mathbb{R}^{d+2} \setminus \{0\}/\mathbb{R}^*$.
Consider the map

$$\begin{aligned} \iota : \mathbb{R}^d &\to \mathbb{RP}^{d+1} \\ X^\mu &\mapsto \left[\frac{1}{2}(1+|X|^2) : X^1 : \cdots : X^d : \frac{1}{2}(1-|X|^2)\right] =: X_{\mathbb{P}} \end{aligned}$$

Properties:

(i) $\iota(\mathbb{R})$ is on the *projective light cone*:

$$\{0 = \eta_{\mathbb{P}}(X_{\mathbb{P}}, X_{\mathbb{P}})\} \subset \mathbb{RP}^{d+1}$$

where

$$\eta_{\mathbb{P}} = \operatorname{diag}(\underbrace{-1, \ldots, -1}_{n+1}, \underbrace{1, \ldots, 1}_{m+1})$$

(ii) $SO(n+1, m+1)$ acts on $\mathbb{RP}^{d+1}$ in the canonical way

$$X_{\mathbb{P}}^A = M_{\mathbb{P}}{}^A{}_B X_{\mathbb{P}}^B$$

where

$$M_{\mathbb{P}}{}^A{}_B \in SO(n+1, m+1)$$

(iii) Transformations in $SO(n+1, m+1)$ induce the following conformal transformations on $\mathbb{R}^d$

a) Rotations:

$$M_B^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & M_\nu^\mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with $M_\nu^\mu \in SO(n,m) \subset SO(n+1, m+1)$.

b) Translations ($X^\mu \to X^\mu + a^\mu$)

$$M_B^A = \begin{bmatrix} 1 + \frac{1}{2}|a| & -a^\mu & -\frac{1}{2}|a| \\ a_\mu & \mathbb{1} & -a^\mu \\ -\frac{1}{2}|a| & a^\mu & 1 - \frac{1}{2}|a| \end{bmatrix}$$

Poincaré Subgroup

c) Dilations ($X^\mu \to rX^\mu$)

$$M_B^A = \begin{bmatrix} \frac{1+r^2}{2r} & 0 & \frac{1-r^2}{2r} \\ 0 & \mathbb{1} & 0 \\ \frac{1-r^2}{2r} & 0 & \frac{1+r^2}{2r} \end{bmatrix}$$

d) SCT's ($X^\mu \to \frac{X^\mu - b^\mu |X|^2}{1 - 2b_\mu X^\mu + |b|^2 |X|^2}$)

$$M_B^A = \begin{bmatrix} 1 + \frac{1}{2}|b| & -b^\mu & -\frac{1}{2}|b| \\ -b_\mu & \mathbb{1} & b_\mu \\ 1 + \frac{1}{2}|b| & -b^\mu & 1 - \frac{1}{2}|b| \end{bmatrix}$$

Conformal Invariants

For points $x_i \in \mathbb{R}^d$

- If we require translational and rotational invariance, we can take as our invariant
$$|x_1 - x_2|$$
- If we additionally require scale invariance, it suffices to add another point:
$$\frac{|x_1 - x_2|}{|x_1 - x_3|}$$
- Applying an SCT, we get
$$|x_1 - x_2| \to \left(1 - 2bx_1 + |b|^2|x_1|^2\right)^{-\frac{1}{2}} \left(1 - 2bx_2 + |b|^2|x_2|^2\right)^{-\frac{1}{2}} |x_1 - x_2|$$
Therefore, conformal invariants are *anharmonic ratios*[5]:
$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|}$$

[5] Also called *cross ratios*

Remark. There are a total of $\frac{N}{2}(N-3)$ distinct anharmonic invariants for a set of N points[6].

[6] Warning: There can be complicated algebraic relations among such invariants.

1.2 Correlation function of Quantum Fields ($d > 2$)

We take the *Euclidean* signature in d dimensions, so that our conformal group is $SO(1, d+1)$. We then get the commutation relations

$$[J_{AB}, J_{CD}] = i\left(\eta_{AB}J_{BC} + \eta_{BC}J_{AB} - \eta_{AC}J_{BD} - \eta_{BD}J_{AC}\right)$$

And $J_{AB} = -J_{BA}$,

$$\eta = \mathrm{diag}(-1, \underbrace{1, 1, \ldots, 1}_{d+1})$$

and A, B, C, D range through $-1, 0, 1, 2, \ldots, D$. For the *Conformal Generators*[7]:

[7] μ, ν are assumed to be strictly positive in the definitions below.

$$\left.\begin{array}{l} \bullet\ L_{\mu\nu} = J_{\mu\nu} \text{ Rotations} \\ \bullet\ P_\mu = J_{-1,\mu} \text{ Translations} \end{array}\right\} \text{Poincaré Algebra}$$

- $D = J_{-1,0}$ Dilations
- $K_\mu = -J_{-1,\mu} + J_{0,\mu}$ SCTs

The *Conformal algebra* is then the Poincaré algebra plus the commutation relations

$$\begin{aligned}
[D, P_\mu] &= iP_\mu \\
[D, K_\mu] &= -iK + \mu \\
[D, L_{\mu,\nu}] &= 0 \\
[K_\mu, P_\nu] &= 2i(\eta_{\mu\nu} D - L_{\mu\nu})
\end{aligned}$$

From these algebraic definitions, we can then get conformal transformations, eg:

- $\exp\left(-ib^\mu K_\mu\right) \in SO(1, d+1)$ SCT
- $\exp\left(-ia^\mu P_\mu\right) \in SO(1, d+1)$ Translations
- $\exp\left(-i\lambda D\right) \in SO(1, d+1)$ Dilation
- $\exp\left(-\frac{i}{2}\omega^{\mu,\nu} L_{\mu\nu}\right) \in SO(1, d+1)$ Rotation

Quantum Fields as '∞-dimensional representations' of conformal group

Example. We can think of functions $\{g\}$ as representations of $SO(1, d+1)$ via the map

$$\begin{aligned}
\rho_g : SO(1, d+1) \times \mathrm{Maps}(\mathbb{R}^d \to \mathbb{R}) &\to \mathrm{Maps}(\mathbb{R}^d \to \mathbb{R}) \\
(\Lambda, g) &\mapsto g \circ f_{\Lambda^{-1}}
\end{aligned}$$

where

$$\begin{aligned}
f_\Lambda : x^\mu &\mapsto f \circ \Lambda(x^\mu) \\
f_{\Lambda^{-1}} : x^\mu &\mapsto f \circ \Lambda^{-1}(x^\mu)
\end{aligned}$$

We then have the infinitesmal generators of the conformal group in the representation ρ_g

$$\begin{aligned}
f_\Lambda : x^\mu &\mapsto x^\mu + \epsilon^\mu(x) \\
f_{\Lambda^{-1}} : x^\mu &\mapsto x^\mu - \epsilon^\mu(x)
\end{aligned}$$

Example. As an example, we can look at the infinitesmal dilation lambda Λ_λ

$$f_{\Lambda_\lambda} : x^\mu \mapsto x^\mu + \lambda x^\mu$$

So we then have

$$\rho_g(\Lambda_\lambda)(g)(x^\mu) = g(x^\mu - \epsilon^\mu(x)) = g(x^\mu) - \lambda x^\mu \frac{\partial g}{\partial x^\mu} \overset{!}{=} e^{-i\lambda D} g(x^\mu)$$

which implies $D_\mu = -ix^\mu d_\mu$

In a similar fashion, we can find that

$$\begin{aligned} P_\mu &= -i\partial_\mu \\ L_{\mu\nu} &= i(x_\mu \partial_\mu - x_\nu \partial_\mu) \\ K_\mu &= -i(2x_\mu x^\nu \partial_\nu - |x|^2 \partial_\mu) \end{aligned}$$

A *Quantum Field* ψ is then given by ϕ^A such that, for a finite dimensional vector space V,

$$\begin{aligned} \rho_\psi : SO(1, d+1) \times \mathrm{Maps}(\mathbb{R}^d \to V) &\to \mathrm{Maps}(\mathbb{R}^d \to V) \\ (\Lambda, \psi^A) &\mapsto \rho_V(\Lambda)^A_B \psi^B \circ f_{\Lambda^{-1}} \end{aligned}$$

QUASI-PRIMARY FIELDS We consider a field $\psi(x)$ at $x = 0$ [8]. We then have

$$\begin{aligned} L_{\mu\nu}\psi(0) &= S_{\mu,\nu}\psi(0) \\ D\psi(0) &= -i\Delta\psi(0) \\ K_\mu \psi(0) &= -K_\mu \psi(0) \end{aligned}$$

We make the further assumption that we have an irreducible representation of the Lorentz group, to get

$$[\Delta, S_{\mu\nu}] = 0$$

Which, by Schur's lemma, implies $\Delta \sim \mathbb{1}$, ie a number, which we will call the *scaling dimension*. We also get

$$[\Delta, K_\mu] = -K_\mu$$

so that $K_\mu = 0$

We can also compute the 'scaling dimension at other locations'

$$\begin{aligned} D\psi(x^\mu) &= De^{-ix^\mu P_\mu}\psi(0) \\ &= \left(\left[D, e^{-ix^\mu P_\mu}\right] + e^{-ix^\mu P_\mu} D\right)\psi(0) \\ &\overset{[D,P_\mu]=iP_\mu}{=} \left(e^{-ix^\mu P_\mu}(x^\mu P_\mu) - i\Delta e^{-ix^\mu P_\mu}\right)\psi(0) \\ &= (X^\mu P_\mu - i\Delta)\psi(x^\mu) \end{aligned}$$

If we assume our field is spinless, ie $S_{\mu\nu} = 0$, then the field is characterized entirely by the scaling dimension D. We can then derive the finite transformation behavior of a spinless field $\phi(x)$.

$$\begin{aligned} \phi(x^\mu) \xrightarrow{\Lambda} \tilde{\phi}(\tilde{x}^\mu) &= \left|\frac{\partial \tilde{x}^\mu}{\partial x^\nu}\right|^{\Delta/d} \phi(x^\mu) \\ x^\mu \mapsto \tilde{x}^\mu &= f_\Lambda(x^\mu) \end{aligned}$$

we call such fields *Quasi-Primary Scalar Fields*.

[8] $x = 0$ can be thought of as the invariant locus with respect to the action of the lorentz group, dilations, and SCTs. See, eg, [1].

Correlation Functions of Quasi-Primary Fields

From a CFT, we get the path integral measure

$$[d\Phi]\, e^{S[\Phi]}$$

Where Φ is the fields in our theory, and S is the action[9] functional.

The correlation function is then

$$\langle \phi_1(x_1)\cdots\phi_N(x_N)\rangle = \frac{1}{Z}\int [d\Phi] e^{-S[\Phi]}\Phi_1(x_1)\cdots\Phi_N(x_N)$$

where

$$Z = \int [d\Phi] e^{-S[\Phi]}$$

Applying a conformal transformation,

$$\langle \phi_1(\tilde{x}_1)\cdots\phi_N(\tilde{x}_N)\rangle = \frac{1}{Z}\int [d\tilde{\Phi}] e^{-S[\tilde{\Phi}]}\tilde{\Phi}_1(\tilde{x}_1)\cdots\tilde{\Phi}_N(\tilde{x}_N)$$

which implies

$$\langle \phi_1(\tilde{x}_1)\cdots\phi_N(\tilde{x}_N)\rangle = \left|\frac{\partial \tilde{x}_1^\mu}{\partial \tilde{x}_1^\nu}\right|^{-\Delta_1/d}\cdots\left|\frac{\partial \tilde{x}_N^\mu}{\partial \tilde{x}_N^\nu}\right|^{-\Delta_N/d}\langle \phi_1(x_1)\cdots\phi_N(x_N)\rangle$$

2-point Correlation Functions

Take two fields ϕ_1 and ϕ_2. From the requisite symmetries, we can deduce some of the form of the correlation function

- Rotation+Translation invariance implies

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|)$$

- Dilation $\tilde{x}^\mu \mapsto \lambda x^\mu$ invariance implies

$$\frac{\langle \phi_1(\tilde{x}_1)\phi_2(\tilde{x}_2)\rangle}{\langle \phi_1(x_1)\phi_2(x_2)\rangle} = \lambda^{-\Delta_1-\Delta_2}$$

Together, these computations show us that

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \frac{C_1}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

Now applying invariance under SCTs we notice that

$$\left|\frac{\partial \tilde{x}_k}{\partial x_\mu}\right| = \frac{1}{(1 - 2b_\mu x_k^\mu + |b|^2|x_k|^2)^d} =: S_k(b^\mu)$$

and then compute

$$\begin{aligned}\langle \phi_1(\tilde{x}_1)\phi_2(\tilde{x}_2)\rangle &= \frac{1}{S_1(b^\mu)^{\Delta_1}S_2(b^\mu)^{\Delta_2}}\langle \phi_1(x_1)\phi_2(x_2)\rangle \\ &= \frac{C_{12}}{S_1^{\Delta_1}S_2^{\Delta_2}|x_1 - x_2|^{\Delta_1+\Delta_2}}\end{aligned}$$

[9] Which must be conformally invariant.

but, on the other hand

$$\begin{aligned}\langle\phi_1(\tilde{x}_1)\phi_2(\tilde{x}_2)\rangle &= \frac{C_{12}}{|\tilde{x}_1-\tilde{x}_2|^{\Delta_1+\Delta_2}}\\ &= \frac{C_{12}}{S_1^{(\Delta_1+\Delta_2)/2}S_2^{(\Delta_1+\Delta_2)/2}|x_1-x_2|^{\Delta_1-\Delta_2}}\end{aligned}$$

Hence $C_{12}\neq 0$ only if

$$\frac{1}{2}(\Delta_1+\Delta_2)=\Delta_1=\Delta_2$$

or, more precisely

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases}\frac{C_{12}}{|x_1-x_2|^{2\Delta}} & \Delta_1=\Delta_2=\Delta\\ 0 & \text{else}\end{cases}$$

One can perform similar simplifications for 3 and 4 point correlation functions. It is left as an exercise to see

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle =$$
$$\frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_1-x_3|^{\Delta_1+\Delta_3-\Delta_2}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}}$$

and

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle =$$
$$f(\text{anh. ratios})\prod_{1\leq k<\ell\leq 4}|x_k-x_\ell|^{\Delta/3-\Delta_k-\Delta_\ell}$$

where $\Delta=\sum_i\Delta_i$.

2 Conformal Field Theories in 2 Dimensions

2.1 Conformal Algebra in 2 Dimensions

Local conformal transformations in $\mathbb{R}^2$ with metric $\delta_{\alpha\beta}=\eta_{\alpha\beta}$ are given by differentiable maps

$$\varphi: U\subset\mathbb{R}^2\to V\subset\mathbb{R}^2$$

where

$$\varphi^*\eta=\Lambda\eta,\quad \Lambda: U\mathbb{R}_{>0}$$

This implies

$$\eta_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^\mu}\frac{\partial\varphi^\beta}{\partial x^\nu}=\Lambda\eta_{\mu\nu}$$

where we assume the transformation is orientation preserving, ie

$$\left|\frac{\partial\varphi(x^1,x^2)}{\partial(x^1,x^2)}\right|>0$$

We then obtain from η_{11} and η_{22}

$$\Lambda = (\partial_1\varphi^1)^2 + (\partial_1\varphi^2)^2 = (\partial_2\varphi^1)^2 + (\partial_2\varphi^2)^2$$

and from η_{12} and η_{21}

$$0 = (\partial_1\varphi^1)(\partial_2\varphi^1) + (\partial_1\varphi^2)(\partial_2\varphi^2)$$

We can combine these two equations into a single complex equation to get

$$\big((\partial_1\varphi^1) - i(\partial_2\varphi^1)\big)^2 = \big((\partial_2\varphi^2) + i(\partial_1\varphi^2)\big)^2$$

or, equivalently

$$\partial_1\varphi^1 - i\partial_2\varphi^1 = \pm(\partial_2\varphi^1 + i\partial_1\varphi^2)$$

We then get solutions

Orientation Preserving	Orientation Reversing
(+)	(-)
$\partial_1\varphi^1 = \partial_2\varphi^2$	$\partial_1\varphi^1 = -\partial_2\varphi^2$
$\partial_2\varphi^1 = -\partial_1\varphi^2$	$\partial_1\varphi^2 = \partial_2\varphi^1$
$\left\lvert\frac{\partial\varphi(x^1,x^2)}{\partial(x^1,x^2)}\right\rvert > 0$	$\left\lvert\frac{\partial\varphi(x^1,x^2)}{\partial(x^1,x^2)}\right\rvert < 0$

Metric in homolorphic coordinates

The (+) solution gives precisely the Cauchy-Riemann Differential Equation, so that local conformal transformations in complex coordinates $z = x^1 + ix^2$ are local biholomorphic functions $\varphi(z)$.

In terms of the holomorphic and antiholomorphic coordinates $z, \overline{z}$, we can rewrite the metric

$$ds^2 = (dx^1)^2 + (dx^2)^2 = dzd\overline{z}$$

where $dz = dx + idy$ and $d\overline{z} = dx - idy$. Under biholomorphic mappings, we get

$$dz \mapsto \frac{\partial\varphi}{\partial z}dz, \quad d\overline{z} \mapsto \frac{\partial\overline{\varphi}}{\partial\overline{z}}d\overline{z}$$

We then have that the metric transforms as

$$ds^2 = dzd\overline{z} \overset{\phi}{\mapsto} \left\lvert\frac{\partial\varphi}{\partial z}\right\rvert^2 dzd\overline{z}$$

so that

$$\Delta(z,\overline{z}) = \left\lvert\frac{\partial\varphi}{\partial z}\right\rvert^2$$

We thereby see that local conformal transformations are local holomorphic coordinate changes.

Remark. • Often we regard z and $\overline{z}$ as independent coordinates, and enhance $(x^1, x^2) \in \mathbb{R}^2$ to complex coordinates $(x^1, x^2) \in \mathbb{C}^2$.

• 'Physical Condition'. We impose $\overline{z} = z^*$, the complex conjugate.

Conformal Algebra

If we take an infinitesmal conformal transformation $z \mapsto \tilde{z} = z + \epsilon(z)$, and take a Laurent expansion

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} c_n z^{n+1}$$

where the c_n are assumed to be infinitesmal, we can compute the action on functions $\varphi(z, \overline{z}) \in \mathrm{Map}(\mathbb{C} \to \mathbb{R})$.

$$e^{c_n \ell_n + \overline{c}_n \overline{\ell}_n} \varphi(z, \overline{z}) = \varphi(z - \epsilon(z), \overline{z} - \overline{\epsilon}(\overline{z}))$$

which is equivalent to

$$(1 + c_n \ell_n + \overline{c}_n \overline{\ell}_n)\varphi(z, \overline{z}) = (1 - \epsilon(z)\partial_z - \overline{\epsilon}(\overline{z})\partial_{\overline{z}})\varphi(z, \overline{z})$$

So the generators of the conformal algebra (in 'function representation') are

$$\ell_n = -z^{n+1}\partial_z, \;\; \overline{\ell}_n = -\overline{z}^{n+1}\partial_{\overline{z}}$$

The commutation relations these generators satisfy (*Witt Algebra* relations)

$$[\ell_n, \ell_m] = (n - m)\ell_{n+m}$$
$$[\overline{\ell}_n, \overline{\ell}_m] = (n - m)\overline{\ell}_{n+m}$$

Remark.
- Treating z and $\overline{z}$ as independent variables yields two copies of the Witt Algebra $A \oplus \overline{A}$.
- Imposing the 'physical condition' leaves us with the subalgebra of $A \oplus \overline{A}$ generated by $\ell_n + \overline{\ell}_n$ and $i(\ell_n - \overline{\ell}_n$ for all n
- For a Quantum Theory, we need a central extension of the Witt Algebra, which is the so-called *Virasoro Algebra*.

Global Conformal Transformations

On $S^2 = \mathbb{C} \cup \{\infty\}$, global holomorphic transformations are generated by global vector fields.

We take as an **Ansatz**

$$\begin{aligned} v(z) &= -\sum_{n=-1}^{\infty} a_n \ell_n \\ &= \sum_{n=-1}^{\infty} a_n z^{n+1}\partial_z \\ &\overset{z \to w = \frac{1}{z}}{\underset{\partial_z \to w^2 \partial_w}{\mapsto}} -\sum_{n=-1}^{\infty} a_n w^{-n+1}\partial_w \end{aligned}$$

With a global vector field

$$v(z) = -(a_{-1}\ell_{-1} + a_0 \ell_0 + a_1 \ell_1)$$

We then have generators (applying the physical condition)

- $\ell_{-1} + \bar{\ell}_{-1}$ and $i(\ell_{-1} - \bar{\ell}_{-1})$ translations
- $\ell_0 + \bar{\ell}_0$ dilation
- $i\ell_0 - i\bar{\ell}_0$ rotation
- $\ell_1 - \bar{\ell}_1$ and $i(\ell_1 - i\bar{\ell}_1)$ SCT

Finite Global Conformal Transformations

For $S^2 = \mathbb{C} \cup \{\infty\}$, $a, b, c, d \in \mathbb{C}$, $ad - bc = 1$ we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2 = \mathrm{SO}_+(1,3)$$

which is the conformal group in $d = 2$ dimensions. From this, we have the group of Möbius transformations of the sphere

$$z \mapsto \frac{az + b}{cz + d}$$

Here, our generators are

- Translations
$$\begin{bmatrix} 1 & \tilde{a} \\ & 1 \end{bmatrix}$$
- Dilations
$$\begin{bmatrix} \lambda^{1/2} & \\ & \lambda^{-1/2} \end{bmatrix}$$
- Rotations
$$\begin{bmatrix} e^{i\frac{\theta}{2}} & \\ & e^{-i\frac{\theta}{2}} \end{bmatrix}$$
- SCT
$$\begin{bmatrix} 1 & 0 \\ \tilde{b} & 1 \end{bmatrix}$$

Remark. • For four points $z_1, \ldots, z_4$, we can define invariant *4-point cross ratios*
$$\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$
There are relations among these cross ratios. In fact, for four points, there is only one independent cross ratio. This can be seen by noting that there is always a transformation in $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ which maps
$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \mapsto \begin{bmatrix} \infty \\ 1 \\ \eta \\ 0 \end{bmatrix}$$

Physical States are characterized by the eigenvalues of the dilation operator $\ell_0 + \overline{\ell}_0$ and the rotation operator $i\ell_0 - i\overline{\ell}_0$. These can be written in terms of the quantities $h, \overline{h}$[10]

[10] Sometimes known as the *conformal weights* of the state $|\psi\rangle$.

$$\ell_0|\psi\rangle = h|\psi\rangle$$
$$\overline{\ell}_0|\psi\rangle = \overline{h}|\psi\rangle$$

The scaling dimension is the $\Delta = h + \overline{h}$ (the dilation operator eigenvalue) and the spin[11] is $s = h - \overline{h}$ (the 'rotation operator eigenvalue').

[11] As one might expect, for bosonic states, we have $s \in \mathbb{Z}$. For Fermionic states, $s \in \mathbb{Z} + \frac{1}{2}$. We could also take, more generally, a *parafermionic state* $s \in \mathbb{Q}$.

2.2 Correlation functions of (quasi-)primary fields

Definition. A field $\phi(z, \overline{z})$ of conformal weight $(h, \overline{h})$ is a *primary field* if it transforms under local conformal transformations

$$z \mapsto \phi(z) = \tilde{z}$$

locally as

$$\phi(z,\overline{z}) \mapsto \tilde{\phi}(\tilde{z},\overline{\tilde{z}}) = \left(\frac{\partial\phi}{\partial z}\right)^{-h} \left(\frac{\partial\overline{\phi}}{\partial\overline{z}}\right)^{-\overline{h}} \phi(z,\overline{z}) \tag{$*$}$$

Remark. • A Field is called a *quasi-primary field* if $(*)$ holds for *global* conformal transformations[12]

[12] Sometimes also called $SL(2,\mathbb{C})$-primaries

- A field wtih a different transformation behavior is called a secondary field.

If we take an infinitesmal variation of a primary field

$$\tilde{z} = z + \epsilon(z)$$
$$\overline{\tilde{z}} = \overline{z} + \overline{\epsilon}(\overline{z})$$

We can compute

$$\begin{aligned}\delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z}) &= \tilde{\phi}(z,\overline{z}) - \phi(z,\overline{z}) \\ &= \tilde{\phi}(\tilde{z} - \epsilon(z), \overline{\tilde{z}} - \overline{\epsilon}(\overline{z})) - \phi(z,\overline{z})\end{aligned}$$

Taylor expanding to first order we get

$$\begin{aligned}&= \tilde{\phi}(\tilde{z},\overline{\tilde{z}}) - \epsilon(z)\partial_z\tilde{\phi} - \overline{\epsilon}(\overline{z})\partial_{\overline{z}}\tilde{\phi} - \phi(z,\overline{z}) \\ &= \left[(1 + \partial_z(\epsilon(z))^{-h}(1 + \partial_{\overline{z}}\overline{\epsilon}(\overline{z}))^{-\overline{h}} - \epsilon(z)\partial_z - \overline{\epsilon}(\overline{z})\partial_{\overline{z}} - 1\right]\phi(z,\overline{z})\end{aligned}$$

which then becomes

$$\boxed{\delta_{\epsilon,\overline{\epsilon}}\phi(z,\overline{z}) = \left[-(h(\partial_z\epsilon(\overline{z})) + \epsilon(z)\partial_z) - (\overline{h}(\partial_{\overline{z}} - \overline{\epsilon}(\overline{z})) + \overline{\epsilon}(\overline{z})\partial_{\overline{z}})\right] \times \phi(z,\overline{z})} \tag{$**$}$$

2-POINT CORRELATION FUNCTION OF QUASI-PRIMARY FIELDS

We consider a quasi-primary field as above[13]. We can then compute

$$\begin{aligned} 0 &= \delta_{\epsilon,\bar{\epsilon}}\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle \\ &= \langle\delta_{\epsilon,\bar{\epsilon}}\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle + \langle\phi_1(z_1,\bar{z}_1)\delta_{\epsilon,\bar{\epsilon}}\phi_2(z_2,\bar{z}_2)\rangle \end{aligned}$$

[13] The equation (∗∗) holds for quasi-primary fields where ϵ is infinitesmal and a generator of a global conformal transformation.

We can write our infinitesmal conformal transformations as

$$\begin{aligned} \epsilon &= c_{-1} + c_0 z + c_1 z^2 \\ \bar{\epsilon} &= \bar{c}_{-1} + \bar{c}_0\bar{z} + \bar{c}_1\bar{z}^2 \end{aligned}$$

for these ϵ and $\bar{\epsilon}$, (∗∗) holds for any quasi-primary field. This implies that

$$0 = [h_1(\partial_{z_1}\epsilon(z_1)) + \epsilon(z_1)\partial_{z_1} + h_2(\partial_{z_2}\epsilon(z_2)) + \epsilon(z_2)\partial_{z_2} + c] \times \langle\phi_1\phi_2\rangle$$

We can then use the coefficients of

$$\epsilon(z) = c_{-1} + c_0 z + c_1 z^2$$

c_{-1}: We have $(\partial_{z_1} + \partial_{z_2})\langle\phi_1\phi_2\rangle = 0$ So

$$\langle\phi_1\phi_2\rangle = c(z_1 - z_2)$$

c_0: We have

$$(h_1 + h_2 + z_1\partial_{z_1} + z_2\partial_{z_2})\langle\phi_1\phi_2\rangle = 0$$

so, substituting our first result, we get

$$(h_1 + h_2)c(z_1 - z_2) + (z_1 - z_2)c'(z_1 - z_2) = 0$$

and solving this differential equation:

$$c(z_1 - z_2) = \frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}}$$

c_1: We have

$$(2h_1 z_1 + 2h_2 z_2 + z_1^2\partial_{z_1} + z_2^2\partial_{z_2})\langle\phi_1\phi_2\rangle = 0$$

substituting in

$$(2h_1z_1+2h_2z_2)\frac{C_{12}}{(z_1 - z_2)^{h_1+h_2}} - (h_1+h_2)(z_1^2-z_2^2)\frac{C_{12}}{(z_1 - z_2)^{h_1+h_2+1}} = 0$$

so

$$\frac{C_{12}(h_1 - h_2)}{(z_1 - z_2)^{h_1+h_2+1}} = 0$$

which means that either $C_{12} = 0$ or $(h_1 = h_2)$.

If we add in complex conjugation, we see that

$$\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2\overline{z}_2)\rangle = \begin{cases} \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}(\overline{z}_1-\overline{z}_2)^{\overline{h}_1+\overline{h}_2}} & h_1 = h_2, \overline{h}_1 = \overline{h}_2 \\ 0 & else \end{cases}$$

Remark. • We have a single-values correlator for $s = h - \overline{h} \in \mathbb{Z}$ or in $\mathbb{Z} + \frac{1}{2}$. This is owing to the face that

$$(z_1 - z_2)^{-2h}(\overline{z}_1 - \overline{z}_2)^{-2\overline{h}} = |z_1 - z_2|^{-2h}(\overline{z}_1 - \overline{z}_2)^{2s}$$

If we let $s \in \mathbb{Q}$, we get a multiple-valued correlation function for parafermions.

• If we let spin equal zero, ie $h = \overline{h}$, we recover our previous d-dimensional result.

More generally, we can apply the same techniques to compute 3- and 4- point correlation functions, finding, when we do, that

$$\begin{aligned}\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\rangle &= \frac{C_{123}}{(z_1-z_2)^{h_1+h_2-h_3}(z_1-z_3)^{h_1+h_3-h_2}(z_1-z_2)^{h_2+h_3-h_1}} \\ &\times \frac{1}{(\overline{z}_1-\overline{z}_2)^{\overline{h}_1+\overline{h}_2-\overline{h}_3}(\overline{z}_1-\overline{z}_3)^{\overline{h}_1+\overline{h}_3-\overline{h}_2}(\overline{z}_2-\overline{z}_3)^{\overline{h}_2+\overline{h}_3-\overline{h}_1}}\end{aligned}$$

and

$$\begin{aligned}&\langle \phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\phi_4(z_4,\overline{z}_4)\rangle \\ &= f(\eta,\overline{\eta}) \prod_{1\leq \leq \ell \leq 4} (z_k - z_\ell)^{h/3-h_k-h_\ell}(\overline{z}_k - \overline{z}_\ell)^{\overline{h}/3-\overline{h}_k-\overline{h}_\ell}\end{aligned}$$

where

$$\eta = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}$$

and

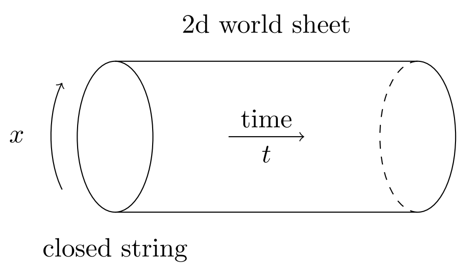
$$h = \sum_{i=1}^{4} h_i$$

$$\overline{h} = \sum_{i=1}^{4} \overline{h}_i$$

2.3 Radial Quantization

First Example

We now return to the context of string theory, with 2-dimensional Minkowski space. Suppose we have a conformally flat world-sheet



where we have coordinates $t \in \mathbb{R}$ ('time') and $x \in S^1 = \mathbb{R}/(L \cdot \mathbb{Z})$ where L is the length of the string.

In 2d Minkowski space we can take the light cone coordinates $t \pm x$, and apply a *Wick Rotation*

$$\tau := it$$

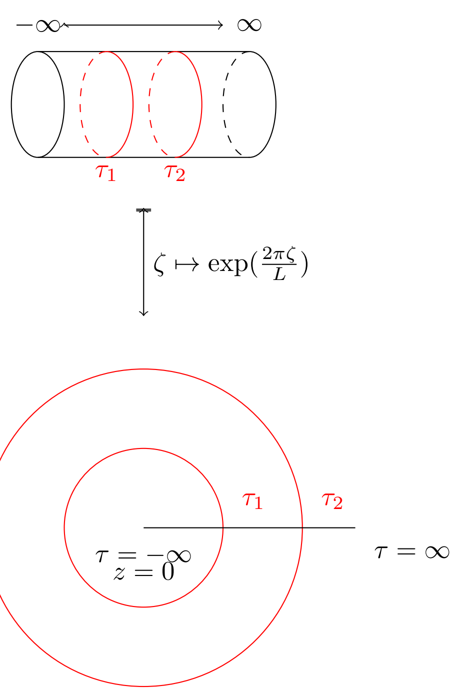
So that we can define complex coordinates

$$\zeta = \tau + ix$$

$$\bar{\zeta} = \tau - ix$$

with the identification $\zeta \sim \zeta + iL$

We then have a conformal map to the punctured conformal plane $\mathbb{C}^*$



In Quantization we have

- A Hilbert state space at each fixed spatial slice (ie, fixed time τ)
- Dynamics given by a propagator amounst such slices in time.

On the conformal plane, we have

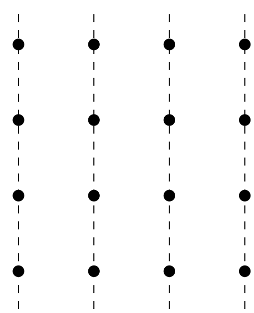
- Hilbert state space defined on circles about the origin
- Propagation of states in the radial direction
  i) Dilation operator is the Hamiltonian for the string
  ii) Rotation operator is a spatial translation along the string.

For such strings, a radial quantization scheme is very natural.

Second Example

We again consider the Euclidean space of statistical mechanics. The standard quantization is comprised of:

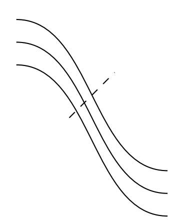
- Hilbert Spaces of states along 1d slices (for example in a 2d lattice).



- Transfer matrices describing a correlation orthogonal to the quantized slices.

In the limit where the lattice spacing goes to zero, and at criticality, we have a conformal symmetry. So taking radial slices of the plane and a radial propagator gives the same result[14]. Radial quantization is a convenient choice because of the use of the radial operator product expansion.

[14] We could also use something stranger, like, for example



but in practice, the radial and euclidean schemes prove easiest to work with.

In and Out States

We make the assumptions:

i) There is a vacuum state $|0\rangle$ from which we can construct the Hilbert space of states in terms of creation operators (positive frequency modes)

ii) As $\tau \to \pm\infty$, that is, as $(z, \overline{z}) \to \infty, 0$, the Hilbert space of states looks like the Hilbert space of states for a theory of free fields.

We can define an *in-state*

$$\boxed{|\phi_{in}\rangle = \lim_{z,\overline{z}\to 0} \phi(z,\overline{z})|0\rangle}$$

This equality is known as the *state-field correspondence*

Hermitian Product

In string theory, Hermition conjugation has no effect on the 2d Minkowski space

- Hermitian conjugation on Euclidean time:

$$\tau = it : \tau \mapsto -\tau$$

- Hermitian conjugation on radial coordinates

$$z \mapsto \frac{1}{z^*}$$

On quasi primary fields, the Hermitian conjugation is given by:

$$\boxed{\phi(z,\overline{z})^\dagger = \overline{z}^{-2h} z^{-2\overline{h}} \phi(\frac{1}{z}, \frac{1}{\overline{z}})}$$

We can then define an *out-state* by

$$\langle \phi_{out}| := |\phi_{in}\rangle$$

that is

$$\langle \phi_{out}| = \lim_{z,\overline{z}\to 0} \langle 0|\phi(z,\overline{z})^\dagger$$

This defines a good Hermitian product for quasi-primary fields.

As a first example, we can compute:

$$\begin{aligned}
\langle \phi_{out}|\phi_{in}\rangle &= \lim_{z,\overline{z},w,\overline{w}\to 0} \langle 0|\phi(w,\overline{w})^\dagger \phi(z,\overline{z})|0\rangle \\
&= \lim \overline{w}^{-2h} w^{-2\overline{h}} \langle 0|(\phi(\frac{1}{w}, \frac{1}{\overline{w}})\phi(z,\overline{z})|0\rangle \\
&\overset{\star}{=} \lim \overline{w}^{-2h} w^{-2\overline{h}} \langle 0|\mathcal{R}\left(\phi(\frac{1}{w}, \frac{1}{\overline{w}})\phi(z,\overline{z})\right)|0\rangle \\
&= \lim \overline{w}^{-2h} w^{-2\overline{h}} \langle (\phi(\frac{1}{w}, \frac{1}{\overline{w}})\phi(z,\overline{z})\rangle \\
&= \lim \frac{C}{\overline{w}^{2h} w^{2\overline{h}} \left(\frac{1}{\overline{w}} - z\right)^{2h} \left(\frac{1}{w} - \overline{z}\right)^{2\overline{h}}}
\end{aligned}$$

So that

$$\boxed{\langle \phi_{out}|\phi_{in}\rangle = C}$$

More generally, if we consider multiple fields, we find

$$\boxed{\langle \phi_{1,out}|\phi_{2,in}\rangle = \begin{cases} C_{12} & h_1 = h_2,\ \overline{h}_1 = \overline{h}_2 \\ 0 & \text{else} \end{cases}}$$

In our computation above, one step went unexplained: The radial ordering ($\star$). We now delve into its meaning

Recall that in QFT, the N-point correlator of quantum fields is given by a time-ordering prescription.

$$\langle \phi_1(x_1,t_1)\cdots\phi_N(x_N,t_N)\rangle = \langle 0|\mathcal{T}\left(\phi_1(x_1,t_1)\cdots\phi_N(x_N,t_N)\right)|0\rangle$$

where

$$\mathcal{T}\left(\phi_1(x_1,t_1)\cdots\phi_N(x_N,t_N)\right) = \pm\phi_{\sigma(1)}(x_{\sigma(1)},t_{\sigma(1)})\cdots\phi_{\sigma(N)}(x_{\sigma(N)},t_{\sigma(N)})$$

The sign is due to Bose/Fermi statistics, and the permutation σ is such that

$$t_{\sigma(1)} > t_{\sigma(2)} > \cdots > t_{\sigma(N)}$$

We can perform a similar ordering for radial quantization: *the radial ordering*.

$$\mathcal{R}\left(\phi_1(z,\overline{z})\phi_2(w,\overline{w})\right) = \begin{cases}\phi_1(z,\overline{z})\phi_2(w,\overline{w}) & |z|>|w| \\ \pm\phi_2(w,\overline{w})\phi_1(z,\overline{z}) & |z|<|w|\end{cases}$$

The positive sign comes in the case where at least one of the fields in question is bosonic, and the minus sign in the case where both fields are fermionic.

With the radial ordering, we have the *operator-state correspondence*

$$\langle \phi_1(z_1,\overline{z}_1)\cdots\phi_N(z_N,\overline{z}_N)\rangle = \langle 0|\mathcal{R}\left(\phi_1(z_1,\overline{z}_1)\cdots\phi_N(z_N,\overline{z}_N)\right)|0\rangle$$

Similarly to a Fourier expansion, we have the *Mode expansion* for quasi-primary fields of dimension $(h,\overline{h})$.

$$\phi(z,\overline{z}) = \sum_{m,n\in\mathbb{Z}} z^{-m-h}\overline{z}^{-n-\overline{h}}\phi_{m,n}$$

where the $\phi_{m,n}$ are operators.

Using the calculus of residues, we can compute the $\phi_{m,n}$

$$\phi_{m,n} = \oint\frac{dz}{2\pi i}z^{m+h-1}\oint\frac{d\overline{z}}{2\pi i}\overline{z}^{n+\overline{h}-1}\phi(z,\overline{z})$$

which then gives us the operator product expansion for the Hermitian conjugate

$$\phi(z,\overline{z})^\dagger = \overline{z}^{-2h}z^{-2\overline{h}}\phi(\frac{1}{z},\frac{1}{\overline{z}}) = \sum_{n,m}\overline{z}^{-m-h}z^{-n-\overline{h}}\phi_{-m,-n}$$

Looking at

$$\phi(z,\overline{z})^\dagger = \sum_{m,n}\overline{z}^{-m-h}z^{-n-\overline{h}}\phi^\dagger_{m,n}$$

we see

$$\boxed{\phi^\dagger_{m,n} = \phi_{-m,-n}}$$

A well-defined in-state implies that

$$\lim_{z,\overline{z}\to 0}\phi(z,\overline{z})|0\rangle$$

must be finite. This, in turn, requires that we have

$$\boxed{\phi_{m,n}|0\rangle = 0 \quad \text{for} \;\; m > -h, \;\; n > -\overline{h}}$$

Remark. The mode expansion gives rise to string Fourier modes on the cylinder

$$z = \exp\left(\frac{2\pi i}{L}(t+x)\right) \qquad \overline{z} = \exp\left(\frac{2\pi i}{L}(t-x)\right)$$

giving the expansion

$$\phi(z,\overline{z}) = \sum_{m,n\in\mathbb{Z}} \phi_{m,n}\exp\left(-\frac{2\pi i}{L}(\Delta+m+n)t\right)\exp\left(-\frac{2\pi i}{L}(s+m+n)x\right)$$

where $s = h - \overline{h}$ is the spin, $\Delta + m + n$ is the energy eigenstate of the mode, and $s + m - n$ is the wave propagation along the spatial direction of the string.

2.4 Operator product expansions

Correlators of 2 or more fields typically exhibit singularities as their insertion points coincide. For example, for quasi-primaries

$$\begin{aligned}\langle\phi_k(z_1,\overline{z_2})\phi_\ell(z_2,\overline{z}_2)\rangle &= \langle 0|\mathcal{R}(\phi_k(z_1,\overline{z}_1)\phi_\ell(z_2,\overline{z}_2)|0\rangle \\ &= \frac{\delta_{\ell k}}{(z_1-z_2)^{2h_k}(\overline{z}_1-\overline{z}_2)^{2\overline{h}_k}}\end{aligned}$$

the last expression is called the *canonical norm for quasi-primaries.*

The *operator product expansion* (OPE): describes the behavior of radially ordered quantum fields as the coincide. For example, for $A(z,\overline{z})$ and $B(z,\overline{z})$

$$\mathcal{R}(A(z,\overline{z})B(w,\overline{w})) = \sum_{n=-\infty}^{N}\sum_{m=-\infty}^{M}\frac{\{A,B\}_{n,m}(w,\overline{w})}{(z-w)^n(\overline{z}-\overline{w})^m}$$

with $N, M \geq 0$. In this case, we get a finite number of singular terms.

These singular terms play a special role:

$$\mathcal{R}(A(z,\overline{z})B(w,\overline{w})) \sim \sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\{A,B\}_{n,m}(w,\overline{w})}{(z-w)^n(\overline{z}-\overline{w})^m} = \overline{A(z,\overline{z})B(w,\overline{w})}$$

The degree zero term, known as the *normal order product*, is written

$$:A(w,\overline{w})B(w,\overline{w}): \;= \{A,B\}_{0,0}(w,\overline{w}) = \oint_w \frac{dz}{2\pi i}\oint_{\overline{w}}\frac{d\overline{z}}{2\pi i}\frac{\mathcal{R}(A(z,\overline{z})B(w,\overline{w}))}{(w-z)(\overline{w}-\overline{z})}$$

Thus

$$\mathcal{R}(A(z,\overline{z})B(w,\overline{W})) = \overbrace{A(z,\overline{z})B(w,\overline{w})}^{\text{singular terms}} + \overbrace{:A(w,\overline{w})B(w,\overline{w}):}^{\text{normal order product}} + \mathcal{O}(z-w,\overline{z}-\overline{w})$$

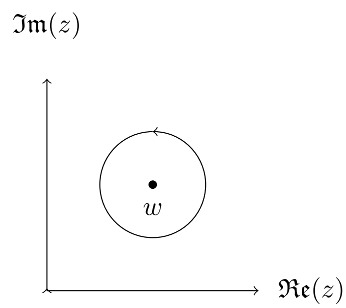
Remark. • Fields $\phi(z)$ that depend only on z are called *chiral fields.* Fields $\phi(\overline{z})$ that depend only on $\overline{z}$ are called *anti-chiral fields*

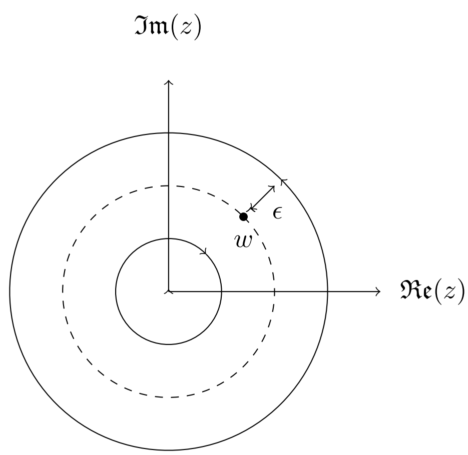
- OPEs of chiral fields will play an important role.
- Notation: In OPEs the radial order symbol $\mathcal{R}$ is often dropped

$$\mathcal{R}(A(z,\overline{z})B(w,\overline{w})) = A(z,\overline{z})B(w,\overline{w})$$

OPEs and commutators

Let $A(z)$ and $B(z)$ be bosonic chiral fields. We can deform the contour of a small circle about w as seen below





So we can rearrange the contour integral of the radially ordered product:

$$\begin{aligned}\oint_w \frac{dz}{2\pi i}\mathcal{R}(A(z),B(w)) &= \oint_{|w|+\epsilon}\frac{dz}{2\pi i}A(z)B(w) - \oint_{|w|-\epsilon}\frac{dz}{2\pi i}B(w)A(z)\\ &= Q_A B(w) - B(w)Q_A\\ &= [Q_A, B(w)]\end{aligned}$$

where $Q_A = \oint \frac{dz}{2\pi i} A(z)$.

The result is that contour integrals encode (anti-)commutators at equal time[15]

$$\oint_\chi \frac{dz}{2\pi i} \mathcal{R}(A(z)B(w) = [Q_A, B(w)]_\pm$$

More generally, we can take

$$\oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} \mathcal{R}(A(z), B(w)) = [Q_A, Q_B]_\pm$$

with Q_A and Q_B as above.

[15] Whether we get a commutator or an anti-commutator is dependent on the Bose/Fermi statistics. If at least one of the operators is bosonic, we have

$$[-,-]_+ = [-,-]$$

If both are Fermionic, we have

$$[-,-]_- = \{-,-\}$$

2.5 Energy-momentum tensor and conformal Ward identities

Interlude: Conserved charges and infinitesmal transformations of fields.

If we have a $(d+1)$ dimensional QFT with a conserved Noether current j^μ, (ie, $\partial_\mu j^\mu = 0$) then we get a conserved charge (*the Noether Charge*)

$$Q = \int dx^d \, [j_0(x)]$$

For infinitesmal symmetry transformations acting on a quantum field $\phi(x)$,

$$\delta_\epsilon \phi(x) = \tilde{\phi}(x) - \phi(x) = -\epsilon[Q, \phi(x)]$$

where the last term is the equal time commutator.

In the context of a 2d CFT with radial quantization, the conserved charge Q of the radial component of a conserved chiral current yields

$$\delta_\epsilon \phi(w, \overline{w}) = -\epsilon[Q, \phi(w, \overline{w}) = -\epsilon \oint_w \frac{dz}{2\pi i} \mathcal{R}(A, \phi)$$

Energy-Momentum tensor

The *energy-momentum tensor* $T_{\mu,\nu}$ generates local coordinate transformations[16]. For Lorentz-invariant theories it is conserved

$$\partial^\mu T_{\mu\nu} = 0$$

so we get a conserved charge

$$P\mu = \int dx^d T_{\mu,nu}$$

which are the *momentum operators*, and are translationally symmetric. By rotational invariance, $T_{\mu\nu}$ can also be chosen to be symmetric

$$T_{\mu\nu} = T_{\nu\mu}$$

In conformal theories, we call the conserved current

$$j^D_\mu = x^\nu T_{\nu\mu}$$

[16] In Lagrangian theories, it is the response of the Lagrangian to the variation of space-time.

the *dilation current* because the associated charge D is the dilation operator.

Scale invariance (generated by the dilation operator) leads to

$$0 = \partial^\mu j^D_\mu = T^\mu_\mu$$

so that the energy-momentum tensor is traceless in CFTs.

Back in the 2d setting, we can take holomorphic coordinates

$$\begin{aligned} z &= x^1 + ix^2 & \overline{z} &= x^1 - ix^2 \\ \partial_z &= \frac{1}{2}(\partial_1 - i\partial_2) & \partial_{\overline{z}} &= \frac{1}{2}(\partial_1 + i\partial_2) \end{aligned}$$

We can then compute the energy-momentum tensor

$$\begin{aligned} T_{z,z} &= \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial z}T_{1,1} + \frac{\partial x^1}{\partial z}\frac{\partial x^2}{\partial z}T_{2,2} + \frac{\partial x^2}{\partial z}\frac{\partial x^2}{\partial z}T_{2,2} \\ &= \frac{1}{4}(T_{1,1} - 2iT_{1,2} - T_{2,2}) \\ T_{\overline{z},\overline{z}} &= \frac{1}{4}(T_{1,1} + 2iT_{1,2} - T_{2,2}) \\ T_{z,\overline{z}} = T_{\overline{z},z} &= \frac{\partial x^1}{\partial z}\frac{\partial x^1}{\partial \overline{z}}T_{1,1} + \left(\frac{\partial x^1}{\partial z}\frac{\partial x^2}{\partial \overline{z}} + \frac{\partial x^2}{\partial z}\frac{\partial x^1}{\partial \overline{z}}\right)T_{1,2} + \frac{\partial x^2}{\partial z}\frac{\partial x^2}{\partial \overline{z}}T_{2,2} \\ &= \frac{1}{4}(T_{1,1} + T_{2,2}) \end{aligned}$$

From the metric

$$\begin{aligned} g_{z,z} &= g_{\overline{z},\overline{z}} = \eta(\partial_z, \partial_z) = \eta(\partial_{\overline{z}}, \partial_{\overline{z}}) = 0 \\ g_{z,\overline{z}} &= g_{\overline{z},z} = \eta(\partial_z, \partial_{\overline{z}}) = \frac{1}{2} \end{aligned}$$

we can compute conservation laws:

i) Translation invariance gives

$$\begin{aligned} \partial_{\overline{z}}T_{z,z} + \partial_z T_{\overline{z},z} &= 0 \\ \partial_{\overline{z}}T_{z,\overline{z}} + \partial_z T_{\overline{z},\overline{z}} &= 0 \end{aligned}$$

ii) Because of scale invariance

$$T_{z,\overline{z}} = T_{\overline{z},z} = 0$$

so that

$$\begin{aligned} \partial_{\overline{z}}T_{z,z} &= 0 \\ \partial_z T_{\overline{z},\overline{z}} &= 0 \end{aligned}$$

Implying that $T_{z,z}$ and $T_{\overline{z},\overline{z}}$ are chiral and anti-chiral currents[17], respectively.

[17] Notation: the chiral energy-momentum current is

$$T(z) := -2\pi T_{z,z}(z)$$

and the anti-chiral is

$$\overline{T}(\overline{z}) := -2\pi T_{\overline{z},\overline{z}}(\overline{z})$$

Conformal Transformations of primary fields

Let $\phi(w,\overline{w})$ be a chiral field of conformal weight $(h,\overline{h})$. Taking a local infinitesmal coordinate transformation

$$z \mapsto \tilde{z} = z + \epsilon(z)$$

we get

$$\begin{aligned}
\delta_\epsilon\phi(w,\overline{w}) &= -[Q_\epsilon, \phi(w,\overline{w})] \\
&= -\oint_w \frac{dz}{2\pi i}\epsilon(z)\mathcal{R}(T(z),\phi(w,\overline{w})) \\
&= -\left(h(\partial_w\epsilon(w))\phi(w,\overline{w}) + \epsilon(w)\partial_w\phi(w,\overline{w})\right) \qquad (*)
\end{aligned}$$

We notice that

$$\begin{aligned}
\oint_w \frac{dz}{2\pi i}\frac{\epsilon(z)\phi(w,\overline{w})}{(z-w)^2} &= \partial_w(\epsilon(w))\phi(w,\overline{w}) \\
\oint_w \frac{dz}{2\pi i}\frac{\epsilon(z)\phi(w,\overline{w})}{(z-w)} &= \epsilon(w)\partial_w\phi(w,\overline{w})
\end{aligned}$$

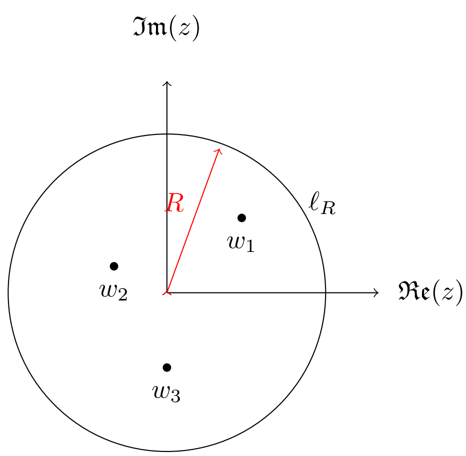
so that, with $(*)$ we find that a primary field $\phi(w,\overline{w})$ is characterized by its OPE with $T(z)$ and $\overline{T}(\overline{z})$.

$$\boxed{\begin{aligned}
\mathcal{R}(T(z)\phi(w,\overline{w})) &\sim \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w\phi(w,\overline{w}) \\
\mathcal{R}(\overline{T}(\overline{z})\phi(w,\overline{w})) &\sim \frac{\overline{h}}{(\overline{z}-\overline{w})^2}\phi(w,\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\partial_{\overline{w}}\phi(w,\overline{w})
\end{aligned}}$$

Remark. Theres OPEs of $\phi(z,\overline{z})$ with $T(z)$ and $\overline{T}(\overline{z})$ are often used as the definition of a primary field.

Conformal Ward identities

Let $\phi_k(w_k,\overline{w}_k)$ be any primary fields located at $|w_k| < R$, $k = 1,\dots,N$. For $z \mapsto z+\epsilon(z)$, we can use the contour



to compute

$$\left\langle \oint_{\ell_R} \frac{dz}{2\pi i}\epsilon(z)T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_N(w_N,\overline{w}_N)\right\rangle$$
$$= \sum_{k=1}^{N}\left\langle \phi_1(w_1,\overline{w}_1)\cdots\left(\oint_{w_k}\frac{dz}{2\pi i}\epsilon(z)T(z)\phi_k(w_k,\overline{w}_k)\right)\cdots\phi_N(w_N,\overline{w}_N)\right\rangle$$

which implies

$$\langle T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_N(w_N,\overline{w}_N)\rangle$$
$$\sim \sum_{k=1}^{N}\left(\frac{h_k}{(z-w_k)^2}+\frac{1}{z-w_k}\frac{\partial}{\partial w_k}\right)\langle\phi_1(w_1,\overline{w}_1)\cdots\phi_N(w_N,\overline{w}_N)\rangle$$

which is known as the *conformal Ward identity for primary fields.*

The conformal Ward identities imply

$$\delta_\epsilon\langle\phi_1\cdots\phi_N\rangle = -\oint_\ell \frac{dz}{2\pi i}\epsilon(z)\langle T(z)\phi_1(w_1,\overline{w}_1)\cdots\phi_N(w_N,\overline{w}_N)\rangle$$
$$\sum_{k=1}^{N}\oint_\ell\frac{dz}{2\pi i}\epsilon(z)\left(\frac{h_k}{(z-w_k)^2}+\frac{1}{z-w_k}\frac{\partial}{\partial w_k}\right)\langle\phi_1\cdots\phi_N\rangle$$

in particular

$$\delta_\epsilon\langle\phi_1\cdots\phi_N\rangle = 0$$

for an infinitesmal global conformal transformation

$$\epsilon(z) = c_{-1} + c_0 z + c_1 z^2$$

allows us to recover differential equations for correlators as discussed before.

There are some technical assumptions that were needed to derive the Ward identities:

- $T(z)$ must be regular in the conformal plane[18].
- Poles must only arise when the fields coincide.
- $T(z)$ is an energy density, that is

$$T_{\mu\nu} \sim (\text{length})^{-2}$$

From which we can calculate the scaling dimension

$$\frac{\Delta}{2} = 2$$

which implies $T(z)$ has conformal weight $h = 2$, and $\overline{T}(\overline{z})$ has conformal weight $\overline{h} = 2$.

[18] It is worth recalling that $T(z)$ should not be thought of as a function, but rather as a holomorphic section of a bundle specified by the scaling dimension. As a result, we can require more strongly that $T(z)$ be regular on $\mathbb{P}^1$ without necessarily imposing the condition that $T(z)$ be constant.

If we take the coordinate transformation

$$z \mapsto w = -\frac{1}{z}$$

we see that

$$T(z) \mapsto \tilde{T}(w) = \left(\frac{dw}{dz}\right)^{-2} T(z) = z^4 T(z)$$

which tells us that $T(z)$ must decay as z^{-4} if regularity is to hold at $z = \infty$.

2.6 Examples of CFTs (Free CFTs)

Recall from QFT:

(A) Noether's Theorem (See, eg, di Francesco et al): For a QFT with an action

$$S[\phi^A] = \int d^d x \mathcal{L}[\phi^A]$$

and an infinitesmal symmetry transformations

$$\tilde{x}^\mu = x^\mu + \frac{\delta x^\mu}{\delta w_a}$$

$$\tilde{\phi}^A(\tilde{x}^\mu) + \frac{\delta F^A[\phi^A]}{\delta w_a} w_a$$

there is a conserved current

$$\partial^\mu j^a_\mu = 0$$

given by

$$j^a_\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^A)} \partial_\nu \phi^A - \eta_{\mu\nu} \mathcal{L}\right) \frac{\delta x^\nu}{\delta w_a} + \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi^A)} \frac{\delta F^A}{\delta w_a}$$

(B) Wick's Theorem: A time-ordered product[19] of fields equals the sum of normal ordered products with all possible contractions, eg

$$\begin{aligned}\tau(\phi_1, \phi_2, \phi_3, \phi_4) =&: \phi_1\phi_2\phi_3\phi_4 : + : \overbracket{\phi_1\phi_2}\,\phi_3\phi_4 : + \cdots + : \phi_1\phi_2\,\overbracket{\phi_3\phi_4} : \\ &+ : \overbracket{\phi_1\phi_2}\,\overbracket{\phi_3\phi_4} : + \cdots + : \overbracket{\phi_1\, \overbracket{\phi_2\phi_3}\, \phi_4} :\end{aligned}$$

[19] For CFT, we replace the time-ordered product with the radially ordered product, ie, $\tau(\cdots)$ with $\mathcal{R}(\cdots)$.

where the contractions are given by[20]

$$: \phi_1 \cdots \overbracket{\phi_{k_1} \cdots \phi_{k_2}} \cdots \phi_n : = (\pm) : \phi_1 \cdots \hat{\phi}_{k_1} \cdots \hat{\phi}_{k_2} \cdots \phi_N : \times \langle \phi_{k_1} \phi_{k_2} \rangle$$

[20] Here I use the notational convention that $\hat{\phi}_i$ means omit the entry ϕ_i

Example I: The Free Boson

Take the free, real field $\phi : \mathbb{R}^2 \to \mathbb{R}$ with

$$S[\phi] = \frac{g}{2}\int d^2x \partial_\mu\phi(x)\partial^\mu\phi(y)$$
$$= \frac{1}{2}\int d^2x \int d^2y\phi(x)A(x,y)\phi(y)$$

as the action[21] where

$$A(x,y) = -g\delta(x^\mu - y^\mu)\partial_\mu\partial^\mu$$

[21] Notice that this is a scalar field whose action is obviously invariant with respect to translations, rotations, and scale trasformations. A little work shows that it is also SCT invariant.

The two point correlation function is based upon

$$\mathcal{R}(\phi(x)\phi(y)) = G_F(x,y)$$

the *Feynman propagator*, which is defined to be

$$G_f(x,y) = A^{-1}(x,y)$$

in the sense that[22]

$$\int d^2yA(x,y)G_F(y,z) = \delta(x^\mu - z^\mu)$$

[22] That is, in a very loose sense, inverse under the inner product on distributions of $\mathbb{R}^2$.

This implies that

$$\boxed{\delta(x^\mu - z^\mu) = -g\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}G_f(x,z)} \qquad (*)$$

As we saw previously, rotational and translational invariance means

$$G_F(x,y) = G_F(|x-y|) = G_F(r)$$

and, expressing the equation $(*)$ in polar coordinates and integrating, we get

$$1 = 2\pi\int dr\left[r\left(-g\frac{1}{r}\partial_r r\partial_r G_F(r)\right)\right]$$

which, when solved, yields

$$G_F'(r) = -\frac{1}{2\pi g r}$$

so that

$$\boxed{G_F(x,y) = -\frac{1}{2\pi g}\log|x-y| + \text{const}}$$

In complex coordinates[23], we can rewrite the action as

$$S[\phi] = 2g\int d^2z\partial_z\phi(z,\overline{z})\partial_{\overline{z}}\phi(z,\overline{z})$$

[23] Where
$$d^2z = \frac{i}{2}dzd\overline{z} = dx^2$$

so that

$$\boxed{\mathcal{R}(\phi(z,\overline{z})\phi(w,\overline{w})) = -\frac{1}{4\pi g}\left(\log(z-w) + \log(\overline{z}-\overline{w}) + \text{const}\right)}$$

For a chiral (holomorphic) field, we have $\partial_{\overline{z}}\phi(z,\overline{z}) = 0$, so[24]

$$\boxed{\mathcal{R}(\partial_z\phi(z,\overline{z})\partial_w\phi(w,\overline{w})) \sim -\frac{1}{4\pi g}\frac{1}{(z-w)^2}}$$

[24] Note that, for free fields there is only a single singular term in the OPE.

Energy-Momentum tensor

We take an infinitesmal symmetry transformation

$$\tilde{x}^\mu = x^\mu + \epsilon^\mu$$

acting on $S[\phi]$.

$$\tilde{\phi}(\tilde{x}^\mu) = \phi(x^\mu)$$

Then we get

$$\begin{aligned} T_{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - \eta_{\mu\nu}\mathcal{L} \\ &= g \partial_\mu \phi \partial_\nu \phi - \frac{g}{2}\eta_{\mu\nu}\partial_\rho \phi \partial^\rho \phi \end{aligned}$$

so the tensor is symmetric and traceless (as expected from a scale invariant theory).

In complex coordinates, we can compute (cf section 2.5)

$$\begin{aligned} T(z) &= -2\pi \frac{1}{4}(T_{1,1} - 2iT_{1,2} - T_{2,2}) \\ &= -2\pi g \partial_z \phi(z,\overline{z})\partial_z \phi(z,\overline{z}) \end{aligned}$$

From quantum theory, we have

$$\langle 0 \mid T(z) \mid 0 \rangle = 0$$

so that

$$\boxed{T(z) = -2\pi g \colon \partial_z \phi(z,\overline{z})\partial_z \phi(z,\overline{z} \colon}$$

OPEs

(A) We compute the radial ordering

$$\begin{aligned} \mathcal{R}(T(z)\partial_w \phi(w,\overline{w})) &= -2\pi g \colon \partial_z\phi\partial_z\phi \colon \partial_w \phi \\ &\sim -2\pi g \left(\colon \partial_z \phi \partial_z \phi \colon \partial_w \phi + \colon \partial_z \phi \partial_z \phi \colon \partial_w \phi \right) \\ &\sim \frac{\partial_z \phi(z,\overline{z})}{(z-w)^2} \\ &\sim \frac{\partial_z \phi(w,\overline{w})}{(z-w)^2} + \frac{\partial_z \phi(w,\overline{w})}{(z-w)} \end{aligned}$$

Which tells us that $\partial_z \phi(z,\overline{z})$ is a chiral primary field with conformal weight $h = 1$. Similarly, we have that $\partial_{\overline{z}}\phi(z,\overline{z})$ is an anti-chiral primary with conformal weight $\overline{h} = 1$.

(B) We now work on the energy-momentum tensor

$$
\begin{aligned}
\mathcal{R}(T(w)T(w)) &\sim 4\pi^2 g^2 \left(:\partial_z\phi\partial_z\phi:\ :\partial_w\phi\partial_w\phi:\right) \\
&\sim 4\pi g\left(2(:\partial_z\phi\partial_z\phi\partial_w\phi\partial_w\phi:) + 4:\partial_z\phi\partial_z\phi\partial_w\phi\partial_w\phi:\right) \\
&\sim \frac{1/2}{(z-w)^4} - 4\pi g\frac{:\partial_z\phi\partial_z\phi:}{(z-w)^2} \\
&\sim \frac{1/2}{(z-w)^4} + \frac{2(-2\pi g:\partial_w\phi\partial_w\phi:)}{(z-w)^2} - 4\pi g\frac{:\partial_w^2\phi\partial_w^{\phi}:}{(z-w)}
\end{aligned}
$$

Where the last step comes by Fourier expanding. In conclusion, we then have

$$
\boxed{\mathcal{R}(T(z)T(z)) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}}
$$

where $c = 1$.

Remark. Note that $T(z)$ is <u>not</u> primary because of the circled term. It is 'almost' a chiral primary field of conformal weight $h = 0$

c is called the *central charge*. For the free boson, as we have seen, $c = 1$.

EXAMPLE II: THE FREE FERMION CFT We take the Euclidean action of a real (Majorana) fermion[25]

$$
\begin{aligned}
S[\psi] &= \frac{g}{2}\int d^2x\Psi^\dagger\gamma^2\gamma^\mu\partial_\mu\Psi \\
&= g\int d^2x\Psi^\dagger\begin{bmatrix}\partial_z & & \\ & ddots & \\ & & \partial_z\end{bmatrix}\Psi \\
&= g\int d^2z\left(\psi\partial_z\psi + \overline{\psi}\partial_z\overline{\psi}\right)
\end{aligned}
$$

where $\Psi = (\overline{\psi}, \psi)$.

The equations of motion are then

$$
\begin{aligned}
&\partial_{\overline{z}}\psi = 0 \Rightarrow \psi \text{ is a chiral (holomorphic) field} \\
&\partial_z\overline{\psi} = 0 \Rightarrow \overline{\psi} \text{ is an anti-chiral field}
\end{aligned}
$$

We can then calculate the Feynman Propagator[26]

$$
\begin{aligned}
S[\Psi] &= \frac{1}{2}\int d^2x\int d^2y\Psi^\dagger(x)A(x,y)\Psi(y) \\
A(x,y) &= g\delta(x^\mu - y^\mu)\gamma^2\gamma^\mu\partial_\mu
\end{aligned}
$$

[25] Where the γ^μ are Pauli matrices. In principle, we could take any, by here we take

$$\gamma^1 = \begin{bmatrix}0 & -i \\ i & 0\end{bmatrix}$$

and

$$\gamma^2 = \begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix}$$

so that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

[26] As before, the idea is to write the action in terms of a kernel which we then invert

We then obtain the Feynman propagators[27]

$$\boxed{\begin{aligned}
&\mathcal{R}(\psi(z,\overline{z})\psi(w,\overline{w})) \sim \frac{1}{2\pi g}\frac{1}{z-w}\\
&\mathcal{R}(\overline{\psi}(z,\overline{z})\overline{\psi}(w,\overline{w})) \sim \frac{1}{2\pi g}\frac{1}{\overline{z}-\overline{w}}\\
&\mathcal{R}(\psi(z,\overline{z})\overline{\psi}(w,\overline{w})) = 0
\end{aligned}}$$

[27] Notice that the fermi statistics is encoded within the propagator.

We can also compute the energy-momentum tensor using Noether's theorem

$$\begin{aligned}
T(z) &= -\pi g \colon \psi \partial_z \psi \colon\\
\overline{T}(\overline{z}) &= -\pi g \colon \overline{\psi} \partial_z \overline{\psi} \colon
\end{aligned}$$

OPEs

(A) We have two possible contractions to worry about in computing the radial ordering

$$\begin{aligned}
\mathcal{R}(T(z)\psi(w,\overline{w}) &\sim \pi g \colon \psi \partial_z \psi \colon \psi(w,\overline{w})\\
&\sim \underbrace{\frac{1}{2}\frac{\partial_z\psi(z,\overline{z})}{(z-w)}}_{\text{red}} - \underbrace{\frac{1}{2}\psi(z,\overline{z})\partial_z\frac{1}{z-w}}_{\text{orange}}\\
&\sim \frac{1}{2}\frac{\partial_w\psi(w,\overline{w})}{(z-w)^2} + \frac{\partial_w\psi(w,\overline{w})}{(z-w)}
\end{aligned}$$

which implies that $\psi(z,\overline{z})$ is a chiral primary with conformal weight $h=\frac{1}{2}$. Similarly, $\overline{\psi}(z,\overline{z})$ can be shown to be an anti-chiral primary with conformal weight $\overline{h}=\frac{1}{2}$

(B) We also have, for the energy-momentum tensor

$$\mathcal{R}(T(z)T(w)) \sim \frac{\frac{1/2}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}$$

which tells us that the free Majorana fermion has central charge $c=\frac{1}{2}$.

2.7 The central charge c

Observation:

$$\mathcal{R}(T(z)T(w)) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}$$

We call c the *central charge*. As we have already calculated, the free boson has $c=1$, and the free Majorana fermion has $c=\frac{1}{2}$.

PROPERTIES

- For dilations $z \mapsto \lambda z$, we have

$$T(z) \mapsto \lambda^{-2} T(z)$$

which, in turn, implies

$$\mathcal{R}(T(z)T(w)) \mapsto \lambda^{-4}\mathcal{R}(T(z)T(w))$$

so the central charge is consistent with the expected scaling relation.

- For a sum of decoupled CFTs $\mathrm{CFT}_1, \dots, \mathrm{CFT}_N$, the energy-momentum tensor is

$$T_{total}(z) = \sum_{i=1}^{N} T_1(z)$$

And, since the theories are decoupled, this means

$$\mathcal{R}(T_{total}(z), T_{total}(w)) = \sum_{k=1}^{N} \mathcal{R}(T_k(z)T_k(w))$$

In turn, this means that we get

$$c_{total} = c_1 + \cdots + c_N$$

ie, the central charge is an extensive quantity.

- Transformation behavior under infinitesmal conformal transformations

$$x^\mu \mapsto \tilde{x}^\mu = x^\mu + \epsilon^\mu(x)$$

yields

$$\begin{aligned}
\delta_\epsilon T(w) = -[Q_\epsilon, T(w)] &= -\oint \frac{dz}{2\pi i}\epsilon(z)\mathcal{R}(T(z)T(w)) \\
&= \oint \frac{dz}{2\pi i}\left[\frac{\epsilon(z)\frac{c}{2}}{(z-w)^4} + \frac{z\epsilon(z)T(w)}{(z-w)^2} + \frac{\epsilon(z)\partial_w T(w)}{z-w}\right] \\
&= -\frac{c}{12}\left(\partial_w^3\epsilon(w)\right) - 2(\partial_w\epsilon(w))T(w) - \epsilon(w)\partial_w T(w)
\end{aligned}$$

A <u>finite</u> local conformal transformation takes the form

$$z \mapsto \tilde{z} = w(z)$$

so that

$$\tilde{T}(\tilde{z}) = \left(\frac{dw}{dz}\right)^{-2}\left(T(z) - \frac{c}{12}\{w(z); z\}\right)$$

with the *Schwarzian derivative*[28]

[28] Interestingly, while the invariance properties proved below related the Schwarzian derivative to the complex plane, it also appears to bear some relation to solutions of the hypergeometric equation.

$$\{w(z);z\} = \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2$$

For $z \mapsto u(z)$ and $u \mapsto w(u)$ we have the following 'chain rule' for the Schwarzian derivative

$$\{w(u(z));z\} = \{u(z);z\} + \frac{du}{dz}\{w(u);u\} \qquad (*)$$

Now, moving on to global conformal transformations in $PSL(2,\mathbb{C})$, we have

$$z \mapsto \tilde{z} = \frac{az+b}{cz+d}$$

Such transformations are generated by

(i) $z \mapsto \lambda z$, which has Schwarzian derivative

$$\{\lambda z; z\} = 0$$

(ii) $z \mapsto z + c$, which has Schwarzian derivative

$$\{z+c; z\} = 0$$

(iii) $z \mapsto -\frac{1}{z}$, which has Schwarzian derivative

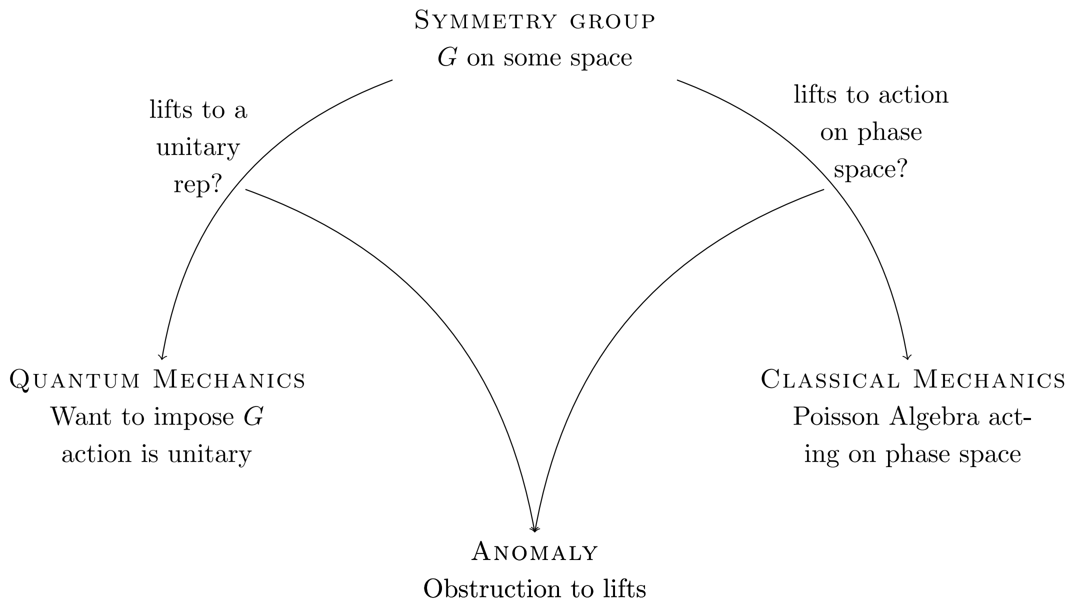
$$\begin{aligned}\{\frac{1}{z};z\} &= \frac{-3!z^{-4}}{-z^{-2}} - \frac{3}{2}\left(\frac{-2z^{-3}}{-z^{-2}}\right)^2 \\ &= 6z^{-2} - 6z^{-2} = 0\end{aligned}$$

As a result, we see that $\{w(z);z\} = 0$ for any $PSL(2,\mathbb{C})$ transformation. This implies that for such transformations

$$\tilde{T}(w) = \left(\frac{dw}{dz}\right)^{-2} T(z)$$

and thus, $T(z)$ is a chiral quasi-primary field with conformal weight $h = 2$.

Conformal anomaly and central extensions



In some cases of anomalies, we can find a group extension $\hat{G}$ of G by some group H[29] such that a representation can be constructed with respect to $\hat{g}$

[29] That is, a $\hat{G}$ fitting into a short exact sequence

$$0 \to H \to \hat{G} \to G \to 0$$

Example. In quantum mechanics in three dimensions, consider the symmetry group $SO(3)$, and the state space of spin $\frac{1}{2}$ states

$$\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$$

The action of

$$\begin{aligned} e^{2\pi i J_3}\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle &= e^{2\pi i(\pm\frac{1}{2})}\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \\ &= -\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle \end{aligned}$$

does not quite agree with identites.

However, there is an extension of $SO(3)$ by $\mathbb{Z}/2$, ie

$$SU(2)/(\mathbb{Z}/2) \cong SO(3)$$

Which implies that, taking $\hat{G} = SU(2)$, and considereding

$$\begin{bmatrix} -1 & \\ & -1 \end{bmatrix} \in SU(2)$$

we get the desired result.

Moreover, this extension is *central.* That is, $\mathbb{Z}/2$ is in the center of $SU(2)$.

Example (Classical Mechanics)**.** We can find an explicit central extension of the Galileo group, see [2] for details.

Central Extensions of the deWitt Algebra

We consider the OPE of $T(z)$, the infinitesmal generator for local conformal transformations.

$$\mathcal{R}(T(z), T(w)) \sim \frac{c/2}{(z-w)^4} + \frac{zT(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}$$

Recall that $T(w)$ has the expansion

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n$$
$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$$

which gives us the commutation relations

$$[L_n, L_m] = \oint_0 \frac{dw}{2\pi i} \oint_w \frac{dz}{2\pi i} w^{m+1} z^{n+1} \mathcal{R}(T(z)T(w))$$

From these relations, we retrieve the Virasoro Algebra $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$[30]

Claim. Vir is a central extension of the deWitt algebra[31] $\mathcal{A}$ by $\mathbb{C}$. That is, there is a short exact sequence of Lie algebra homomorphisms

$$0 \to \mathbb{C} \xrightarrow{f} \mathrm{Vir} \xrightarrow{g} \mathcal{A} \to 0$$

where $\mathbb{C}$ is generated by c, and there includes into the center of Vir, and f, g respect the Lie bracket of Lie algebras.

Remark. Vir is the unique non-trivial[32] central extension of $\mathcal{A}$.

To see why this is the case, we first comment that non-trivial central extensions are classified by the cohomology group

$$H^2(\mathfrak{g}, \mathfrak{a}) = \frac{Z^2(\mathfrak{g}, \mathfrak{a})}{B^2(\mathfrak{g}, \mathfrak{a})}$$

Where

(a) $\Theta \in Z^2(\mathfrak{g}, \mathfrak{a})$ means that

$$\Theta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{a}$$
$$\Theta(X, Y) = -\Theta(Y, X)$$
$$\Theta(X, [Y, X]) + \Theta(Z, [X, Y]) + \Theta(Y, [Z, X]) = 0$$

we call such Θ *cocycles*

(b) $\Theta \in B^2(\mathfrak{G}, \mathfrak{a})$, the *exact cocycles*, are characterized by the property that there exists

$$\mu : \mathfrak{g} \to \mathfrak{a}$$

such that

$$\Theta(X, Y) = \mu([X, Y])$$

[30] This algebra is completely characterized by the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}$$
$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} + \frac{\bar{c}}{12} n(n^2-1)\delta_{n+m,0}$$
$$[L_n, \overline{L}_m] = 0$$

[31] Whose commutation relations are of the form

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}$$

[32] Non-trivial meaning the sequence does not *split*, that is, is not equivalent to the sequence

$$0 \to \mathfrak{a} \to \mathfrak{a} \oplus \mathfrak{g} \to \mathfrak{g} \to 0$$

Given a representative of an equivalence class $[\Theta] \in H^2(\mathfrak{g}, \mathfrak{a})$, we see that Θ defines a Lie bracket for $\mathfrak{h}$ (The direct sum of $\mathfrak{g}$ and $\mathfrak{a}$ as vector spaces) by the formula[33]

$$[L \oplus \Theta, L' \oplus \Theta]_{\mathfrak{h}} = [L, L'] + \Theta([L, L'])$$

Since the Θ is exact, we have

$$\tilde{L} = L \oplus_{\mu} L$$

and hence

$$[\tilde{L}, \tilde{L}']_{\mathfrak{h}} = [\tilde{L}, \tilde{L}']_{\mathfrak{g}}$$

Now, returning to the case of the de Witt algebra $\mathcal{A}$, one can compute that

$$H^2(\mathcal{A}, \mathbb{C}) \cong \mathbb{C} \cong \langle \Theta \rangle$$

where

$$\Theta(L_n, L_m) = \frac{c}{12} n(n^2 - 1)\delta_{n+m,0}$$

This tells us that Vir is unique as a non-trivial central extension[34].

[33] More generally, the second *Lie Algebra Cohomology* classifies extensions of a Lie algebra $\mathfrak{g}$ by a module over $\mathfrak{g}$. See, for example [3] pp. 234 or [4] pp. 153 for a purely mathematical treatment.

[34] For a more thorough treatment, see [5] ch. 4 & 5.

The physical interpretation of c

(1) **Casimir Energy**[35]: Previously we mapped the cylindrical worldsheet of a string to the (punctured) conformal plan. If we now do the reverse, we get a map

$$z \mapsto \zeta = \frac{L}{2\pi} \log(z)$$

where L is the circumference of the cylinder. In these new coordinates,

$$T_{cyl}(\zeta) = \underbrace{\left(\frac{2\pi}{L}\right)^2 z^2}_{\left(\frac{d\zeta}{dz}\right)^2} \left(T(z) - \frac{c}{12} \underbrace{\overbrace{\{\log(z), z\}}^{\text{Schwarz. Deriv.}}}_{\frac{1}{2}z^{-2}} \right)$$

So that

$$\boxed{T_{cyl}(z) = \left(\frac{2\pi}{L}\right)^2 \left(z^2 T(z) - \frac{c}{24}\right)}$$

Assuming that the expectation values satisfy

$$\langle T(z) \rangle = \langle \overline{T}(\overline{z}) \rangle = 0$$

then

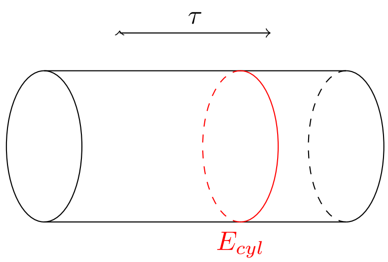
$$\begin{aligned} \langle T_{0,0} \rangle &= -\frac{1}{2\pi} \left(\langle T_{cyl}(\zeta) \rangle + \langle \overline{T}_{cyl}(\overline{\zeta}) \rangle \right) \\ &= \frac{\pi(c + \overline{c})}{12L^2} \end{aligned}$$

[35] The Casimir effect is a phenomenon whereby a force is observed between two plates positioned extremely close together.

This allows us to compute an energy for the cylinder

$$E_{cyl} = -\int_0^L \langle T_{0,0} \rangle = -\frac{\pi(c+\bar{c})}{12L}$$

which is the Casimir energy.



Notice that, as expected, if $L \to \infty$, we have $E_{cyl} \to 0$

Casimir Energy of the Free Scalar Field (Heuristic derivation):

We can compute the ground state energy of a sum of harmonic oscillators

$$E_{|0\rangle} = \sum_{p \in \{\text{Harm. Osc.}\}} \frac{1}{2}\omega_p$$

which yields[36]

$$E_{|0\rangle} = 2\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{2\pi}{L}\cdot k\right)$$

[36] We can think of our harmonic oscillators as merely being waves on a τ-slice. The factor of 2 appearing in the equation can then be said to arise as a result of left- and right-moving waves (sin and cos respectively).

To attempt to remove this divergence, we introduce the UV regulator

$$\epsilon = \frac{L}{\Lambda_{UV}} \ll 1$$

and add an exponential damping term

$$\begin{aligned} E^{\epsilon}_{|0\rangle} &= 2\sum_{k=1}^{\infty} \frac{1}{2}\left(\frac{2\pi}{L}\cdot k\right)e^{-\epsilon k} \\ &= \frac{2\pi}{L}\frac{d}{d\epsilon}\left(\sum_{k=0}^{\infty} e^{-\epsilon k}\right) \\ &= \frac{2\pi}{L}\left(\frac{1}{\epsilon^2} - \frac{1}{12} + \mathcal{O}(\epsilon)\right) \end{aligned}$$

We then define the regularized energy to be

$$E^{reg}_{|0\rangle} = \lim_{\epsilon \to 0}\left(E^{\epsilon}_{|0\rangle} - \frac{2\pi}{L\epsilon^2}\right) = -\frac{\pi}{6L}$$

And, since for the scalar field $c = \bar{c} = 1$,

$$E^{reg}_{|0\rangle} = -\frac{\pi}{12L}(c+\bar{c})$$

(2) **Trace anomaly on a closed oriented Riemann Surface $\mathcal{C}_g$ of genus g:**

On the classical level, $T^\mu_\mu = 0$.

On the quantum level, we get

$$\langle T^\mu_\mu \rangle_{\mathcal{C}_g} \sim c \cdot R$$

where R is the Ricci scalar[37].

The proportionality constant can be computed from the free fermionic CFT. See [6] for more details. This yields

$$\boxed{\langle T^\mu{}_\mu \rangle = \frac{c}{24\pi} R}$$

[37] The term on the left hand side is called the *trace anomaly.* Notice that R is a local quantity which depends locally on the metric and has scaling dimension 2. This dependence on the metric is acceptable since a conformal transformation is also a metric transformation.

2.8 The Hilbert Space of States

As before we consider the OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{zT(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}$$

with

$$T(z) = \sum_n z^{-n-2} L_n$$

and the commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)$$

In this system the vacuum, $T(z)|0\rangle$ for $z \to 0$ must be well defined, which implies that[38]

$$L_n|0\rangle = \overline{L}_n|0\rangle = 0$$

for $n > -2$.

[38] This follows from the fact that $T(z)|0\rangle$ must not have singular terms.

In particular, the vacuum $|0\rangle$ invariant under global conformal transformations, which are generated by $L_{\pm 1}$, L_0, $\overline{L}_{\pm 1}$, and $\overline{L_0}$.

Raising and Lowering operators of primaries

We consider a primary of conformal weights $(h, \overline{h})$

$$\phi(w, \overline{w}) = \sum_{k,m} w^{-m-h} \overline{w}^{-k-\overline{h}} \phi_{m,k}$$

And the commutation relations[39]

$$\begin{aligned}[L_n, \phi(w,\overline{w})] &= \oint_w \frac{dw}{2\pi i} \mathcal{R}(T(z)\phi(w,\overline{w})) z^{n+1} \\ &\quad - h(n+1) w^n \phi(w,\overline{w}) + w^{n+1} \partial_w \phi(w,\overline{w})\end{aligned}$$

[39] Following from the OPE

$$T(z)\phi(w,\overline{w}) \sim \frac{hT(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}$$

Which, in turn, implies[40]

[40] Because $\phi_{-m,-k}|0\rangle$ vanishes for $m < h$, $k < \overline{h}$.

$$\boxed{[L_n, \phi_{m,k}] = (n(h-1) - m)\phi_{n+m,k}}$$

We define an *asymptotic state*

$$|h, \overline{h}\rangle := \phi(0,0)|0\rangle = \phi_{-h,-\overline{h}}|0\rangle$$

And get the action of L_0

$$L_0|h, \overline{h}\rangle = [L_0, \phi(0,0)]|0\rangle = h|h, \overline{h}\rangle$$

analogously, we see that

$$\overline{L}_0|h, \overline{h}\rangle = \overline{h}|h, \overline{h}\rangle$$

We then see that $|h, \overline{h}\rangle$ is the energy eigenstate of the Hamiltonian

$$H = L_0 + \overline{L}_0$$

The commutation relations

$$[L_0, \phi_{m,k}] = -m\phi_{m,k}$$

tell us that $\phi_{m,k}$ are raising operators for $m < 0$ and lowering operators for $m > 0$.

Verma Module

The Virasoro raising operators L_{-m} $(m > 0)$ acting on $|h, \overline{h}\rangle$ yield

$$\begin{aligned} [L_0, L_{-m}] &= mL_{-m} \\ \Rightarrow L_0 L_{-m}|h, \overline{h}\rangle &= (m+h)|h, \overline{h}\rangle \end{aligned}$$

By similar arguments, we see that L_m for $m > 0$ are lowering operators

$$\begin{aligned} L_m|h, \overline{h}\rangle &= L_m \phi_{-h,-\overline{h}}|0\rangle \\ &= (m(h+1) + h)\phi_{-h+m,-\overline{h}}|0\rangle \\ &= 0 \end{aligned}$$

where the last equality holds because $\phi_{-m,-h}|0\rangle = 0$ for $m < h$.

Definition. A *descendant state* is given by acting with a chain of raising operators $L_{-m_1}, \ldots, L_{-k_n}$ on the asymptotic state $|h, \overline{h}\rangle$. Any descendant state can be written in the form

$$L_{-k_1} \cdots L_{-k_n}|h, \overline{h}\rangle$$

with the ordering convention $1 \leq k_1 \leq k_2 \cdots \leq k_n$

The Hilbert space of states arising from $|h, \overline{h}\rangle$ is called a *Verma modules*. It forms a representation of the Virasoro algebra[41]

[41] ie a module over the Virasoro algebra.

2.9 Conformal Families and Operator Algebra

Recall that a descendant state[42]

$$
\begin{aligned}
L_{-n}|h,\overline{h}\rangle &= [L_{-n},\phi(0,0)]|\rangle \\
&= \oint_0 \frac{dz}{2\pi i}\frac{\mathcal{R}(T(z)\phi(0,0)}{z^{n-1}}|0\rangle
\end{aligned}
$$

for $n\geq 1$ yields a definition of a *descendant field* (secondary field):

$$\phi^{\{u\}}(w,\overline{w}) = \oint_w \frac{dz}{2\pi i}\frac{\mathcal{R}(T(z)\phi(w,\overline{w}))}{(z-w)^{n-1}}$$

for $n\geq 1$. Recursively, we have descendant states:

$$\phi^{\{k_s,k_{s-1},\dots,k_1\}}(w,\overline{w}) = \oint_w \frac{dz}{2\pi i}\frac{\mathcal{R}(T(z)\phi^{\{k_{s-1},\dots,k_1\}}(w,\overline{w})}{(z-w)^{k_{s-1}}}$$

where $k_i\geq 1$

Remark.

- We use the short-hand notation
$$\phi^{\{\vec{k}\}}(w,\overline{w}) = \phi^{\{k_s,k_{s-1},\dots,k_1\}}(w,\overline{w})$$
- If we include anti-holomorphic descendants[43], we get states
$$\phi^{\{\vec{k}\},\{\vec{\overline{k}}\}}(w,\overline{w})$$

We can also compute correlation functions of descendants, for instance:

$$
\begin{aligned}
\langle\phi_0^{\{k\}}(w_0)\phi_1(w_1)\cdots\phi_N(w_N)\rangle &= \oint_{w_0}\frac{dz}{2\pi i}\frac{1}{(z-w_0)^{k-1}}\langle T(z)\phi_0(w_0)\cdots\phi_N(w_N)\rangle \\
&= \oint_{w_0}\frac{dz}{2\pi i}\frac{1}{(z-w_0)^{k-1}} \\
&\times\left(\sum_{n=0}^{N}\left(\frac{1}{(z-w_n)}\partial_{w_n}+\frac{h_n}{(z-w_n)^2}\right)\langle\phi_0(w_0)\cdots\phi_N(w_N)\rangle\right)
\end{aligned}
$$

where the second equality follows from the Ward identity (cf section 2.5).

The correllators

$$\langle T(z)\phi_0(w_0)\cdots\phi_N(w_N)\rangle$$

can have poles **only** at $z\to w_k$. This implies that

$$
\begin{aligned}
\langle\phi_0^{\{k\}}(w_0)\phi_1(w_1)\cdots\phi_N(w_N)\rangle &= -\sum_{\ell=1}^{N}\oint_{w_\ell}\frac{dz}{2\pi i}\frac{1}{(z-w_0)^{k-1}} \\
&\times\left(\sum_{n=0}^{N}\left(\frac{1}{z-w_n}\partial_{w_n}+\frac{h_n}{(z-w_n)^2}\right)\langle\phi_0\cdots\phi_N\rangle\right) \\
&= -\sum_{\ell=1}^{N}\left(\frac{\partial_{w_\ell}}{(w_\ell-w_0)^{k-1}}+(-(n-1))\frac{h_e ll}{(w_\ell-w_0)^k}\right) \\
&\times\langle\phi_0\cdots\phi_N\rangle
\end{aligned}
$$

[42] Recall:
$$T(z)=\sum_n z^{-n-2}L_n$$

[43] Using the operator $\overline{L}_{-n}$ instead of L_{-n}.

So that

$$\boxed{\begin{gathered}\langle \phi^{\{k\}}(w_0)\cdots\phi_N(w_N)\rangle = \mathcal{L}_{-k}\langle \phi_0(w_0)\cdots\phi_N(w_N)\rangle \\ \text{where} \\ \mathcal{L}_{-k} = \sum_{\ell=1}^{N}\left(\frac{(k-1)h_\ell}{(w_\ell - w_0)^k} - \frac{1}{(w_\ell - w_0)^{k-1}}\partial_\ell\right)\end{gathered}}$$

In general, any correlator

$$\langle \phi_0^{\{\vec{k_0}\}}\cdots\phi_N^{\{\vec{k_N}\}}\rangle$$

can be rewritten in terms of differential operators acting on

$$\langle \phi_0\cdots\phi_N\rangle$$

with the help of the Ward identity. Therefore,

$$\langle \phi_0\cdots\phi_N\rangle$$

determines **all** descendant correlators.

Definition. The set of a primary field ϕ together with all its descendant fields is called the *conformal family* of ϕ, which we denote by $[\phi]$.

Conformal families have the following properties:

- Members of the conformal family $[\phi]$ transform under (local) conformal transformations among themselves. That is, OPEs of $T(z)$ with any field of $[\phi]$ will involve only fields of $[\phi]$.
- We have the *operator state correspondence*

$$\{\text{Conformal families}\} \overset{1:1}{\leftrightarrow} \{\text{Verma Modules}\}$$

The Operator Algebra

For a given CFT, the OPEs among all its primaries (including regular terms) form the so-called *operator algebra.* The knowledge of the operator algebra determines all correlators of the CFT (it 'solves' the CFT).

Given a CFT (with countably many primaries), we normalize primaries such that

$$\langle \phi_k(w,\overline{w})\phi_\ell(z,\overline{z})\rangle = \frac{\delta_{k\ell}}{(z-w)^{2h_k}(\overline{z}-\overline{w})^{2\overline{h}_\ell}}$$

Then[44],

$$\lim_{\substack{(w,\overline{w})\to\infty \\ (z,\overline{z})\to 0}} w^{2h_k}\overline{w}^{2h_k}\langle \phi_k(w,\overline{w})\phi_\ell(z,\overline{z})\rangle = \langle h_k,\overline{h}_k|h_\ell,\overline{h}_\ell\rangle = \delta_{k\ell}$$

[44] See, for instance, the definitions of in and out states in section 2.3.

The orthogonality of primaries therefore implies the orthogonality of entire Verma modules. To see this, we can use the Virasora algebra for descendant states. For example:

$$L_{-n}|h_\ell, \overline{h}_\ell \text{ and } L_{-n_2}L_{-n_3}|h_k, \overline{h}_k\rangle \qquad (n_i > 0)$$

gives us

$$\begin{aligned}
\langle h_k, \overline{h}_k|L^\dagger_{-n_3}L^\dagger_{-n_2}|h_\ell, \overline{h}_\ell\rangle &= \langle h_k, \overline{h}_k|L_{n_3}L_{n_2}L_{-n_1}|h_\ell, \overline{h}_\ell\rangle \\
&= \langle h_k, \overline{h}_k|L_{n_3}[L_{n_2}, L_{-n_1}]|h_\ell, \overline{h}_\ell\rangle \\
&= \langle h_k, \overline{h}_k|L_{n_3}((n_2+n_1)L_{n_2-n_1}) \\
&\quad + \frac{c}{12}n_2^2(n_2^2-1)\delta_{n_2-n_1,0}|h_\ell, \overline{h}_\ell\rangle \\
&= (n_2+n_1)\langle h_k, \overline{h}_k|L_{n_3}L_{n_2-n_1}|h_\ell, \overline{h}_\ell\rangle \quad \substack{\text{non-vanishing} \\ \text{for } n_1>n_2} \\
&\overset{n_1>n_2}{=} (n_2+n_1)\langle h_k, \overline{h}_k|[L_{n_3}, L_{n_2-n_1}]|h_\ell, \overline{h}_\ell\rangle \\
&= (n_2+n_1)\langle h_k, \overline{h}_k|(n_3+n_1-n_2)L_{n_2-n_3-n_1} \\
&\quad + \frac{c}{12}n_3(n_3^2-1)\delta_{n_2+n_3-n_1,0}|h_\ell, \overline{h}_\ell\rangle \\
&= (n_2+n_1)\delta_{n_2+n_3,n_1}((n_3+n_1-n_2)h_\ell \\
&\quad + \frac{c}{12}n_3(n_3^2-1))\langle h_k, \overline{h}_k|h_\ell, \overline{h}_\ell\rangle \\
&= 0
\end{aligned}$$

Scale invariance determines the structure of the operator algebra:

$$\boxed{\begin{aligned}
\phi_k(z,\overline{z})\phi_\ell(0,0) = \sum_{[\phi_s]}\sum_{\{\vec{k}\}}\sum_{\{\overline{\vec{k}}\}} \Big[& C_{k\ell}^{s\{\vec{k}\}\{\overline{\vec{k}}\}} z^{h_s-h_k-h_\ell+|\vec{k}|} \\
& \times z^{\overline{h}_s-\overline{h}_k-\overline{h}_\ell+|\vec{k}} \phi_s^{\{\vec{k}\}\{\overline{\vec{k}}\}}(0,0)\Big]
\end{aligned}}$$

where

$$|\vec{k}| = \sum_n k_n$$

and

$$\vec{k} = (k_1, k_2, \ldots) \quad k_1 \le k_2 \le k_3 \cdots$$

The constants $C_{k\ell}^{s\{\vec{k}\}\{\overline{\vec{k}}\}}$ are the structure constants of the operator algebra.

Three-Point Correlator

We can compute the correlator

$$
\begin{aligned}
\langle\phi_s|\phi_k(z,\overline{z})|\phi_\ell\rangle &= \lim_{w,\overline{w}\to\infty} w^{2h_s}\overline{w}^{2h_s}\langle\phi_s(w,\overline{w})\phi_k(z,\overline{z})\phi_\ell(0,0)\rangle \\
&= \lim_{w,\overline{w}\to\infty} C_{sk\ell}\frac{w^{2h_s}}{z^{h_k+h_\ell-h_s}w^{h_s+h_\ell-h_k}(w-z)^{h_s+h_k-h_\ell}} \\
&\qquad \times \binom{\text{anti-holomorphic}}{\text{part}} \\
&= \frac{C_{sk\ell}}{z^{h_k+h_\ell-h_s}\overline{z}^{\overline{h}_k+\overline{h}_\ell-\overline{h}_s}}
\end{aligned}
$$

or, alternately

$$
\begin{aligned}
\langle\phi_s|\phi_k(z,\overline{z})|\phi_\ell\rangle &= \langle\phi_s|\sum_{[\phi_\ell]}\sum_{\{\vec{k}\}}\sum_{\{\overline{\vec{k}}\}}[\dots]\,|0\rangle \\
&= \sum_r C_{k\ell}^{r\{\}\{\}}\langle\phi_s|\phi_r\rangle z^{h_s-h_k-h_\ell}\overline{z}^{\overline{h}_s-\overline{h}_k-\overline{h}_\ell} \\
&= \frac{\delta_{sr}C_{k\ell}^{r\{\}\{\}}}{z^{h_k+h_\ell-h_s}\overline{z}^{\overline{h}_k+\overline{h}_\ell-\overline{h}_s}}
\end{aligned}
$$

so that

$$
\boxed{C_{k\ell}^{s\{\}\{\}} = C_{k\ell}^{s} = C_{sk\ell}}
$$

Since all correlators of descendants arise from correlators of primaries, one can show that:

$$
\boxed{C_{k\ell}^{s\{\vec{k}\}\{\overline{\vec{k}}\}} = C_{k\ell}^{s}\beta_{k\ell}^{s\{\vec{k}\}}\overline{\beta}_{k\ell}^{s\{\overline{\vec{k}}\}}}
$$

where

$$
\begin{aligned}
\beta_{k\ell}^{s\{\vec{k}\}} &= \beta_{k\ell}^{s\{\vec{k}\}}(h_s,h_k,h_\ell,c) \\
\beta_{k\ell}^{s\{\}} &= 1
\end{aligned}
$$

Example. Consider two chiral primaries $\phi_k(z)$ and $\phi_\ell(z)$ with $h = h_k = h_\ell$ (for simplicity), then

$$
\phi_k(z)\phi_\ell(0) = \sum_s C_{k\ell}^{s} z^{h_s-2h}X_s(z)
$$

where

$$
X_s(z) = \sum_{N=0}^{\infty}\sum_{\substack{\{\vec{k}\}\\ |\vec{k}|=N}} z^N\beta_{k\ell}^{s\{\vec{k}\}}L_{-\{\vec{k}\}}\phi_s(0)
$$

Hence

$$
X_s(z)|0\rangle = \sum_{N=0}^{\infty} z^N|N;h_s\rangle
$$

where $|H;h_s\rangle$ is the level N state descending from

$$
|\phi_s\rangle = |h_s\rangle
$$

We can then compute:

$$\begin{aligned}
L_n\phi_k(z)\phi_\ell(0) &= [L-n,\phi_k(z)]|h_\ell\rangle \\
&= \left(z^{n+1}\partial_z + (n+1)hz^n\right)\phi_k(z)|h_\ell\rangle
\end{aligned}$$

so that

$$\begin{aligned}
L_n\left(X_s(z)|0\rangle\right) &= \sum_{N=0}^{\infty} z^N L_n|N;h_s\rangle \\
&\overset{!}{=} \sum_{N=0}^{\infty}\left((N+h_s-2h)z^{n+n} + (n+1)hz^{N+n}\right)|N;h_s\rangle
\end{aligned}$$

Which gives us that

$$\boxed{L_n|N+n;h_s\rangle = (h_s+(n-1)h,+N)|N;h_s\rangle} \qquad (*)$$

For low N, we can then begin to examine what descendant states are produced:

<u>Level 1</u> There is one descendant state

$$|1;h_s\rangle = \beta_{k\ell}^{s\{1\}}L_{-1}|h_s\rangle$$

(i) $L_1|1;h_s\rangle \overset{(*)}{=} h_s|0;h_s\rangle = h_s|h_s\rangle$

(ii) $L_1|1;h_s\rangle = \beta_{k\ell}^{s\{1\}}\underbrace{[L_1,L_{-1}]}_{2L_0}|h_s\rangle = 2\beta_{k\ell}^{s\{1\}}h_2|h_2\rangle$

Hence,

$$\boxed{\beta_{kl}^{s\{1\}} = \frac{1}{2}}$$

<u>Level 2</u> There are 2 descendant states

$$|2,h_s\rangle = \beta_{k\ell}^{s\{2\}}L_{-2}|h_s\rangle + \beta_{k\ell}^{s\{1,1\}}L_{-1}L_{-1}|h_s\rangle$$

(i) We have

$$\begin{aligned}
L_1|2;h_s\rangle &\overset{(*)}{=} (h_s+1)|1;h_s\rangle = \frac{1}{2}(h_s+1)L_{-1}|h_s\rangle \\
L_s|2;h_s\rangle &\overset{(*)}{=} (h_s+h)|h_s\rangle
\end{aligned}$$

From (i) and (ii), we find a pair of linearly independent equations for $\beta_{k\ell}^{s\{1,1\}}$ and $\beta_{k\ell}^{s\{2\}}$:

$$\begin{aligned}
\beta_{k\ell}^{s\{1,1\}} &= \frac{c-12h-4h_s+ch_s+2h_s^2}{4(c-10h_s+2ch_s+16h_s^2)} \\
\beta_{k\ell}^{s,\{2\}} &= \frac{2h-h_s+4hh_s+h_s^2}{c-10h_s+2ch_s+16h_s^2}
\end{aligned}$$

Remark.

- All of the coefficients $\beta^{s\{\vec{k}\}}_{k\ell}(h_k, h_\ell, h_s, c)$ can, in principle, be recursively determined. As a result, the operator algebra of a CFT is determined completely by

  a) conformal families $[\phi_s]$

  b) three-point correlators of the primaries, ie $C_{sk\ell}$

- $C_{sk\ell}$ must be obtained separately (eg, by dynamic input or crossing symmetries).

2.10 Conformal Blacks & Crossing Symmetries

Having now treated three-point correlation functions, we an in a position to revisit four-point and higher correlation functions, ie

$$\langle \phi_k(z_1, \overline{z}_1)\phi_\ell(z_2, \overline{z}_2)\phi_m(z_3, \overline{z}_3)\phi_n(z_4, \overline{z}_4)\rangle$$

We apply a conformal transformationwhich sends

$$z_1 \to \infty, \quad z_2 \to 1, \quad z_3 \to \eta, \quad z_4 \to 0$$

where, η is, as usual, the cross ratio

$$\eta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$$

We can then write down a correlation function matrix element[45]

$$\begin{aligned} G^{\ell k}_{mn}(\eta, \overline{\eta}) &= \lim_{z, \overline{z} \to \infty} z^{2h_k} \overline{z}^{2h_k} \langle \phi_k(z, \overline{z})\phi_\ell(1,1)\phi_m(\eta, \overline{\eta})\phi_n(0,0)\rangle \\ &= \langle h_k, \overline{h}_k | \phi_\ell(1,1)\phi_m(\eta, \overline{\eta}) | h_n, \overline{h}_n\rangle \end{aligned}$$

[45] Where we use the indexing convention that the indices are read counterclockwise around the symbol, as in

$$G^{\ell k}_{mn}$$

If we then insert the operator product expansion of $\phi_m(\eta, \overline{\eta})$ and $\phi_n(0,0)$, we get

$$\begin{aligned} G^{\ell k}_{mn}(\eta, \overline{\eta}) = \Bigg\langle h_k, \overline{h})_k \mid \phi_\ell(1,1) \sum_{[\phi_p]} \sum_{\vec{k}, \vec{\overline{k}}} C^{p\{\vec{k}\}\{\vec{\overline{k}}\}}_{mn} \eta^{h_p - h_m - h_n - |\vec{k}|} \\ \times \overline{eta}^{\overline{h}_p - \overline{h}_m - \overline{h}_n - |\vec{\overline{k}}|} \times L_{-\{\vec{k}\}} \overline{L}_{-\{\vec{\overline{k}}\}} \mid h_p, \overline{h}_p \Bigg\rangle \end{aligned}$$

We can then compute

$$G^{\ell k}_{mn}(\eta, \overline{\eta}) = \sum_{[\phi_p]} C^p_{mn} C_{pk\ell} \mathcal{F}^{\ell k}_{mn}(p|\eta) \overline{\mathcal{F}}^{\ell k}_{mn}(p|\overline{\eta})$$

where

$$\mathcal{F}^{\ell k}_{mn}(p|\eta) = \eta^{h_p - h_m - h_n} \sum_{\{\vec{k}\}} \eta^{|\vec{k}|} \beta^{p\{\vec{k}\}}_{mn} \frac{\langle h_i|\phi_j(1) L_{-\{\vec{k}\}}|h_p\rangle}{\langle h_i|\phi_j(1)|h_p\rangle} \times L_{-\{\vec{k}\}} \overline{L}_{-\{\bar{\vec{k}}\}} |h_p, \overline{h}_p\rangle$$

The

$$\mathcal{F}^{\ell k}_{mn}(p|\eta)$$

are called *conformal blocks* of the conformal family $[\phi_p]$.

Remark.
- $\mathcal{F}^{\ell k}_{mn}(p|\eta)\eta^{h_n + h_m - h_p}$ are regular holomorphic functions at $\eta = 0$.
- The coefficients of the expansion of $\mathcal{F}^{\ell k}_{mn}(p|\eta)$ depend only on

$$c,\ h_p,\ h_\ell,\ h_k,\ h_m,\ h_n$$

  that is, there is **no** dependence on the 3-pt structure constants.
- For explicit expressions for $\mathcal{F}^{\ell k}_{mn}(p|\eta)$, see, eg, [7]

Crossing Symmetries

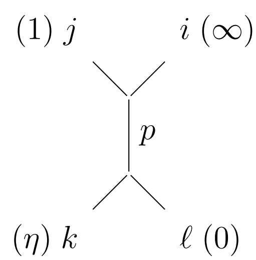
The $G^{ij}_{k\ell}$ depend on a particular order of primaries, which results in one particular OPE. We could, however, think of reordering the primaries to get a different expression.

We can diagrammatically represent the conformal blocks in the following way, permuting the indices each time[46]:

[46] Only the index circled in orange must be fixed under this permutation. This is because it determines both the out state and the $\eta^{h_i \cdots}$ regularization factor in the limit.

1)

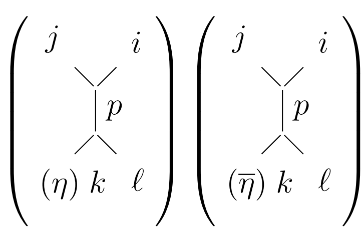
$$\mathcal{F}^{ji}_{k\ell}(p|x) =$$



which, in turn, implies that we can write the *s-channel* expression[47]

[47] From quantum field theory one would also expect a T-channel and a U-channel, which, as we will shortly see, also exist.

$$G^{ji}_{k\ell}(x) = \sum_p C^p_{ij} C_{pk\ell}$$

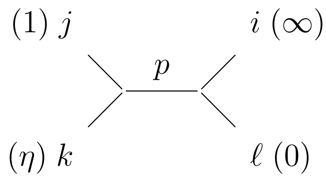


Here, we have

$$x = \frac{(\infty - 1)(\eta - 0)}{(\infty - \eta)(1 - 0)} = \eta$$

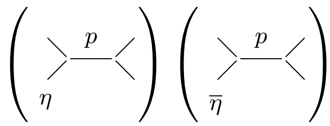
2)

$$\mathcal{F}^{ji}_{k\ell}(p|x) \ =$$



giving the *t-channel*

$$G^{\ell i}_{kj}(1-\eta, 1-\overline{\eta}) = \sum_p C^p_{i\ell} C_{pkj}$$

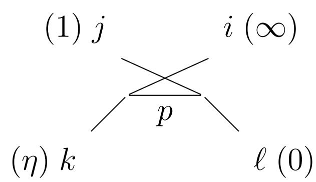


Here we have

$$x = \frac{(\infty - 0)(\eta - 1)}{(\infty - \eta)(0-1)} = 1 - \eta$$

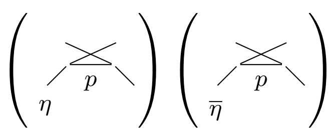
3)

$$\mathcal{F}^{ki}_{\ell j}(p|x) \ =$$



giving the *u-channel*

$$G^{\ell i}_{kj}\left(\frac{1}{\eta - 1}, \frac{1}{\overline{\eta} - 1}\right) = \sum_p C^p_{ik} C_{p\ell j}$$



here,

$$x = \frac{(\infty - \eta)(0-1)}{(\infty - 0)(\eta - 1)} = \frac{1}{\eta - 1}$$

We would expect these different interpretations to agree, that is, we impose the *crossing symmetries*

$$\boxed{\underbrace{G^{ji}_{k\ell}(\eta, \overline{\eta})}_{\text{s-channel}} = \underbrace{G^{\ell i}_{kj}(1-\eta, 1-\overline{\eta})}_{\text{t-channel}} = \underbrace{G^{ki}_{\ell j}\left(\frac{1}{\eta-1}, \frac{1}{\overline{\eta}-1}\right)}_{\text{u-channel}}}$$

Bootstrap approach: Specify the dynamics of a CFT by consistency using crossing symmetries.

Given a CFT with central charge c and N conformal families we have

$$\underbrace{N^3}_{C_{k\ell m}} + \underbrace{N}_{h_\ell}$$

parameters, for $k, \ell, m \in \{1, \ldots, N\}$. The crossing symmetries provide N^4 constraints, so naive counting suggests that crossing symmetries fix all unknown parameters. This is true in some cases, eg for minimal models.

Remark. Higher point correlators can be calculated analogously to the 4 point case.

3 Minimal Models

Definition. A *minimal model conformal field theory* is a conformal field theory with a finite numbe rof conformal families.

Example. The 2d Ising model at criticality.

Remark. For the purposes of this lecture, we will focus on unitary minimal models.

3.1 Warm-up: Representations of $\mathfrak{su}(2)$

The Lie algebra[48] $\mathfrak{su}(2)$ has generators J_3 and $J_\pm$[49], satisfying

$$\begin{aligned}[J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2J_3\end{aligned}$$

[48] More commonly studied are representations of the Lie *group* $SU(2)$, but for our purposes, we need the corresponding Lie algebra.

[49] It also admits a Casimir Operator

$$J^2 := J_3 + \frac{1}{2}\{J_+, J_-\}$$

To find representations of $\mathfrak{su}(2)$, we start with a highest weight state $|j\rangle$ with

$$\begin{aligned}J_3|j\rangle &= j|j\rangle \\ J_+|1\rangle &= 0\end{aligned}$$

We can then define a decendant state J_-^k, where

$$\begin{aligned}J_3(J_-^k|j\rangle) &= (j-k)|J_-^k|j\rangle \\ J_+(J_-^k|j\rangle) &= 2\sum_{\ell=0}^{k-1} J_-^{k-\ell-1} J_3 J_-^\ell |j\rangle \\ &= 2\left(\sum_{\ell=0}^{k-1}(j-\ell)\right) J_-^{k-1}|j\rangle \\ &= k(2j+1-k)J_-^{k-1}|j\rangle\end{aligned}$$

This implies that, for $j \in 2\mathbb{N}_0$, the value $k = 2j+1$ yields a highest weight state, ie

$$J_+\left(J_-^{2j+1}|j\rangle\right) = 0$$

The state

$$|\chi_{2j+1}\rangle := J_-^2 j + 1|j\rangle$$

is called a *singular vector*. $|\chi_{2j+1}\rangle$ generates a subrepresentation of the highest weight representation generated by $|j\rangle$.

Hilbert Space of States & Unitary Representations

Let $|\psi\rangle$ be a state with conjugate $\langle\psi|$,

$$J_3 = J_3^\dagger$$
$$J_\pm^\dagger = J_\mp$$

and take the normalization

$$\langle j|j\rangle$$

(i) For $2j \in \mathbb{N}_0$, if we have

$$\langle j|J_+^{k_1} J_-^{k_2}|j\rangle \neq 0$$

then $k_1 = k_2$. Additionally

$$\langle\chi_{2j+1}|\chi_{2j+1}\rangle = 0$$

and

$$\langle\chi_{2j+1}|J_+^k J_-^k|\chi_{2j+1}\rangle = 0$$

so that the subrepresentation generated by $|\chi_{2j+1}\rangle$ yields only null states.

We can therefore take our Hilber space of states to be the quotient of our representation by the relations

$$|\psi\rangle \sim |\psi\rangle + \alpha J_-^k|\chi_{2j+1}\rangle$$

That is

$$\mathcal{H} = \left\{J_-^k|j\rangle\right\}_{k=0,\dots,2j} = \langle|j\rangle\rangle \,/\, \langle|\chi_{2j+1}\rangle]\rangle$$

So we get a finite spin representation of $\mathfrak{su}(2)$[50]

[50] Which is, in addition, unitary.

(ii) Let $j > 0$, $2j \notin \mathbb{N}_0$: We then get a non-finite representation of $\mathfrak{su}(2)$. Pick $k = \lceil 2j+1 \rceil$, then

$$\langle j|J_+^k J_-^k|j\rangle = \underbrace{\langle j|J_+^{k-1} J_-^{k-1}|j\rangle}_{>0} \times k\, \underbrace{(2j+1-k)}_{<0}$$

We easily see that we get a negative norm state at level $\lceil 2j+1 \rceil$, giving us a non-unitary representation of $\mathfrak{su}(2)$.

Observation. • Singular vectors give rise to unitary (finite dimensional) representations of $\mathfrak{su}(2)$.

• Highest weight representations without singular vectors yield (infinite dimensional) non-unitary representations of $\mathfrak{su}(2)$.

3.2 Reducible Verma Modules & Singular Vectors

We explore highest weight representations of the Virasora algebra with central charge c. Let $|h\rangle$ be a highest weight state:

$$\begin{aligned} L_n|h\rangle &= 0 \quad \text{for } n > 0 \\ L_0|h\rangle &= h|h\rangle \end{aligned}$$

The *space of states* will then be

$$L_{-\{\vec{k}\}}|h\rangle$$

with inner product.

Definition. A *singular vector* is a descendant state $|\chi\rangle$ with $L_n|\chi\rangle = 0$ for all $n > 0$, that this, $|\chi\rangle$ is a highest weight state.

Observation. A singular vector and its descendants are orthogonal to any other state.

Proof. Let $|\chi\rangle$ be a singular vector at level N, so that it has an expansion as

$$|\chi\rangle = \sum_{\substack{\vec{k} \\ |\vec{k}|=N}} c_{\vec{k}} L_{-\{\vec{k}\}}|h\rangle$$

Then the state

$$\langle\psi|L_{-\{\vec{\ell}\}}|\chi\rangle$$

can only be non-vanishing if ψ is a state at level $N + |\vec{\ell}|$.

Let

$$|\psi\rangle = L_{-\{\vec{m}\}}|h\rangle$$

where $|\vec{m}| = N + |\vec{\ell}|$. Then

$$\begin{aligned} \langle\psi|L_{\{vec\ell\}}|\chi\rangle &= \langle h|L_{\{\vec{m}\}}L_{-\{\vec{\ell}\}}|\chi\rangle \\ &= \langle h| \left[L_{\{\vec{m}\}}, L_{-\{\vec{\ell}\}}\right] |\chi\rangle = 0 \end{aligned}$$

because

$$\left[L_{\{\vec{m}\}}, L_{-\{\vec{\ell}\}}\right] = \sum_{\vec{s}} D_{\vec{s}} L_{\vec{s}}$$

where $|\vec{s}| = |\vec{m}| - |\vec{\ell}| = N > 0$. □

Reducible Verma Module

Definition. A Verma module $V(c, h)$ is *reducible*if there is a singular vector at some level N, ie

$$|\chi\rangle = \sum_{\substack{\vec{k} \\ |\vec{k}|=N}} c_{\vec{k}} L_{-\{\vec{k}\}}|h\rangle$$

such that, for all $n > 0$, $L_n|\chi\rangle = 0$.

In this case, $|\chi\rangle$ is a highest weight state generating a Verma submodule V_χ of $V(c,h)$.

Remark.
- $|\chi\rangle$ is a primary state of conformal weight $h+N$, and it is also a descendant state of $|h\rangle$
- V_χ is orthogonal to $V(c,h)$ with respect to the defined inner product.
- Let $V(c,h)$ be a Verma Module with singular vectors $|\chi_i\rangle$, then we can construct *irreducible*[51] Verma module $M(c,h)$ by quotienting out all Verma submodules
$$M(c,h) = V(c,h)/\sim$$
where
$$|\phi\rangle \sim |\phi\rangle + |\psi\rangle$$
for all $|\phi\rangle \in V(c,h)$ and for all $|\psi\rangle \in \bigoplus_i V_{\chi_i}$.
- $M(c,h)$ are the building blocks of minimal models.

[51] See previous note.

Singular Vectors & Negative Norm States at Low Levels

Level 0 $\langle h|h\rangle = 1$: normalization condition.

Level 1 $L_{-1}|h\rangle$ has norm
$$\langle h|L_1L_{-1}|h\rangle = 2h\langle h|h\rangle = 2h$$
so that the *Gram matrix at level 1* is
$$M^{(1)} = (\langle h|L_1L_{-1}|h\rangle) = (2h)$$
That means we get a null state for $h = 0$[52], and a necessary condition for unitary representations is that
$$\boxed{h \geq 0}$$

[52] That is, $L_{-1}|0\rangle$ is a singular vector. This null state corresponds to the vacuum.

Level 2 We now have two possible states
$$|\psi_{1,1}\rangle = L_{-1}^2|h\rangle$$
$$|\psi_2\rangle - L_{-2}|h\rangle$$
We can compute the 2×2 Gram matrix at level 2,
$$M^{(2)}(c,h) = \begin{bmatrix} \langle\psi_{1,1}|\psi_{1,1}\rangle & \langle\psi_{1,1}|\psi_2\rangle \\ \langle\psi_2|\psi_{1,1}\rangle & \langle\psi_2|\psi_2\rangle \end{bmatrix} = \begin{bmatrix} 4h(2h+1) & 6h \\ 6h & 4h+\frac{1}{2}c \end{bmatrix}$$

We compute that the trace of this matrix is

$$\mathrm{tr} M^{(2)}(c,h) = 8h(h+1) + \frac{1}{2}c$$

and the determinant[53]

$$\det M^{(2)}(c,h) = 32(h-h_{1,1})(h-h_{1,2})(h-h_{2,1})$$

Each zero of this determinant will correspond to a singular vector. the first,

$$h_{1,1,} = 0$$

corresponds to the singular vector we have already found. The other two

$$h_{1,2} = \frac{1}{16}\left(5-c-\sqrt{(1-c)(25-c)}\right)$$
$$h_{2,1} = \frac{1}{16}\left(5-c+\sqrt{(1-c)(25-c)}\right)$$

are new

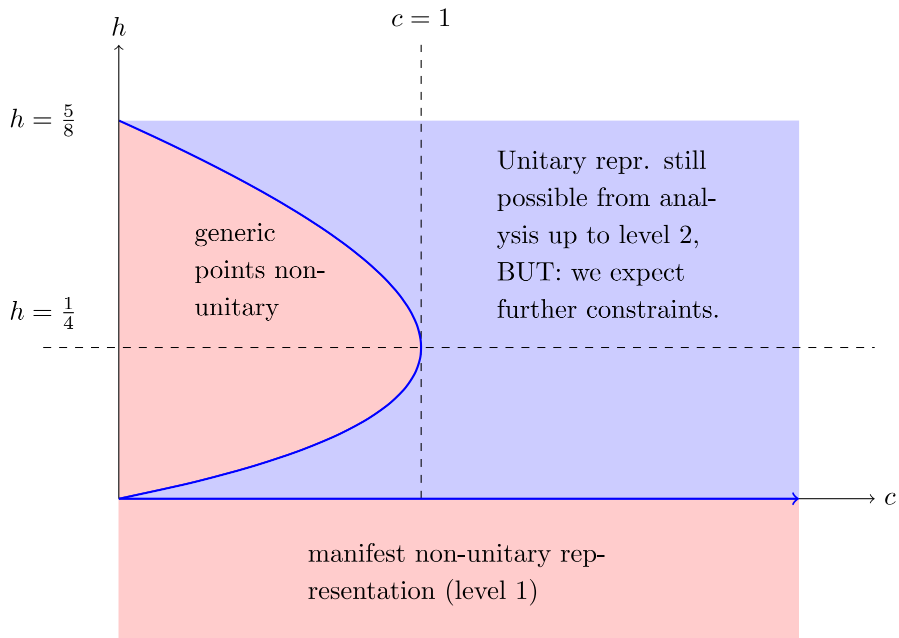
[53] Known as the *Kac determinant*.

Exercise. There are two singular vectors at level 2, $|\chi_{1,2}\rangle$ and $|\chi_{2,1}\rangle$ associated to $h = h_{1,2}$ and $h = h_{2,1}$ respectively.

A First Glance at Unitary Representations

We have singular vectors $h = h_{1,1} = 0$, and

$$h = h_{1,2}, h_{2,1} \Leftrightarrow 1 = \left(\frac{4}{3}h + \frac{2}{3}\right)\left(-\frac{4}{3}h - \frac{c}{6} + \frac{3}{2}\right)$$



3.3 Kac Determinants

The *Kac determinants* are the determinants of the Gram Matrices, and display a number of useful properties:

(i) A singular vector $|\chi\rangle$ at level K yields null states at level $N > K$:

$$L_{-\{\vec{k}\}}|\chi\rangle$$

with $|\vec{k}| = N - K$. Each singular vector at level K therefore yields $p(N-K)$[54] null states at level N.

(ii) The order of the entries of the Gram determinants $M^{(N)}(c,h)$ at level N in h is given by

- Since each element in $L^{\dagger}_{-\{\vec{k}\}} = L_{\{\vec{k}\}}$ 'commutes to L_0',

$$\mathrm{ord}_h\left(\langle h|L^{\dagger}_{-\{\vec{k}\}}L_{-\{\vec{k}\}}|h\rangle\right) = \mathrm{length}(\vec{k})$$

- For $\vec{k}' \neq \vec{k}$

$$\mathrm{ord}_h\left(\langle h|L^{\dagger}_{-\{\vec{k}'\}}L_{-\{\vec{k}\}}|h\rangle\right) < \mathrm{length}(\vec{k})$$

Since now not all generators in $L^{\dagger}_{-\{\vec{k}\}} = L_{\{\vec{k}\}}$ do not commute to L_0.

This tells us that the diagonal terms in the Gram matrix at level N give rise to the leading contribution in h[55]:

$$\begin{aligned}\mathrm{ord}_h\left(\det M^{(N)}(c,h)\right) &= \sum_{\substack{\vec{k}\\|\vec{k}=N}} \mathrm{length}(\vec{k})\\ &= \sum_{\substack{r,s\in\mathbb{Z}\\1\leq r,s\\r\cdot s\leq N}} p(N - r\cdot s)\end{aligned}$$

(iii) At level N we have

$$\#\left((r,s)\mid 1\leq r,s \text{ and } r\cdot s = N\right)$$

new singular vectors, labelled by $h_{r,s}$

We can compute a formula for the Kac determinants

$$\boxed{\det M^{(N)}(c,h) = \alpha_N \prod_{\substack{r,s\geq 1\\r\cdot s\leq N}} (h - h_{r,s}(c))^{p(N-r\cdot s)}}$$

where α_N is some numerical (non-vanishing) constant independent of h and c.

[54] Where $p(m)$ is the number of *partitions* of m. More precisely, it is the number of distinct non-decreasing sequences of natural numbers which add up to m.

[55] The second step of this calculation is a purely number theoretic identity, not a physical principle.

Conjecture (Kac). [56] *The values of $h_{r,s}$ and c for minimal models are given by*

$$h_{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}$$

$$c(m) = 1 - \frac{6}{m(m+1)}$$

[56] Later proved by Feign and Fuchs.

Note. There are two possible solutions for the second equation

$$m_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{25-c}{1-c}}$$

However,

$$h_{r,s}(m_1(c)) = h_{s,r}(m_2(c))$$

We take the Convention that, for m real, we choose $m \in \mathbb{R}_{\geq 0}$

UNITARY REPRESENTATIONS OF Vir

We begin with a few observations

(i) For level 1, if

$$\det M^{(1)}(c,h) \geq 0$$

then $h \geq 0$

(ii) $\langle h | L_{-n}^{\dagger} L_{-n} | h \rangle = 2nh + \frac{c}{12} n(n^2-1) > 0$

This must hold for all n, which implies that $c \geq 0$

(iii) If we consider the Kac determinants $1 < c < 25$, we get that $m_{1/2} \notin \mathbb{R}$, which implies that $h_{r,s} \notin \mathbb{R}$ for $r \neq s$. If $r = s$, then $h_{r,r} < 0$ for $r >$. Similarly, for $c \geq 25$, $h_{r,s} < 0$, so there are no positive solutions for h.

Taken all together, this implies that $\det M^{(k)}$ is non-vanishing and positive definite for $c > 1$ and $h \geq 0$ (see exercises).

(iv) In the region $0 \leq c \leq 1$, $h \geq 0$ we have the Kac determinant

$$96 h_{r,s} + 4(1-c) = \left(\sqrt{1-c}(r+s) \pm \sqrt{25-c}(r-s)\right)^2 \geq 0$$

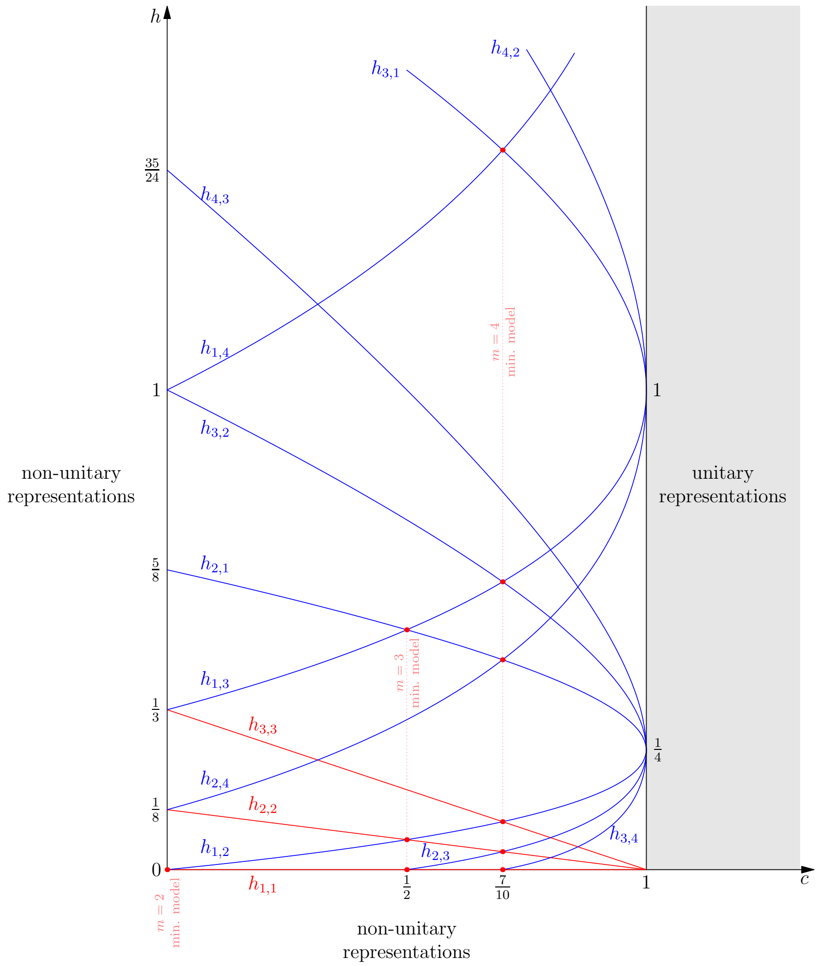
a) near $c = 1$, take $c = 1 - 6\epsilon$. For $r \neq s$

$$h_{r,s}(\epsilon) = \frac{1}{4}(r-s)^2 \pm \frac{1}{4}(r^2 - s^2)\sqrt{\epsilon} + \mathcal{O}(\epsilon)$$

For $r = s$

$$h_{r,r} = \frac{1-c}{24}(r^2 - 1)$$

Diagrammatically, we can represent what we know so far about unitary representations and minimal models in a somewhat finer version of the graph from last section:



Returning to the region $0 \leq c < 1, \quad h \geq 0$,

- A generic point in this region gives rise to non-unitary representations only.

- Frienan, Qiu, Shenkar, in [8] found unitary representations on 'first intersections'. The idea is that the resulting singular vectors at 'first intersections' are sufficient to make the associated Verma module Unitary. These 'first intersections' are given by precisely the formulae from Kac's conjecture above, where $1 \leq r < m$ and $1 \leq s < r$

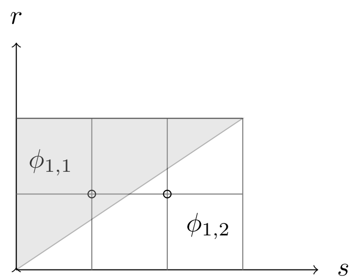
Note. • $h_{r,s}(m) = h_{m-r,m+1-s}(m)$ for $m \in \mathbb{Z}_{\geq 2}$

- We denote the associated conformal families

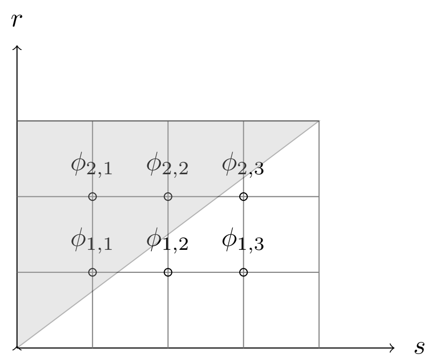
$$[\phi_{r,s}] \equiv [\phi_{m-r,m+1-s}]$$

We can codify the distinct conformal families with the use of *conformal diagrams*. For example, when $m = 2$, we have $c = 0$ $h_{1,2} = h_{1,2} = 0$[57] or, diagrammatically

[57] That is, this minimal model gives the vacuum CFT.



In a more complicated case, we can take $m = 3$, so that $c = \frac{1}{2}$. Looking at the diagram



we see that

$$\begin{aligned} [\phi_{1,1}] &= [\phi_{2,3}], & h_{1,1} &= 0 \\ [\phi_{2,1}] &= [\phi_{1,3}], & h_{2,1} &= \frac{1}{2} \\ [\phi_{2,2}] &= [\phi_{1,2}], & h_{2,2} &= \frac{1}{16} \end{aligned}$$

The conformal diagrams list unitary represenations for a given central charge $c = c(m)$.

Goal: Construct, for a given $c(m)$, unitary CFTs constructed frm the conformal families associated to unitary representations of the Virasora algebra for the given $c(m)$. The resulting CFT's are *unitary minimal models*.

3.4 Fusion Rules

Singular vectors give rise to selection rules for non-vanishing three point correlators.

Example. We work at level 2 (cf Exercise 8.3). We have a singular vector

$$\left(L_{-2} - \frac{3}{2(2h_{2,1}+1)}L_{-1}^2\right)|h_{2,1}\rangle = |\chi_{2,1}\rangle$$

with the null field

$$\chi_{2,1}(z) = \phi_{2,1}^{\{2\}}(z) - \frac{3}{2(2h_{2,1}+1)}\phi_{2,1}^{\{1,1\}}$$

where[58]

$$\phi_{2,1}^{1,1}(z) = \partial_z^2\phi_{2,1}(z)$$

[58] This holds because

$$\begin{aligned}\phi_{2,1}^{\{1\}}(w) &= \oint_w \frac{dz}{2\pi i}\mathcal{R}(T(z)\phi_{2,1}(w))\\ &= \partial_w\phi_{2,1}(w)\end{aligned}$$

Since $|\chi_{2,1}\rangle$ is singular, we have

$$\begin{aligned}0 &= \langle\chi_{2,1}(z)\phi_1(z_1)\phi_2(z_2)\rangle\\ &= \left(\mathcal{L}_{-2} - \frac{3}{2(2h_{2,1}+1)}\partial_z^2\right)\overbrace{\langle\phi_{2,1}(z)\phi_1(z_1)\phi_2(z_2)\rangle}^{\mathbf{C}(z,z_1,z_2)}\end{aligned}$$

where

$$\mathbf{C}(z,z_1,z_2) = C(z-z_1)^{h_2-h_1-h_{2,1}}(z_1-z_2)^{h_{2,1}-h_1-h_2}(z_2-z)^{h_1-h_{2,1}-h_2}$$

(cf section 2.9). We then have the differential equation for $\mathbf{C}(z,z_1,z_2)$, given by

$$0 = \left(\frac{h_1}{(z_1-z)^2} - \frac{\partial_{z_1}}{z_1-z} + \frac{h_2}{(z_2-z)^2} - \frac{\partial_{z_2}}{(z_2-z)} - \frac{3}{2(2h_{2,1}+1)}\partial_z^2\right) \times \mathbf{C}(z,z_1,z_2)$$

This gives us a constraint for $C \neq 0$:

$$2(h_{2,1}+1)(h_{2,1}+2h_2-h_1) = 3(h_{2,1}-h_1+h_2)(h_{2,1}-h_1+h_2+1)$$

Inserting $h_{2,1} = h_{2,1}(m)$, $h_1 = h_{r,s}(m)$ and $h_2 = h_{r',s'}(m)$ yields the solution

$$r' = r\pm 1 \qquad s' = s$$

So the *selection rule* is

$$\begin{aligned}\langle\phi_{2,1}\phi_{r,s}\phi_{r',s'}\rangle \neq 0 \quad&\Rightarrow\quad r' = r\pm 1, \quad s' = s\\ \langle\phi_{1,2}\phi_{r,s}\phi_{r',s'}\rangle \neq 0 \quad&\Rightarrow\quad r' = r, \quad s' = s\pm 1\end{aligned}$$

Fusion rules, on the other hand, summarize the conformal families appearing in the OPE. For a CFT with states $[\phi_p]$, we can define the *fusion product*:

$$[\phi_p]\times[\phi_j] = \sum N_{kj}^{\ell}[\phi_\ell]$$

where $N_{kj}^{\ell} \in \mathbb{Z}_{\geq 0}$. For minimal models in particular, we can go further, and say

$$N_{ij}^{\ell} \in \{0,1\}$$

The three point selection rules at level 2 then tell us

$$N_{(2,1),(r,s)}^{(r',s')} = 0 \text{ for } (r',s') \neq (r \pm 1, s)$$

and

$$N_{(1,2),(r,s)}^{(r',s')} = 0 \text{ for } (r',s') \neq (r, s \pm 1)$$

A dynamical analysis of minimal models yields[59] for a unitary minimal model $c = c(m)$

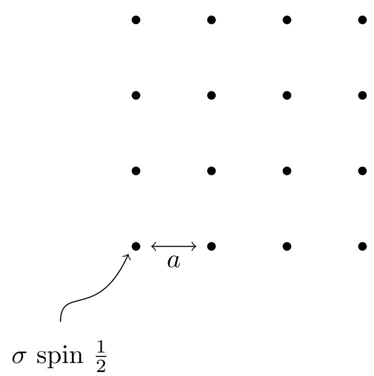
[59] See, for example, [6] or [9].

$$\boxed{[\phi_{(r,s)}] \times [\phi_{(r',s')}] = \sum_{\substack{k=1+|r-r'| \\ k+r+r' \equiv 1 \bmod 2}}^{k=r+r'-1} \sum_{\substack{\ell=1+|s-s'| \\ \ell+s+s' \equiv 1 \bmod 2}}^{\ell=s+s'-1} \left[\phi_{(k,\ell)}\right]}$$

3.5 The Critical Ising Model

Recall the Ising model from the first lecture:

We have $N \times N$ lattice of critical sites:



These spins are equipped with a nearest neighbor interaction. The energy for the system is given by

$$E(\{\sigma\}) = -\epsilon \sum_{\substack{\text{adjacent} \\ \text{lattice sites}}} \sigma_i \sigma_j$$

where $\sigma_i \in \pm\frac{1}{2}$. There are ground states for the system: all states having the same spin (ie either all $|\uparrow\rangle$ or all $|\downarrow\rangle$). The partition function is

$$Z = \sum_{\{\sigma\}} \exp\left(-E(\{\sigma\})\beta\right)$$

where $\beta = \frac{1}{T}$ is the inverse temperature. The energy at a site is

$$\epsilon_k = \frac{\epsilon}{q} \sum_{i \in \left\{\substack{\text{nearest neigh.} \\ \text{of } k}\right\}} \sigma_i \sigma_k$$

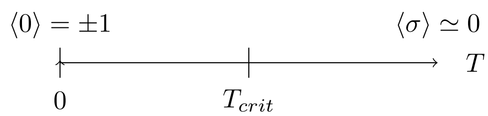
We can take a continuum limit, $a \to 0$, where $Na = \text{const}$. In this case, we get

$$\sigma(z, \overline{z}) \text{ spin field}$$
$$\epsilon(z, \overline{z}) \text{ energy density field}$$

The correlation functions for these fields will be given by

$$\langle \sigma(z, \overline{z}))\sigma(0,0)\rangle = \lim_{\substack{a \to 0 \\ Na = \text{const}}} Z^{-1} \sum_{\{\sigma\}} \sigma_k \sigma_0 e^{-\beta E(\{\sigma\})}$$
$$\langle \epsilon(z, \overline{z}))\epsilon(0,0)\rangle = \lim_{\substack{a \to 0 \\ Na = \text{const}}} Z^{-1} \sum_{\{\sigma\}} \epsilon_k \epsilon_0 e^{-\beta E(\{\sigma\})}$$

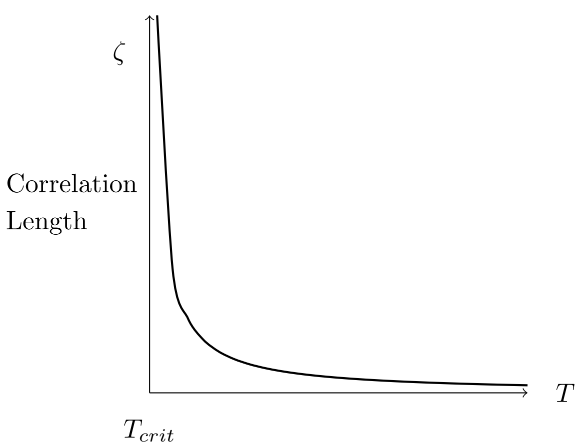
The Ising model has a critical temperature T_{crit}



and the correlator goes as

$$\langle \sigma(z, \overline{z})\sigma(0,0)\rangle \sim e^{-|z|/\zeta(T)}$$

for $T > T_{crit}$. Where ζ goes as:



At T_{crit}, Onsager showed local conformal invariance, and computed the correlators

$$\langle \sigma(z, \overline{z})\sigma(0,0)\rangle = \frac{1}{|z|^{1/4}}$$
$$\langle \epsilon(z, \overline{z})\epsilon(0,0)\rangle = \frac{1}{|z|^2}$$

with $c = \overline{c} = \frac{1}{2}$.

He also computed the field content of the Ising model[60]

[60] It is a spinless theory, ie $h = \overline{h}$.

$$\begin{aligned}
\sigma(z,\overline{z}): &\quad (h,\overline{h}) = \left(\frac{1}{16},\frac{1}{16}\right) \quad \text{spin operator}\\
\epsilon(z,\overline{z}): &\quad (h,\overline{h}) = \left(\frac{1}{2},\frac{1}{2}\right) \quad \text{energy operator}\\
\mathbb{1}: &\quad (h,\overline{h}) = (0,0) \quad \text{(in any CFT)}
\end{aligned}$$

The $m = 3$ minimal model has $c = \frac{1}{2}$, and

$$\begin{aligned}
[\phi_{2,2}] = [\phi_{1,2}]: &\quad h_{2,2} = \frac{1}{16}\\
[\phi_{1,1}] = [\phi_{2,3}]: &\quad h_{1,1} = 0\\
[\phi_{2,1}] = [\phi_{1,2}]: &\quad h_{2,1} = \frac{1}{2}
\end{aligned}$$

The physical 'Ising minimal model' includes the anti-holomorphic sector to match with the operators

$$\begin{aligned}
\sigma(z,\overline{z}) &= \phi_{2,2}(z)\phi_{2,2}(\overline{z})\\
\epsilon(z,\overline{z}) &= \phi_{2,1}(z)\phi_{2,1}(\overline{z})
\end{aligned}$$

The fusion rules for the Ising model can be determined using the order-disorder symmetry and the symmetry

$$\sigma \mapsto -\sigma$$

This yields

$$\begin{aligned}
[\sigma]\times[\sigma] &= [\mathbb{1}] + [\epsilon]\\
[\sigma]\times[\epsilon] &= [\sigma]\\
[\epsilon]\times[\epsilon]&[\mathbb{1}]
\end{aligned}$$

We would like to determine the 3 point structure constants. From the above, we have

$$C_{\sigma\sigma\mathbb{1}} = C_{\epsilon\epsilon\mathbb{1}} = 1$$

We need to determine $C_{\sigma\sigma\epsilon}$ by dynamics.

To do this, we first examine the leading terms in the Operator Product

$$\boxed{\sigma(z)\sigma(0) = \frac{1}{|z|^{\frac{1}{4}}}\left(\mathbb{1} + \cdots + |z| C_{\sigma\sigma\epsilon}\epsilon(0,0) + \cdots\right)} \qquad (*)$$

The $m = 3$ minimal model has a singular vector at level 2

$$|\chi_{1,2}\rangle = \left(L_{-2} - \frac{4,3^{\,2}}{L_{\ \ -1}}\right)|h_{2,2}\rangle$$

$$(|\chi_{2,2}\rangle = |\chi_{1,2}\rangle)$$

Which implies that

$$\left(\mathcal{L}_{-2} - \frac{4}{3}\mathcal{L}_{-1}^2\right)\langle\sigma(z,\overline{z})\phi_1(z_1,\overline{z}_1)\cdots\phi_N(z_N,\overline{z}_N)\rangle = 0 \qquad (**)$$

We would like to determine the four-point correlation function

$$G^{\sigma\sigma}_{\sigma\sigma} = \lim_{z,\overline{z}\to\infty} |z|^{\frac{1}{4}} \langle \sigma(z,\overline{z})\sigma(1,1)\sigma(\eta,\eta)\sigma(0,0)\rangle$$

the idea is to take $(**)$ with $\phi_1 = \phi_2 = \phi_3 = \sigma$ and derive a differential equation in η. We obtain the equation

$$\boxed{\left(\eta(1-\eta)\frac{d^2}{d\eta^2} + (\frac{1}{2}-\eta)\frac{d}{d\eta} + \frac{1}{16}\right) f_k(\eta) = 0}$$

for $k = 1, 2$[61]. The matrix element can be expressed in terms of these f_k[62] as:

$$G^{\sigma\sigma}_{\sigma\sigma}(\eta,\overline{\eta}) = \frac{1}{|\eta(1-\eta)|^{\frac{1}{4}}} \sum_{k,\ell=1}^{2} c_{k\ell} f_k(\eta)\overline{f}_\ell(\eta)$$

[61] That is, since this is a second order differential equation (with singular points at 0,1, and ∞) we have two linearly independent solutions.

[62] And the $\overline{f}_k$, which turn out to satisfy the same differential equations.

The solutions f_k are given by

$$f_{1/2} = \left(1 \pm \sqrt{1-\eta}\right)^{\frac{1}{2}}$$

However, there are multiple branch cuts involved in this definition. Since we want $G^{\sigma\sigma}_{\sigma\sigma}$ to be single valued, there is only one choice we can make for the constants $c_{k\ell}$, which gives us

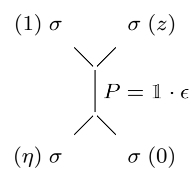
$$G^{\sigma\sigma}_{\sigma\sigma}(\eta,\overline{\eta}) = \frac{C}{|\eta(1-\eta)|^{\frac{1}{4}}} \left(|1+\sqrt{1-\eta}| + |1-\sqrt{1-\eta}|\right)$$

Expanding around $\eta = 0$, we get (to leading order)

$$G^{\sigma\sigma}_{\sigma\sigma} = \frac{C}{|\eta|^{\frac{1}{4}}} \left(2 + \frac{1}{2}|\eta| + \cdots\right) \qquad (***)$$

We can also obtain an expression for $G^{\sigma\sigma}_{\sigma\sigma}$ using the OPE[63], If we apply $(*)$, we find that

[63] Corresponding to the diagram



$$\begin{aligned}
G^{\sigma\sigma}_{\sigma\sigma} &= \lim_{z,\overline{z}\to\infty} |z|^{\frac{1}{4}} \langle \sigma(z,\overline{z})\sigma(1,1)\sigma(\eta,\overline{\eta})\sigma(0,0)\rangle \\
&= \lim_{z,\overline{z}\to\infty} \frac{|z|^{\frac{1}{4}}}{|\eta|^{\frac{1}{4}}} \Bigg(\overbrace{\langle \sigma(z,\overline{z}\sigma(1,1)\mathbb{1}\rangle}^{\langle\sigma(z,\overline{z})\sigma(1,1)\rangle} + \\
&+ |\eta| C_{\sigma\sigma\epsilon} \langle \sigma(z,\overline{z})\sigma(1,1)\epsilon(0,0)\rangle + \cdots \Bigg) \\
&= \lim_{z,\overline{z}\to\infty} \frac{|z|^{\frac{1}{4}}}{|\eta|^{\frac{1}{4}}} \left(\frac{1}{|z-1|^{\frac{1}{4}}} + |\eta| \frac{C^2_{\sigma\sigma\epsilon}}{|z-1|^{\frac{1}{8}+\frac{1}{8}-1}|z|^{\frac{1}{8}+1-\frac{1}{8}}} \right) \\
&= \frac{1}{|\eta|^{\frac{1}{4}}} \left(1 + |\eta| C^2_{\sigma\sigma\epsilon}\right)
\end{aligned}$$

Comparing this result with $(***)$, we get that

$$c = \frac{1}{2} \qquad C_{\sigma\sigma\epsilon} = \frac{1}{2}$$

As a result, we have solved the Ising model by consistency, and the CFT we have found completely determines the dynamics of the system.

Remark. Remarkably enough, we did not at any point need a Lagrangian of our theory!

3.6 Minimal Model Characters

The character of a (potentially reducible) Verma module $V(c,h)$ is defined to be[64]

$$\chi_{V(c,h)} = q^{h-\frac{c}{24}} \sum_{N=0}^{\infty} \#(N) q^N = q^{h-\frac{c}{24}} \prod_{N=1}^{\infty} (1-q^N)^{-1}$$

[64] Where $\#(N)$ is the number of states at level N.

This last equality means that we can represent the character in terms of the Euler function

$$\phi(q) = \prod_{n=1}^{\infty} (1-q^N)$$

as

$$\chi_{V(c,h)} = \frac{q^{h-\frac{c}{24}}}{\phi(q)}$$

Turning our attention to the characters of a unitary minimal model representation $M(c,h)$, we first note that

$$h_{r,-s}(m) - h_{r,s}(m) = h_{-r,s}(m) - h_{r,s}(m) = r \cdot s$$

Additionaly, we have a 'symmetry of indices' so that

$$h_{r,s}(m) = h_{-r,-s}(m) = h_{r+m,s+(m+1)}(m)$$

We can use these to find out more about our singular vectors from first intersections

Singular vectors	Level	Conf. Weight
$\lvert\chi_{r,s}\rangle$	$r \cdot s$	$h_{r,s} + r \cdot s = h_{-r,s} = h_{m+r,m+1-s}$
$\lvert\chi_{m-r,m+1-s}\rangle$	$(m-r)(m+1-s)$	$h_{r,s} + (m-r)(m+1-s) = h_{m-r,s-m-1}$ $= h_{r,2(m+1)-s}$

The irreducible Verma module is therefore given by

$$M(c(m), h_{r,s}(m)) = V_{r,s} / \left(V_{r+m,-s+m+1} \cup V_{r,2(m+1)-s}\right)$$

To compute the character, we want to subtract contributions from each submodule, but this leads to the possibility of double counting sub-submodules, and so on. We therefore need to look at the structure of submodules:

$$\begin{aligned} V_{r,s} &= V_{-r,-s} = V_{r+m,s+m+1} \\ V_{r+m,m+1-s} &= V_{-r-m,s-m-1} = V_{m-r,s+(m+1)} \\ V_{r,2(m+1)-s} &= V_{-r,s-2(m+1)} = V_{2m-r,s} \end{aligned}$$

so we have

$$
\begin{array}{ccccc}
V_{r+m,m+1-s} \supset & & V_{r+2m,s} \cup & & V_{2m-r,-s} \\
 & & \rVert\!\rVert\!\rVert & & \rVert\!\rVert\!\rVert \\
V_{r,2(m+1)-s} & \supset & V_{r+m,s-(m+1)} & \cup & V_{3m-r,(m+1)-s}
\end{array}
$$

These are the only common submodules, but they might have non-trivial intersections.

Considering higher submodule chain structure gives us[65]

[65] Where each arrow in the diagram goes from a module to a submodule.

$$
\begin{tikzcd}
 & V_{r+m,m+1-s} \arrow[r] \arrow[ddr] & V_{r+2m,s} \arrow[r] & \cdots \arrow[r] \arrow[ddr] & V_{r+km,(-1)^k s+\frac{1-(-1)^k}{2}(m+1)} \arrow[ddr] & \\
V_{r,s} \arrow[ur] \arrow[dr] & & & & & \\
 & V_{r,2(m+1)-s} \arrow[r] \arrow[uur] & V_{r,2(m+1)+s} \arrow[r] & \cdots \arrow[r] \arrow[uur] & V_{r,k(m+1)+(-1)^k s+\frac{1-(-1)^k}{2}(m+1)} \arrow[uur] &
\end{tikzcd}
$$

The character of $M(c(m), h_{r,s}(m))$ thus becomes

$$
\chi_{r,s}^{(m)}(q) = \frac{q^{-c/2\pi}}{\phi(q)} \left(q^{h_{r,s}} + \sum_{k=1}^{\infty} (-1)^k \cdot \left(q^{h_{r+km,(-1)^k s+\frac{1-(-1)^k}{2}}} \right.\right.
$$
$$
\left.\left. + \, q^{r,k(m+1)+(-1)^k s+\frac{1-(-1)^k}{2}(m+1)} \right) \right)
$$

Or, if we define the functions

$$
K_{r,s}^{(m)} = \frac{q^{-1/24}}{\phi(q)} \sum_{k \in \mathbb{Z}} q^{(2m(m+1)k + r(m+1) - ms)^2/(4m(m+1))}
$$

then

$$
\chi_{r,s}^{(m)} = K_{r,s}^{(m)} - K_{r,-s}^{(m)}
$$

Remark. $\chi_{r,s}^{(m)}$ are the generating functions for the (physical) states at level N[66].

[66] Shifted by

$$q^{h_{r,s}-c/24}$$

4 Modular Invariance

Up to this point, we have studied CFTs on the conformal plane. We now aim to study CFT's on the 2-torus.

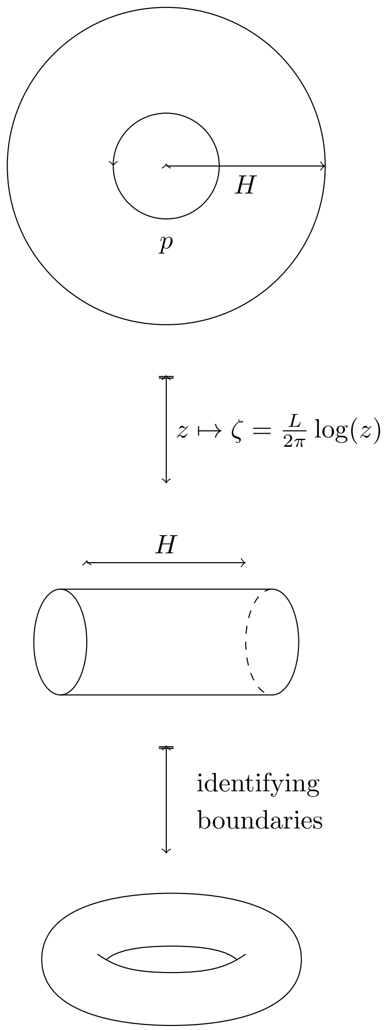
Motivation:

- Consistent 2d CFTs describing critical phenomena should be locally independent of the 2d geometry. In particular, it should be independent of the quantization scheme (on the torus T^2).
- In String Theory, CFTs on Riemann surfaces form part of perturbative string theory. T^2[67] gives rise to a 1 loop correction term.

[67] A genus 1 Riemann surface.

4.1 Partition Function

We can consider our map on the punctured plane from section 2.3, and extend it to the torus:



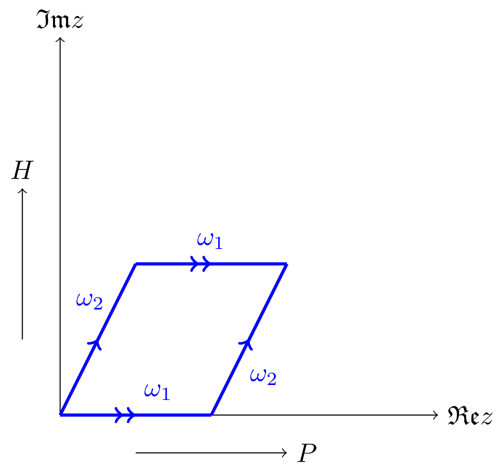
On T^2, our local symmetries are still the Virasora generators L_n. Our global symmetries, however, are generated by L_0 and $\overline{L}_0$[68].

[68] So that the group of symmetries is $U(1) \times U(1)$, the isometries of T^2 with the flat metric.

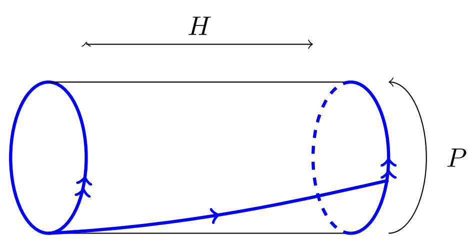
Space-time Structure of the Torus

We can obtain the torus as a quotient of the complex plane by a lattice generated by a pair of vectors ω_1 and ω_2.

Diagrammatically, we take the quotient of:



to get:



Translations along the cycle are given by

$$\exp\left(-\frac{a}{|\omega_2|}(H\cdot\mathfrak{Im}\omega_2 - iP\mathfrak{Re}\omega_2)\right)$$

Recall (cf. section 2.7) that we calculated the energy momentum tensor for the cylinder:

$$T_{cyl}(\zeta) = \left(\frac{2\pi}{L}\right)^2\left(\sum_{n\in\mathbb{Z}} L_n e^{-2\pi n/L\zeta} - \frac{c}{24}\right)$$

This allows us to compute the operators H and P:

$$H = \frac{1}{2\pi}\int_{\omega_1}\left(\frac{2\pi}{L}\right)^2\left(L_0 + \overline{L}_0 - \frac{c}{12}\right) = \frac{2\pi}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right)$$
$$P = \frac{1}{2\pi}\int_{\omega_1}\left(\frac{2\pi}{L}\right)^2\left(L_0 + \overline{L}_0\right) = \frac{2\pi}{L}\left(L_0 + \overline{L}_0\right)$$

The partition function on T^2 is then given by

$$Z(\omega_1,\omega_2) = \mathrm{Tr}\left(\exp\left(-\left(H\mathfrak{Im}\omega_2 - iP\mathfrak{Re}\omega_2\right)\right)\right)$$

Using the modular parameter $\tau = \tau_1 + i\tau_2$ where

$$\tau_1 = \frac{\mathfrak{Re}\omega_2}{\omega_1} = \frac{\mathfrak{Re}\omega_2}{L}$$
$$\tau_2 = \frac{\mathfrak{Im}\omega_2}{\omega_1} = \frac{\mathfrak{Im}\omega_2}{L}$$

the partition function takes the form

$$Z(\tau) = \operatorname{Tr}\left(\exp 2\pi i \left(\tau\left(L_0 - \frac{c}{24}\right) - \overline{\tau}\left(\overline{L}_0 - \frac{c}{24}\right)\right)\right)$$

Or

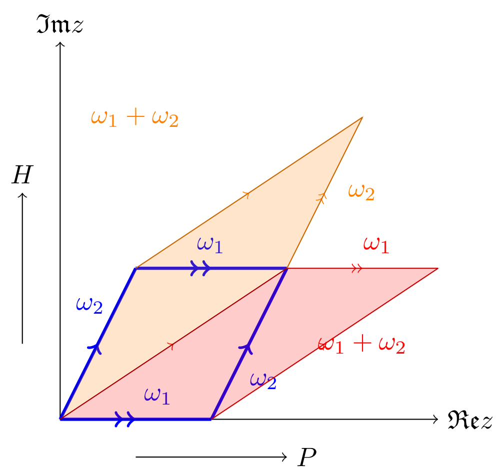
$$\boxed{Z(\tau) = \operatorname{Tr}\left(q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L}_0 - \frac{c}{24}}\right)}$$

where $q = \exp(2\pi i \tau)$.

Observation. The partition function only depends on the modular parameter τ (that is, on the shape of the torus/the twisted gluing condition). No dependence on the size of the torus exists, which is consistent with the conformal property of CFTs.

4.2 Modular Invariance

So far, we have singled out a particular choice of lattice (1-cycles) on T^2, however, we can think somewhat more generally. Consider the picture:



It is not hard to see that the lattice Λ generated by ω_1 and ω_2 will also be generated by $\omega_1 + \omega_2$ and ω_1, and will be generated by $\omega_1 + \omega_2$ and ω_2. Consequently, many different choices of initial cycles will give us the same torus.

In general, if (ω_2, ω_1) defines the lattice $\Lambda \subset \mathbb{C}$, then

$$\begin{bmatrix} \omega_2' \\ \omega_1' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$$

for

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{Z})$$

describes the same lattice Λ.

Under such a transformation, the modular parameter transforms as

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where[69]

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})/(\mathbb{Z}/2)$$

[69] The group

$$PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/(\mathbb{Z}/2)$$

is the modular group of the torus.

Partition functions of consistent CFTs must be modular invariant (ie, independent of the quantization scheme) due to conformal invariance. Therefore we require that

$$\boxed{Z^{T^2}_{CFT}(\tau') = Z^{T^2}_{CFT}(\tau)}$$

for $\tau' = \frac{a\tau+b}{c\tau+d}$ where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{Z})$.

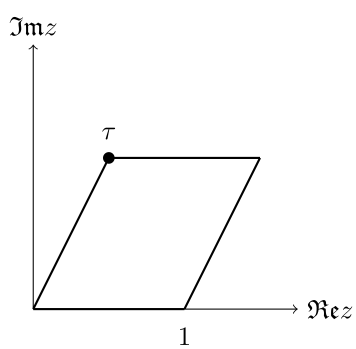
Remarks. (i) $PSL(2,\mathbb{Z})$ is generated by the transformations

$$\boxed{T : \tau \mapsto \tau + 1 \quad S : \tau \mapsto -\frac{1}{\tau}}$$

So $Z(\tau)$ is modular invariant if and only if

$$Z(\tau+1) = Z(\tau)$$
$$Z\left(-\frac{1}{\tau}\right) = Z(\tau)$$

(ii) The shape of a torus is specified by a point in the upper half-plane $\tau \in \mathbb{H} = \{\mathbb{C} \mid \mathfrak{Im}\tau > 0\}$



tori with the 'same shape' are therefore identified by the action of $PSL(2,\mathbb{Z})$ on τ. As a result the moduli space of 'shapes of tori'[70] is given by

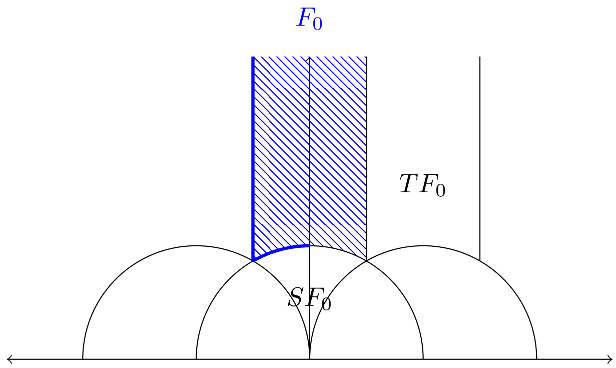
$$\mathscr{M}_{cs} \simeq \mathbb{H}/PSL(2,\mathbb{Z})$$

[70] More explicitly, this is the moduli space of complex structures on the surface of genus 1.

To describe distinct points on $\mathscr{M}_{cs}$, we can look at the fundamental domain $F_0 \subset \mathbb{H}$, which is given explicitly by

$$F_0 = \begin{cases} \mathfrak{Im}\tau > 0, \;\; \frac{1}{2} \leq \mathfrak{Re}\tau \leq 0, \;\; |\tau| \geq 1 \\ \mathfrak{Im}\tau > 0, \;\; 0 < \mathfrak{Re}\tau < \frac{1}{2}, \;\; |\tau| > 1 \end{cases}$$

or, diagrammatically:



4.3 Free Boson Parition Function

Recall (Exercise 5.1)**.** The Free Boson CFT has a continuum of primaries

$$V_\alpha =: e^{i\alpha\phi(z,\bar{z})}:$$

for $\alpha \in \mathbb{R}$. These satisfy the relations[71]

$$L_0|V_\alpha\rangle = \frac{\alpha^2}{2}|V_\alpha\rangle$$
$$\overline{L}_0|V_\alpha\rangle = \frac{\alpha^2}{2}|V_\alpha\rangle$$

The free Boson CFT also has central charge

$$c_{Boson} = 1$$

We now compute the partition function of the free Boson[72]

$$\begin{aligned} Z_{Boson}(\tau) &= \mathrm{Tr}\left(q^{L_0 - \frac{c}{24}}\bar{q}^{\overline{L}_0 - \frac{c}{24}}\right) \\ &= \underbrace{\sqrt{2}}_{\substack{\text{conv.}\\ \text{normal-}\\ \text{ization}}} \underbrace{\int_{-\infty}^{\infty} d\alpha}_{\substack{\text{int.}\\ \text{over}\\ \text{primaries}}} \underbrace{\left(\sum_{\substack{\{\vec{k}\}\\ k_1<k_2<\cdots}} q^{\frac{\alpha^2}{2}+|\vec{k}|-\frac{1}{24}}\right)}_{\text{hol. descendants}} \\ &\times \underbrace{\left(\sum_{\substack{\{\vec{\bar{k}}\}\\ \bar{k}_1<\bar{k}_2<\cdots}} \bar{q}^{\frac{\alpha^2}{2}+|\vec{\bar{k}}|-\frac{1}{24}}\right)}_{\text{anti-hol. desc.}} \end{aligned}$$

[71] Where $g = \frac{1}{4\pi}$ is the coupling constant.

[72] Where, as before,

$$q = e^{2\pi i\tau}$$

$$= \underbrace{\sqrt{2}\left(\int_{-\infty}^{\infty} d\alpha e^{-2pi\mathfrak{Im}(\tau)\alpha^2}\right)}_{(\mathfrak{Im}(\tau))^{\frac{1}{2}}} q^{-\frac{1}{24}}\overline{q}^{-\frac{1}{24}}$$

$$\times \prod_{N=1}^{\infty}\left(1-q^N\right)^{-1} \times \prod_{N=1}^{\infty}\left(1-\overline{q}^N\right)^{-1}$$

So that, with some simplification[73],we get

$$\boxed{Z_{Boson}(\tau) = \frac{1}{\sqrt{\mathfrak{Im}(\tau)}|\eta(\tau)|^2}}$$

[73] Notably, using *Dedekind's eta function*

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{N=1}^{\infty}(1-q^N)$$

to ease writing.

To see modular invariance, we consider the modular properties of $\eta(\tau)$.

T: $\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau)$.

S: $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$ (This can be computed by Poisson resummation).

Returning to Z_{Boson}, we see that

$$T : |\eta(\tau)| \mapsto |\eta(\tau)|$$

and since $\mathfrak{Im}(\tau+1) = \mathfrak{Im}(\tau)$, we see

$$Z_{Boson}(\tau+1) = Z_{Boson}(\tau)$$

Similarly, we note

$$S : \left|\eta\left(-\frac{1}{\tau}\right)\right|^2 = |\tau||\eta(\tau)|^2$$

and

$$\mathfrak{Im}\left(-\frac{1}{\tau}\right) = \mathfrak{Im}\left(-\frac{\overline{\tau}}{|\tau|^2}\right) = \frac{\mathfrak{Im}(\tau)}{|\tau|^2}$$

so that

$$Z_{Boson}\left(-\frac{1}{\tau}\right) = Z_{Boson}(\tau)$$

Taken together, this shows that the free Boson CFT is modular invariant.

4.4 Interlude: Mode Expansions of Free Fermions

The mode expansion of the Boson and the Fermion CFT is given by

$$i\partial_z\phi(z,\overline{z}) = \sum_{n\in\mathbb{Z}} \alpha_n z^{-n-1}$$

The radial ordering is given by

$$(\mathcal{R}\left(\partial_z\phi(z,\overline{z})\partial_w\phi(w,\overline{w})\right)) \sim -\frac{1}{(z-w)^2}$$

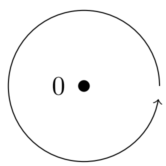
And

$$\psi(z) = \sum_n \psi_n z^{-n-\frac{1}{2}}$$

where $h = \frac{1}{2}$ and

$$\begin{cases} n \in \mathbb{Z} + \frac{1}{2} & \text{periodic boundary cond.} \\ n \in \mathbb{Z} & \text{anti-periodic boundary cond.} \end{cases}$$

We call the periodic boundary conditions the *NS (Nevea-Schwartz)-sector* and the anti-periodic boundary conditions the *R (Raymond) sector*. We can think of these two different sectors in terms of monodromy about the origin:



In the NS sector, we get $\psi(e^{2\pi i}z) = \psi(z)$, whereas in the R sector, we get $\psi(e^{2\pi i}z) = -\psi(z)$. In more explicit geometric terms, we can think of ψ as sections of a spin bundle[74] on the (punctured) complex plane.

The commutation relations for the modes are given by

$$[\alpha_n, \alpha_m] = i^2 \left[\oint \frac{sz}{2\pi i} z^n \partial_z, \oint \frac{dw}{2\pi i} w^m \partial_w \phi\right] = n\delta_{n+m,0}$$

and, taking the anticommutator because of Fermi statistics.

$$\{\psi_n, \psi_m\} = \delta_{n+m,0}$$

We can also consider *twist fields*, which change the boundary conditions of fermions from periodic to anti-periodic.

As an application, let us look at the spin operator of the Ising model:

$$\underline{\text{Ising Model } c = \tfrac{1}{2}} \quad \xleftrightarrow{1:1} \quad \underline{\text{Free Fermion CFT } c = \tfrac{1}{2}}$$

$$\begin{array}{rcl} [\sigma] \times [\sigma] & = & [\mathbb{1}] + [\epsilon] \\ \epsilon(z,\overline{z}) & = & \phi_{(2,1)}(z)\overline{\phi}_{(2,1)}(\overline{z}) \\ h & = & \overline{h} \; = \; \frac{1}{2} \end{array} \quad \longleftrightarrow \quad i \; : \; \psi\overline{\psi}:$$

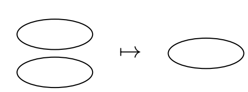
[74] As it turns out, there are precisely two spin bundles on the annulus $\mathbb{C}^* \simeq S^1$ up to isomorphism, corresponding to the NS and R sectors. The structure group for spin bundles is $\mathrm{Spin}(n)$, defined by the short exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to SO(n) \to 0$$

For S^1, we consider $SO(1) \cong 0$, so that $\mathrm{Spin}(1) \cong \mathbb{Z}_2$. The two principal $\mathbb{Z}_2$-bundles over S^1 ($H^1(\mathbb{C}^*, \mathbb{Z}_2) \cong H^1(S^1, \mathbb{Z}_Z) \cong \mathbb{Z}_2$) are then given by:

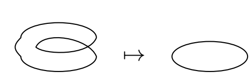
① The map

$$S^1 \times \mathbb{Z}_2 \to S^1$$
$$(e^{i\phi}, \sigma) \mapsto e^{i\phi}$$



② The map

$$S^1 \to S^1$$
$$e^{i\phi} \mapsto e^{2i\phi}$$



From $[\epsilon] \times [\sigma] = [\sigma]$ we infer

$$\epsilon(z,\overline{z})\sigma(w,\overline{w}) \sim \frac{\sigma(w,\overline{z})}{2(z-w)^{\frac{1}{2}}(\overline{z}-\overline{w})^{\frac{1}{2}}}$$

because $C_{\sigma\sigma\epsilon} = \frac{1}{2}$. This implies that[75]

$$\psi(z)\sigma(w,\overline{w}) \sim \frac{\mu(w,\overline{w})}{\sqrt{2}(z-w)^{\frac{1}{2}}}$$

and, dually,

$$\psi(z)\mu(w,\overline{w}) \sim \frac{\sigma(w,\overline{w})}{\sqrt{2}(\overline{z}-\overline{w})^{\frac{1}{2}}}$$

[75] Where we denote by μ the *disorder operator* of the Ising model, ie the dual field to σ.

Important for us: The spin operator $\sigma(w,\overline{w})$ changes the sign of $\psi(z)$.

In the NS and R sectors, we can then compute 2-point correlators. In the NS sector:

$$\begin{aligned}
\langle\psi(z)\psi(w)\rangle_{NS} &= \langle 0| \sum_{n=\frac{1}{2}}^{\infty} \psi_n z^{-n-\frac{1}{2}} \sum_{m=-\frac{1}{2}}^{\infty} \psi_m w^{-m-\frac{1}{2}} |0\rangle \\
\overset{\{\psi_n,\psi_m\}=\delta_{n+m,0}}{} &= \sum_{n=\frac{1}{2}}^{\infty} z^{-n-\frac{1}{2}} w^{n-\frac{1}{2}} \\
&= \frac{1}{z}\sum_{k=0}^{\infty} z^{-k}w^k = \frac{1}{(z-w)}
\end{aligned}$$

In the R sector (from the twist field):

$$\begin{aligned}
\langle\psi(z)\psi(w)\rangle_{R} &= \langle 0|\sigma(\infty)\psi(z)\psi(w)\sigma(0)|0\rangle \\
&= \langle\sigma|\psi(z)\psi(w)|\sigma\rangle \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{-\infty} \langle\sigma|\psi_n\psi_m|\sigma\rangle z^{-n-\frac{1}{2}} w^{-m-\frac{1}{2}} \\
&= \frac{1}{\sqrt{zw}}\langle\sigma|\frac{1}{2}\underbrace{\{\psi_0,\psi_0\}}_{{\psi_0}^2}|\sigma\rangle + \sum_{n=1}^{\infty} z^{-n-\frac{1}{2}} w^{-n-\frac{1}{2}} \\
&= \frac{\frac{1}{2}\left(\sqrt{\frac{z}{w}}+\sqrt{\frac{w}{z}}\right)}{z-w}
\end{aligned}$$

Remarks. • The short distance behavior is unchanged between the sectors[76]

[76] Does this correspond to the local triviality of (spin) bundles?

• The boundary conditions modify the behavior at 0 and ∞

• Similarly, we can introduce $\mu(z,\overline{z})$ as a twist field!

If we now examine the Fermionic zero mode ψ_0 in the R-sector, we have

$$(-1)^F \quad : \quad \text{fermionic mode counting operator}$$

which satisfies the anti-commutation relation

$$\left\{(-1)^F, \psi_n\right\} = 0$$

for all n. This implies that $(-1)^F$ has eigenvalues ± 1 for even/odd numbers of fermionic creation and annihilation operators.

In particular:

$$\begin{aligned} \left\{(-1)^F, \psi_0\right\} &= 0 \\ \left\{\sqrt{2}\psi_0, \sqrt{2}\psi_0\right\} &= 2 \\ \left\{(-1)^F, (-1)^F\right\} &= 2 \end{aligned}$$

furnish a 2 dimensional Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}$$

where

$$g^{\mu\nu} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$$

Furthermore

$$[L_0, \psi_0] = [L_0, (-1)^F] = 0$$

so the R-ground state must be a representation of the 2 dimensional Clifford algebra. The smallest representation is $|\pm\rangle_R$ with

$$\boxed{(-1)^{\pm}|\pm\rangle = \pm|\pm\rangle_R}$$

and

$$\boxed{(\sqrt{2}\psi_0) = |\pm\rangle_R}$$

Remarks. • We identify:

$$\begin{aligned} |+\rangle_R &= \sigma(0,0)|0\rangle \\ |-\rangle_R &= \mu(0,0)|0\rangle \end{aligned}$$

- Considering $\psi(z)$ and $\overline{\psi}(\overline{z})$ together, there are two operators $(-1)^{F_L}$ and $(-1)^{F_R}$.

  We consider: $(-1)^F := (-1)^{F_L+F_R}$ and a two state subset of vacuum states (the *non-chiral subsector*) $|\pm\rangle_R$ with

  $$(-1)^{F_L+F_R}|\pm\rangle_R = \pm|\pm\rangle_R$$

  with $h = \overline{h} = \frac{1}{16}$

If instead of the punctured plane we work on the cylinder, we must recall that primaries transform as

$$\phi(z,\overline{z}) \mapsto \left(\frac{\partial \phi}{\partial z}\right)^{-h} \left(\frac{\partial \overline{\phi}}{\partial \overline{z}}\right)^{-h} \phi(z,\overline{z})$$

The mapping to the cylinder is given by

$$z \mapsto \zeta(z) = \frac{l}{2\pi}\log(z)$$

so $\frac{\partial\zeta}{\partial z} = \frac{L}{2\pi}\frac{1}{z}$ and $e^{\frac{2\pi}{L}\zeta}$ So, for Bosons

$$(i\partial\phi)_{cyl}(\zeta) = \frac{2\pi}{L}\sum_{n\in\mathbb{Z}}\alpha_n e^{\frac{2\pi\zeta}{L}n}$$

and for Fermions

$$\psi_{cyl}(\zeta) = \sqrt{\frac{2\pi}{L}}\sum_{\psi_n}\psi_n e^{-\frac{2\pi\zeta}{L}n}$$

where

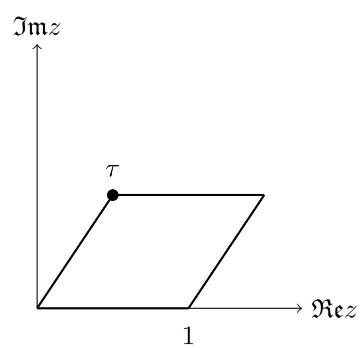
$$n \in \begin{cases} \mathbb{Z}+\frac{1}{2} & \text{NS sector} \\ \mathbb{Z} & \text{R sector} \end{cases}$$

We can summarize the boundary conditions for fermions[77]

	Conformal Plane	Cylinder
<u>NS sector</u>: half-integral modes	Periodic bound- ary conditions. $L_0\lvert 0\rangle = 0$ $(-1)^{\pm}\lvert 0\rangle = 0$	Anti-periodic bound- ary conditions. $(L_0)_{cyl}\lvert 0\rangle = -\frac{1}{48}\lvert 0\rangle$
<u>R sector</u>: ψ_0 zero mode	Anti-periodic boundary conditions. $L_0\lvert\pm\rangle_R = \frac{1}{16}\lvert\pm\rangle_R$ $(-1)^{\pm}\lvert\pm\rangle_R = \pm\lvert\pm\rangle_R$	Periodic bound- ary conditions. $(L_0)_{cyl}\lvert\pm\rangle_R = \frac{1}{24}\lvert\pm\rangle_R$

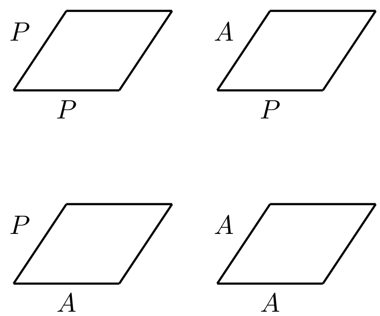
4.5 Partition Function of the Free Fermion

For a torus of shape τ



we have 4 choices of boundary conditions:

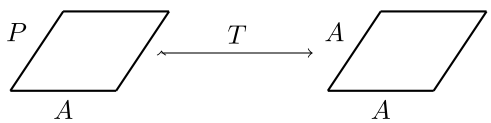
[77] $(L_0)_{cyl} = L_0 - \frac{c}{24}$.



R sector along spatial slices

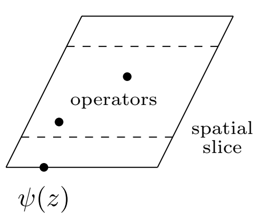
NS sector along spatial slices

Observation. Except for P,P, boundary conditions are <u>not</u> modular invariant



Our expectation, therefore, is that a modular invariant partition function with anti-periodic boundary conditions must involve <u>all</u> sectors involving anti-periodic boundary conditions.

In the operator formalism, we look at $\langle\psi(z)X\rangle$ where X is some product of fields inserted at various positions.



In this case $\langle\psi(z)X\rangle \neq 0$ only if X is fermionic. We can then transport $\psi(z)$ past all operators X. In anti-periodic boundary conditions

$$\psi(z+\tau) = -\psi(z)$$

In periodic boundary conditions, the insertion of $(-1)^F$ operator yields an additional minus sign

$$\langle\psi(z)(-1)^F X\rangle \neq 0$$

for X bosonic.

$$\psi(z+\tau) = \psi(z)$$

Thus, in the holomorphic sector, we get, in the spatial R sector

$$P\underset{P}{\square} = \frac{1}{\sqrt{2}} q^{-\frac{1}{48}} \mathrm{tr}_R((-1)^F q^{L_0})$$

$$A\underset{P}{\square} = \frac{1}{\sqrt{2}} q^{-\frac{1}{48}} \mathrm{tr}_R q^{L_0}$$

And, in the spatial NS sector

$$P\,\underset{A}{\boxed{\quad\quad}} = q^{-\frac{1}{48}}\mathrm{tr}_{NS}((-1)^F q^{L_0})$$

$$A\,\underset{A}{\boxed{\quad\quad}} = q^{-\frac{1}{48}}\mathrm{tr}_{NS} q^{L_0}$$

Where the $\frac{1}{\sqrt{2}}$ in the R sector is a convenient normalization factor related to picking only two out of 4 ground state values.

In the NS sector: $[0] + [h = \frac{1}{2}] \simeq [\phi_{1,1}] + [\phi_{2,1}]$ conformal families.

L_0 eigenvalue	state	
0	$\|0\rangle$	primary of conformal weight $h = 0$
$\frac{1}{2}$	$\psi_{-\frac{1}{2}}\|0\rangle$	primary of conformal weight $h = \frac{1}{2}$
$\frac{3}{2}$	$\psi_{-\frac{3}{2}}\|0\rangle$	desc. of $[\frac{1}{2}]$ level 1
2	$\psi_{-\frac{3}{2}}\psi_{-\frac{1}{2}}\|0\rangle$	desc. of $[0]$ level 2
$\frac{5}{2}$	$\psi_{-\frac{5}{2}}\|0\rangle$	desc. of $[\frac{1}{2}]$ level 2
$\vdots$	$\vdots$	$\vdots$
4	$\psi_{-\frac{5}{2}}\psi_{-\frac{3}{2}}\|0\rangle$	desc. of $[0]$ level 4 (two states)

so we have

$$\begin{aligned}\mathrm{tr}_{NS} q^{L_0} &= 1 + q + q^{\frac{3}{2}} + q^2 + q^{\frac{5}{2}} + q^{\frac{7}{2}} + 2q^4 + \cdots \\ &= \prod_{n=0}^{\infty}\left(1 + q^{n+\frac{1}{2}}\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}_{NS}\left((-1)^F q^{L_0}\right) &= 1 - q^{\frac{1}{2}} - q^{\frac{3}{2}} + q^2 - q^{\frac{5}{2}} + q^3 - q^{\frac{7}{2}} + 2q^4 + \cdots \\ &= \prod_{n=0}^{\infty}\left(1 - q^{n+\frac{1}{2}}\right)\end{aligned}$$

Remarks. • The structure of states is consistent with the singular vector structure:

$[\phi_{1,1}]$: state at level 1 singular vector

$[\phi_{2,1}]$: singular vector at level 2, so there is only one (rather than two) state at level 2.

• Projections to $m = 3$ minimal model characters

$$\chi_{[\phi_{1,1}]} = q^{-\frac{1}{48}}\mathrm{tr}_{NS}\left(\frac{1}{2}(1 + (-1)^f)q^{L_0}\right)$$

$$\chi_{[\phi_{2,1}]} = q^{-\frac{1}{48}}\mathrm{tr}_{NS}\left(\frac{1}{2}(1 - (-1)^F)q^{L_0}\right)$$

In the RR sector, we have $[\sigma]$ and $[\mu]$ dual conformal families $[\phi_{1,2}] \simeq [\phi_{2,2}]$

L_0 eigenvalue	State
$\frac{1}{16} + 0$	$\lvert\pm\rangle_R$
$\frac{1}{16} + 1$	$\psi_{-1}\lvert\pm\rangle_R$
$\frac{1}{16} + 2$	$\psi_{-2}\lvert\pm\rangle_R$

so we have

$$\begin{aligned} \mathrm{tr}_R q^{L_0} &= \prod_{n=0}^{\infty} \\ &= 2 + 2q + \cdots \\ \mathrm{tr}_R(-1)q^{L_0} &= \prod_{n=0}^{\infty}(1-q^n) = 0 \end{aligned}$$

because of the fermionic zero mode.

So, in summary, we find[78]:

$$P\underset{P}{\square} = \frac{1}{\sqrt{2}}q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1-q^n) = \sqrt{\frac{\theta_1(\tau)}{\eta(\tau)}} = 0$$

$$A\underset{P}{\square} = \frac{1}{\sqrt{2}}q^{-\frac{1}{24}}\prod_{n=0}^{\infty}(1+q^n) = \sqrt{\frac{\theta_2(\tau)}{\eta(\tau)}}$$

$$P\underset{A}{\square} = q^{-\frac{1}{48}}\prod_{n=\frac{1}{2}}^{\infty}(1-q^n) = \sqrt{\frac{\theta_3(\tau)}{\eta(\tau)}}$$

$$A\underset{A}{\square} = q^{-\frac{1}{48}}\prod_{n=\frac{1}{2}}^{\infty}(1+q^n) = \sqrt{\frac{\theta_4(\tau)}{\eta(\tau)}}$$

We can then write down the modular invariant partition function of the fermion:

$$Z_{ferm}(\tau) = \left| P\underset{P}{\square} \right|^2 + \left| A\underset{P}{\square} \right|^2 + \left| P\underset{A}{\square} \right|^2 + \left| A\underset{A}{\square} \right|^2$$

Or

$$\boxed{Z_{ferm}(\tau) = \overbrace{\left|\frac{\theta_2(\tau)}{\eta(\tau)}\right| + \underbrace{\left|\frac{\theta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\theta_4(\tau)}{\eta(\tau)}\right|}_{\text{wrt } T}}^{\text{exchange wrt } S}}$$

is modular invariant.

[78] Where θ_i are the *Jacobi theta functions*:

$$\begin{aligned} \theta_2(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2/2} \\ \theta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2/2} \\ \theta_4(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/2} \end{aligned}$$

Which display the modular properties

$$\begin{aligned} \theta_2(\tau+1) &= e^{i\pi/4}\theta_2(\tau) \\ \theta_2(-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_4(\tau) \\ \theta_3(\tau+1) &= \theta_4(\tau) \\ \theta_3(-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_3(\tau) \\ \theta_4(\tau+1) &= \theta_3(\tau) \\ \theta_4(-\frac{1}{\tau}) &= \sqrt{-i\tau}\theta_2(\tau) \end{aligned}$$

Remarks. • The normalization projection onto 'physical' zero mode $\psi_0 + \overline{\psi}_0$ in RR rector means we choose 2 out of 4 values, so as to arrive at the $\frac{1}{\sqrt{2}}$ normalization factor.

• Ising model partition function has only $[\sigma]$ family and no $[\mu]$ family, which implies

$$\boxed{Z_{ferm}(\tau) = 2Z_{Ising}(\tau)}$$

so the fermion description induces the 'dual' Ising model description.

5 Applications

5.1 Affine Kac-Moody algebras and WZW models

Let $\mathfrak{g}$ be a simple Lie algebra

a) A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with an anti-symmetric bilinear pairing

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called the *Lie bracket* satisfying the Jacobi identity. That is, such that for any $x, y, z \in \mathfrak{g}$

$$[[x, y], z] + [[z, x], y] + [[y, z], x] = 0$$

b) a *simple* Lie algebra $\mathfrak{g}$ is one such that there are no proper subsets $S \subset \mathfrak{g}$ such that

$$[S, \mathfrak{g}] \subset S$$

These have be classified, and are precisely the Lie algebras A_n,B_n, C_n, D_n,E_6, E_7, E_8,F_4, and G_2.[79]

c) Let J^a be generators of $\mathfrak{g}$, then

$$[J^a, J^b] = \sum_c i f^{ab}{}_c J^c$$

where the $f^{ab}{}_c$ are the *real structure constants* of $\mathfrak{g}$. The *dimension* of $\mathfrak{g}$ is then given by

$$\dim \mathfrak{g} = \#(\text{generators})$$

d) The (non-degenerate) *Killing Form* for $\mathfrak{g}$ is given by

$$K(x, y) = \frac{1}{2g} \mathrm{Tr}\left(\mathrm{ad}(x) \cdot \mathrm{ad}(y)\right)$$

Where g is the dual Coexter number and

$$\mathrm{ad}(x) : y \mapsto [x, y]$$

[79] The dual Coexter numbers associated with these algebras are, respectively:

	g
A_n	$n+1$
B_n	$2n-1$
C_n	$n+1$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	9
G_2	4

From $\mathfrak{g}$, we can consider *Affine Kac-Moody algebras*:

i) The *loop algebra* of $\mathfrak{g}$ is the vector space of maps

$$\mathrm{Map}(S^1 \to \mathfrak{g})$$

Since these functions are periodic, we can consider the Fourier decomposition of $J(\theta) \in \mathrm{Map}(S^1 \to \mathfrak{g})$

$$J(\theta) = \sum_n e^{in\theta} X_n \quad , X_n \in \mathfrak{g}$$

Identifying $e^{i\theta}$ with t, we get an isomorphism given by the Fourier decomposition

$$\mathrm{Map}(S^1 \to \mathfrak{g}) \leftrightarrow \mathfrak{g} \otimes \mathbb{C}[[t, t^{-1}]]$$

The algebra on the right has generators

$$J^a \otimes t^n =: J^a_n$$

for $n \in \mathbb{Z}$. Multiplying two periodic functions according to the Lie bracket yields another periodic function

$$\begin{aligned} [J^a \otimes t^n, J^b \otimes t^m] &= i f^{ab}{}_c J^c t^{n+m} \\ [J^a_n, J^b_m] &= i f^{ab}{}_c J^c_{n+m} \end{aligned}$$

ii) There is a unique central extension of the loop algebra

$$[J^a_n, J^b_m] = \sum_c i f^{ab}{}_c J^c_{n+m} + \hat{K} n \underbrace{K(J^a, J^b)}_{\delta^{a,b}} \delta_{n+m,0}$$

which satisfies the commutation relations

$$\begin{aligned} [J^a_n, J^b_m] &= \sum_c i f^{ab}{}_c J^c_{n+m} + \hat{K} n \delta^{a,b} \delta_{n+m,0} \\ [J^a_n, \hat{K}] &= 0 \end{aligned}$$

This is the *affine Kac-Moody algebra* of $\mathfrak{g}$.

Associated to an affine Kac-Moody algebra, we have a current algebra

$$\mathcal{R}(J^a(z) J^b(w)) \sim \frac{k\delta^{ab}}{(z-w)^2} + \sum_c i f^{ab}{}_c \frac{J^c(w)}{z-w}$$

where

$$J^a(z) = \sum_{n \in \mathbb{Z}} z^{-n-1} J^a_n$$

In modes, we recover

$$[J^a_n, J^b_m] := \sum_c i f^{ab}{}_c J^c_{n+m} + \hat{K} n \delta_{a,b} \delta_{n+m,0}$$

Suyawara Energy-Momentum Tensor:

$J^a(z)$ are primary fields with respect to the energy-momentum tensor

$$T(z) = \gamma \sum_a : J^a J_a : (z)$$

With coupling constant γ, which is fixed by the OPE

$$\mathcal{R}(T(z)T(w)) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

So one obtains

$$\boxed{\gamma = \frac{1}{2(k+g)} \quad ; \quad c(\mathfrak{g}_k) = \frac{k \cdot \dim \mathfrak{g}}{k+g}}$$

Moreover:

$$\begin{aligned}\mathcal{R}(T(z)J^a(w)) &= \gamma \oint_z \frac{du}{2\pi i} \sum_b \left(\frac{J^b(u)\, J_b(z) J^a(w)}{u-z} \right) \\ &= \cdots = 2\gamma(k+g) \left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w} \right)\end{aligned}$$

Thus, $J^a(w)$ are chiral primary fields of conformal weight $h = 1$.

The mode expansion yields

$$T(z) = \sum_n L_n z^{-n-2}$$

where[80]

$$L_n = \frac{1}{2(k+g)} \sum_a \sum_{n \in \mathbb{Z}} : J^a_m J_{a,n-m} :$$

[80] We define

$$: J^a_n J_{a,\ell} : \; = \begin{cases} J^a_n J_{a\ell} & n < \ell \\ J^a_\ell J_{a,n} & m \geq \ell \end{cases}$$

So we see that

$$\boxed{[L_n, J^a_m] = -m J^a_{n+m}}$$

In particular

$$[L_0, J^a_n] = -n J^a_n$$

We can then consider primary states. The representation theory of our affine Kac-Moody algebra yields affine heighest weight states $|\lambda\rangle$:

- For a simple Lie algebra $\mathfrak{g}$, we have generators $E^{\pm\alpha}$[81] and H^i[82] for $i = 1, \ldots, r$ where $r = \text{rank}(\mathfrak{g})$. They satisfy commutation relations

  $$[H^i, H^j] = 0 \quad [H^i, E^{\pm\alpha}] = \pm\alpha E^{\pm\alpha}$$

  where $\alpha \in \Delta^+$ is a positive root.

  As an example, we can consider $SU(2)$, which has Cartan generator J^3 and raising/lowering generators $J^\pm$ satisfying

  $$[J_3, J_\pm] = \pm J_\pm \;\; , \;\; [J_+, J_-] = +2J_3$$

[81] The *Raising and lowering generators.*

[82] The *Cartan generators.*

Recall that unitary representations are labeled by $j, m\rangle$, $2j \in \mathbb{Z}_{\geq 0}$, and $m \in \{-j, -j+1, \ldots, j-1, j\}$. The highest weight state is then $|j, j\rangle$.

- Affine Kac-Moody algebras have generators $E_n^{\pm\alpha}$ and H_n^i. For a *Wess–Zumino–Witten (WZW) highest weight state*[83]

$$E_n^{\pm\alpha}|\lambda\rangle = H_n^i|\lambda\rangle = E_0^{\alpha}|\lambda\rangle = 0$$

Note that $|\lambda\rangle$ will also be a Virasora primary state, since, for $n \geq 0$

$$L_n|\lambda\rangle = \sum_m : J_m^a J_{a,n-m} : |\lambda\rangle = 0$$

That said, a Virasora primary need not be a WZW primary.

[83] Also known as a *WZW primary.*

Example ($SU(2)_k$)**.** We have generators $J_n^{\pm}$ and J_n^3 with

$$\begin{aligned}
[J_n^3, J_m^{\pm}] &= \pm J_{n+m}^{\pm} \\
[J_n^3, J_m^3] &= kn\delta_{n+m,0} \\
[J_n^+, J_m^-] &= 2J_{n+m}^3 + kn\delta_{n+m,0}
\end{aligned}$$

Observe that $\tilde{J}^+ := J_{+1}^+$, $\tilde{J}^- := J_{-1}^-$ and $\tilde{J}^3 := J_0^3 + \frac{k}{2}$ form a $SU(2)$-subalgebra. The highest weight states is

$$|j, j\rangle$$

which has WZW descendants

$$|j, j-1\rangle, \ldots, |j, -j\rangle$$

The norm of the final descendant is

$$\langle j, -j| \underbrace{J_{+1}^+ J_{-1}^-}_{2\tilde{J}^3} |j, -j\rangle = (-2j + k)\langle j, -j|j, -j\rangle$$

So we derive a necessary condition for our representation to be unitary:

$$\boxed{k \leq \frac{k}{2} \;\;, \;\; k \in \mathbb{Z}_{\geq 0}}$$

which must hold lest we have negative norm states.

To compute the conformal weight of the highest weight state, we note first that $g(SU(2)) = 2$, so that

$$L_0 = \frac{1}{k+2}\left((\vec{J}_0)^2 + \ldots\right)$$

and hence

$$h_{|j,j\rangle} = \frac{j(j+1)}{h+2}$$

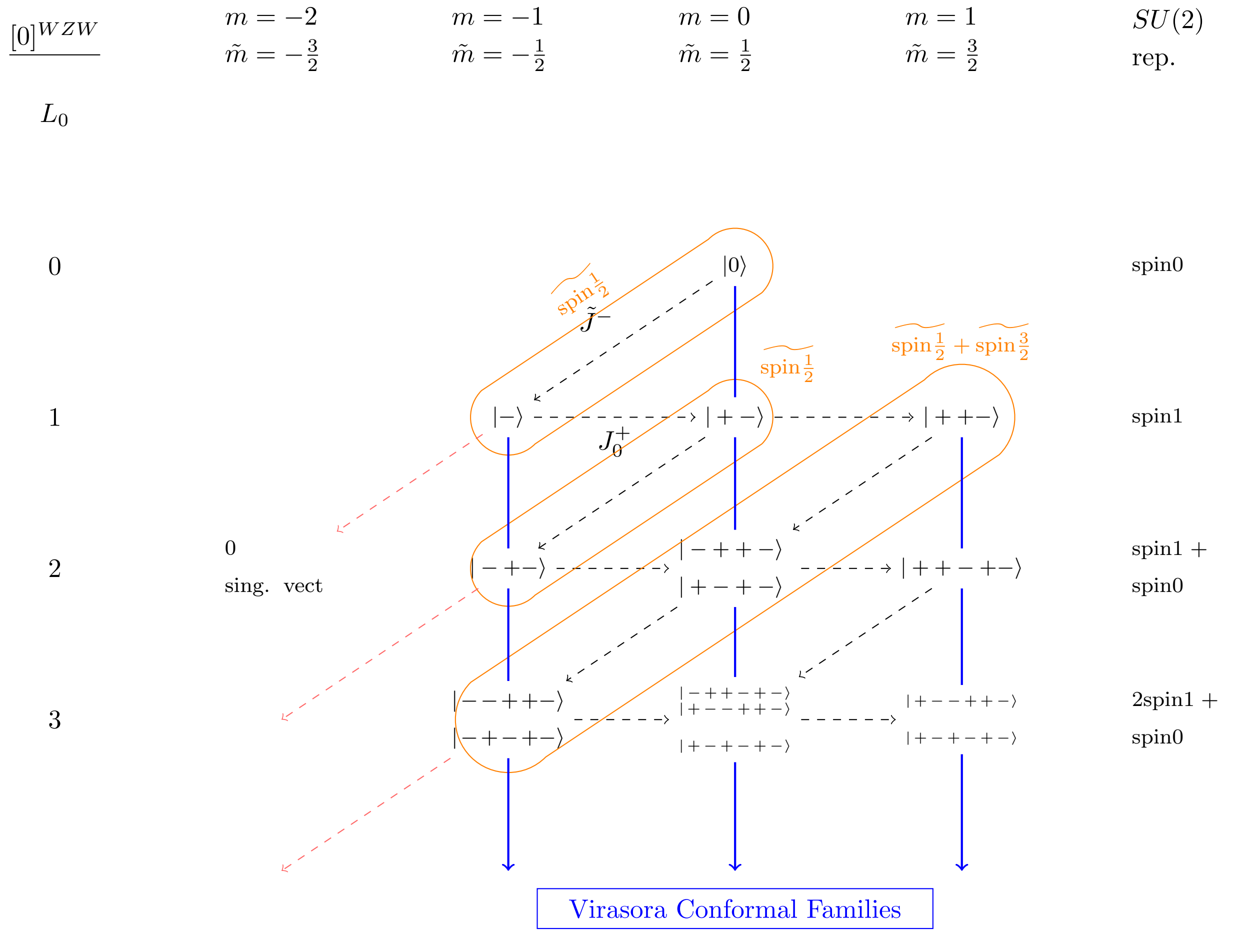
and

$$c = \frac{3}{k+2}$$

According to our constraint derived in the example, the unitary representations of $SU(2)_k$ can be labeled by $[j]^{WZW}$ for $j = 1, \frac{1}{2}, \ldots, \frac{k}{2}$

Example $(SU(2)_1)$**.** We have two unitary representations: $[0]^{WZW}$ and $[\frac{1}{2}]^{WZW}]$, both with $c = 1$[84]. All states are generated by $\tilde{J}^-$ and J_0^+.

[84] For those familiar with string theory, this can be associated with the Boson on the self-dual circle. It arises from T-duality on the bosonic string.



Note. The commutation relation

$$[L_n, J_0^3] = 0$$

ensures that the Virasora generators do not change the m-quantum numbers.[85]

[85] I have not yet included the $[\frac{1}{2}]^{WZW}$ chart.

Remarks. • WZW models are unitary, which can be proven by:

a) Pure representation theoretic arguments on $\mathfrak{g}_k$.

b) WZW models admit a non-linear σ-model description with manifest unitary action[86]

$$S_{WZW} = -\frac{k}{8\pi}\int_{S^2} d^2x \left[K(g^{-1}\partial^\mu g, g\partial_\mu g)\right] + 2\pi k S^{WZ}(g)$$

where, for an extension of g $\tilde{g} : B^3 \to G$[87], the correction term is given by

$$S^{WZ}(g) = -\frac{1}{48\pi^2}\int_{B^3} K\left(\tilde{g}^{-1}\partial_\mu \tilde{g}, \left[\tilde{g}^{-1}\partial_\nu \tilde{g}, \tilde{g}\partial_\rho \tilde{g}\right]\right) dx^\mu \wedge dx^\nu \wedge dx^\rho$$

- A WZW model has a finite number of conformal WZW families, but an infinite number of Virasora conformal families.
- Roughly speaking, a CFT is called a *Rational Conformal Field Theory (RCFT)* if it has a finite number of conformal families with respect to some current algebra (eg, minimal models, WZW models).

[86] For $g : S^2 \to G$, the compact Lie group associated to G.

[87] Such an extension always exists since $\pi_2(G)$ vanishes for G a compact, simply-connected Lie group.

5.2 Coset Constructions

If we consider the affine Lie algebra

$$\mathfrak{g}_k \oplus \mathfrak{g}_\ell$$

we find that $\mathfrak{g}_{k+\ell}$ is a diagonal affine Lie subalgebra, that is

$$\begin{aligned} {}^D J^a_n &= J^{(1)\,a}_n + J^{(2)\,a}_n \\ \left[\,{}^D J^a_n,\ {}^D J^b_m\right] &= i f^{ab}_{\ \ c}\ {}^D J^c_{n+m} + (k+\ell) n \delta^{ab}\delta_{n+m,0} \end{aligned}$$

This allows us to construct the *diagonal coset CFT*.

$$\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_\ell / \hat{\mathfrak{g}}_{k+\ell}$$

is generated by the Virasora generators

$$L^{coset}_n := L^{\mathfrak{g}_k}_n + L^{\mathfrak{g}_\ell}_n - L^{\mathfrak{g}_{k+\ell}}_n$$

which implies that the central charge is

$$c^{coset} - \dim\mathfrak{g}\left(\frac{k}{k+g} + \frac{\ell}{k+g} - \frac{k+\ell}{k+\ell+g}\right)$$

by construction, such coset CFTs are unitary because the $\mathfrak{g}_k$ and $\mathfrak{g}_\ell$ WZW models are unitary.

Minimal model CFTs arise naturally in this framework, via

$$\frac{SU(2)_k \oplus SU(2)_\ell}{SU(2)_{k+\ell}}$$

which has

$$\begin{aligned} c &= 3\left(\frac{k}{k+\ell}+\frac{1}{3}-\frac{k+1}{k+3}\right) \\ &= 1-\frac{6}{(k+1)(k+3)} \end{aligned}$$

We can then identify these with $m = 2, 3, \ldots$ minimal models via $m = k + 2$.

References

[1] G. Mack and A. Salam, "Finite component field representations of the conformal group," *Annals Phys.*, vol. 53, pp. 174–202, 1969.

[2] S. Garcia, "Hidden invariance of the free classical particle," *Submitted to: Amer. J. Phys.*, 1993.

[3] C. A. Weibel, *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.

[4] J. Hilgert and K.-H. Neeb, *Lie-Gruppen und Lie-Algebren (German Edition).* Vieweg+Teubner Verlag, 1991 ed., 1 1991.

[5] M. Schottenloher, "A mathematical introduction to conformal field theory," *Lect. Notes Phys.*, vol. 759, pp. 1–237, 2008.

[6] P. Di Francesco, P. Mathieu, and D. Senechal, *Conformal Field Theory.* Graduate Texts in Contemporary Physics, New York: Springer-Verlag, 1997.

[7] E. Perlmutter, "Virasoro conformal blocks in closed form," *JHEP*, vol. 08, p. 088, 2015.

[8] D. Friedan, S. H. Shenker, and Z.-a. Qiu, "Details of the Nonunitarity Proof for Highest Weight Representations of the Virasoro Algebra," *Commun. Math. Phys.*, vol. 107, p. 535, 1986.

[9] P. H. Ginsparg, "Applied Conformal Field Theory," in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988*, 1988.

[10] W. Lerche, D. J. Smit, and N. P. Warner, "Differential equations for periods and flat coordinates in two-dimensional topological matter theories," *Nucl. Phys.*, vol. B372, pp. 87–112, 1992.

[11] D. Friedan, Z. A. Qiu, and S. H. Shenker, "Conformal Invariance and Critical Exponents in Two Dimensions," 1986.

[12] R. Blumenhagen and E. Plauschinn, "Introduction to conformal field theory," *Lect. Notes Phys.*, vol. 779, pp. 1–256, 2009.

[13] J. Fuchs, "Lectures on conformal field theory and Kac-Moody algebras," 1997. [Lect. Notes Phys.498,1(1997)].