On Graph Sparsity and Structure: Colourings and Graph Decompositions

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vorgelegte von

Konstantinos Stavropoulos, M.Sc.

aus

Athen, Griechenland

Berichter: Univ.-Prof. Dr. Martin Grohe, Univ.-Prof. Dr. Victor Chepoi

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To the memory of my beloved grandmother Lela.

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Declaration of Authorship

I, Konstantinos Stavropoulos, confirm that the research included within this dissertation is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below. I attest that I have exercised reasonable care to ensure that the work is original, and does not to the best of my knowledge break any german law, infringe any third party's copyright or other Intellectual Property Right, or contain any confidential material. I confirm that this thesis has not been previously submitted for the award of a degree by this or any other university.

Signature: Konstantinos Stavropoulos
Date: 9th August, 2016

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The parts included in this dissertation from the first two papers jointly written with my co-authors correspond to my contribution to them. In particular, my contributions to the first paper appear in Sections 3.3 and 3.4. My contribution to the second paper appears in Section 4.2.

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Abstract

We study structural aspects both of sparse and dense graph classes. In particular, we study in detail generalised colourings for sparse classes and their combinatorial applications to other related notions. Furthermore, we extend the known concept of tree decompositions which is central in the theory of Graph Minors and various other classes of sparse graphs, now as a structural tool for classifying dense graph classes.

Bounded expansion and nowhere dense classes of graphs are relatively new families of graph classes generalising many commonly studied sparse graph classes such as classes of graphs of bounded degree and classes defined by excluded (topological) minors. They can be characterised through the generalised colouring numbers, for which we show various lower and upper bounds. We utilise the generalised colouring numbers to prove colouring results related to distance colourings of graphs and to obtain a new characterisation of bounded expansion by the notion of neighbourhood complexity.

We generalise tree decompositions by introducing median decompositions along with their respective medianwidth invariants, where a graph can be modelled after any median graph. Depending on the notion of dimension we consider, this gives rise to hierachies of graph parameters that start from treewidth or pathwidth and converge to the clique number. Another variation of the parameter characterises the chromatic number of a graph. We provide characterisations of the parameters via intersections of tree or path decompositions and by a generalisation of the classical Cops and Robber game, where the robber plays against not just one team of cops, but many teams of cops simultaneously. Contrary to tree decompositions, we demonstrate that the high-dimensional nature of general median decompositions and their medianwidth parameters makes them more suitable for the study of classes of dense graphs.

Zusammenfassung

Wir untersuchen strukturelle Aspekte von Klassen magerer und dichter Graphen. Insbesondere betrachten wir verallgemeinerte Färbungen für magere Klassen und deren kombinatorische Anwendungen. Ausserdem erweitern wir das bekannte Konzept der Baumzerlegungen und erhalten auf diese Weise ein strukturelles Werkzeug zur Klassifikation dichter Graphklassen.

Graphklassen mit beschränkter Expansion und nirgends dichte Graphklassen sind relativ neue Familien von Klassen, die viele bekannte magere Graphklassen verallgemeinern, wie etwa Klassen von Graphen mit beschränktem Grad und Klassen mit verbotenen (topologischen) Minoren. Sie können mit Hilfe der verallgemeinerten Färbungszahlen charakterisiert werden, für die wir untere und obere Schranken zeigen. Wir verwenden die verallgemeinerten Färbungszahlen, um Färbungsresultate mit Bezug zu Abstandsfärbungen von Graphen zu beweisen und um eine neue Charakterisierung von beschränkter Expansion mittels des Konzepts der Nachbarschaftskomplexität zu erhalten.

Wir verallgemeinern Baumzerlegungen, indem wir Medianzerlegungen sowie entsprechende Medianweiten-Invarianten einführen. Bei solchen Zerlegungen wird ein Graph nach einem beliebigen Mediangraphen modelliert. In Abhängigkeit vom betrachteten Dimensionsbegriff ergeben sich dadurch Hierarchien von Graphparametern, die bei Baumweite oder Wegweite beginnen und gegen die Cliquenzahl konvergieren. Eine andere Parametervariante wiederum charakterisiert die chromatische Zahl eines Graphen. Wir geben Charakterisierungen der Parameter durch Schnitte von Baum- bzw. Wegzerlegungen und durch eine Verallgemeinerung des klassischen Räuber-und-Polizisten-Spiels, in dem der Räuber nicht gegen nur ein einziges, sondern gleichzeitig gegen mehrere Teams von Polizisten antritt. Wir zeigen auf, dass allgemeine Medianzerlegungen und die entsprechenden Medianweiten-Parameter, anders als Baumzerlegungen, aufgrund ihrer Hochdimensionalität besser zur Untersuchung von Klassen dichter Graphen geeignet sind.

Anyone who keeps the ability to see beauty never grows old.

Franz Kafka

1

Introduction

Graph theory is one of the classical disciplines of Discrete Mathematics with rapidly growing influence on other disciplines and especially on Computer Science.

One could say that the main goals of Structural Graph Theory is to study (families of) graphs satisfying a certain set of properties and understand what other structural properties these graphs must also satisfy (and vice versa). This can be achieved by attempting to look at the studied graphs through as many sensible viewpoints as possible. Ultimately, we would even like to fully grasp what their structure is by decomposing them in a manageable way into smaller, easier to understand parts. More specifically, when studying a specific question on a certain kind of graphs some of the above lines of research are the following:

- we want to draw comparisons or connect to objects, properties and notions of which we already have a higher level of understanding
- for difficult problems, we sometimes restrict ourselves to special cases to ease some of the difficulty to at least be able to say something meaningful
- for easier questions, we would like to see up to what degree we can abstract or relax certain objects or notions so that they preserve in a similar—and still manageable—fashion all (or most) of their characteristic properties
- hopefully, we would like to be naturally led into new perspectives that are motivated by the nature of the studied graphs and questions themselves.

1.1 From Single Sparse Graphs...

A typical structural approach is the description of many important graph classes by a finite set of forbidden substructures that are not allowed to exist within the graphs of the family in a certain specified way. A prototypical example of this approach is Kuratowski's Theorem, which states that a graph is planar (can be drawn in the plane without crossings) if and only if it does not contain either of two forbidden graphs, the complete graph K_5 and the complete bipartite graph $K_{3,3}$, as *(topological) minors*.

Possibly motivated in pursue of a direct generalisation of the above concept, it was conjectured—known in the literature under the name *Wagner's Conjecture*—that every infinite sequence of graphs contains a pair such that one is a minor of the other. This would directly imply that every minor-closed family of graphs (namely a class closed under taking minors) would be fully characterised by forbidding a finite set of minor-minimal graphs that can not appear as minors of any graph of the class, exactly as planar graphs are characterised by forbidding K_5 and $K_{3,3}$.

In their monumental work throughout a series of twenty three papers, Robertson and Seymour proved Wagner's conjecture, now known as the Graph Minor Theorem [86]. The notion of *tree decompositions* and *treewidth* was first introduced (under different names) by Halin [46] and arose as a natural and central concept in the work of Robertson and Seymour on graph minors, who reintroduced it in its more standard form [81, 82]. Treewidth, denoted by $\mathrm{tw}(G)$, can be seen as a measure of how 'treelike' a graph is and has turned out to be a connectivity measure of graphs (see [78]). Maybe the most important cornerstone of their theory is a decomposition theorem, also known as 'Excluded Minor Theorem', characterising the structure of all graphs that exclude a fixed minor. Let us only note for the moment that, roughly, this decomposition is a tree decomposition of such a graph essentially into its '2-dimensional' parts.

Moreover, the whole theory on graph minors that Robertson and Seymour developed for classes of graphs excluding a fixed minor is a very powerful structure theory which has found a large number of algorithmic consequences. But even more generally, Structural Graph Theory has proven to be a powerful tool for coping with computational intractability. The wealth of concepts and results that it provides can be used to design efficient algorithms for hard computational problems when restricted to specific classes of graphs that occur naturally in applications.

In general, from the algorithmic point of view we want to understand what kind of problems can be solved efficiently on which classes of graphs. That is, for natural classes of problems we want to understand their general tractability frontier, i.e. the 'most general' classes of graphs on which these problems become tractable.

As diverse as the examples of graph classes with a rich algorithmic theory may appear, such as the already thoroughly studied classes of bounded degree, classes of bounded genus, classes defined by excluded (topological) minors, a feature all these classes have in common is that they are relatively sparse, i.e. graphs in these classes have a relatively low number of edges compared to the number of vertices. This suggests that this 'sparsity' might be an underlying reason why many problems can be solved efficiently on these classes of graphs, even though they otherwise do not have much in common.

This leads to the question, both from the structural and the algorithmic perspective, how to define a reasonable concept of 'sparse graphs', a concept that distinguishes between sparse and dense graphs with no fuzzy boundaries. Such a task is far from obvious, especially when one has to come up with a robust framework that settles a dichotomy between what is sparse and what is dense. In contrast to graphs excluding a fixed minor, whose sparsity is absolute in the sense that we classify a graph as sparse just by looking only at the graph itself, it turns out that the key intuition towards achieving such a dichotomy is that a sparse graph should not be defined by itself, but relative to other graphs. It should be a notion not just applying to single graphs, but rather to whole *classes of graphs*.

1.2 ...to Graph Classes and the Limit of Sparseness...

Graph classes of *bounded expansion* (and their further generalisation, *nowhere dense classes*) have been introduced by Nešetřil and Ossona de Mendez [68, 69, 72] as a general model of *structurally sparse* graph classes. They include and generalise many other natural sparse graph classes, among them all classes of bounded degree, classes of bounded genus, and classes defined by excluded (topological) minors. Nowhere dense classes even include more general classes such as classes that locally exclude a minor.

Bounded expansion and nowhere dense classes roughly capture the following intuition: the *deeper* one has to look into graphs to find a dense substructure, the *sparser* the graphs should be and one should note that this notion of sparsity makes sense only within the context of a whole graph class and not a single graph any more.

The appeal of the above sparsity notions and their applications stems from the fact that they have turned out to fulfil exactly the requirement that we would look for, i.e. they are very robust properties of graph classes with various seemingly unrelated characterisations (see [43, 72]). These include characterisations through the density of shallow minors [68], *quasi-wideness* [25], *low treedepth colourings* [68], and *generalised colouring numbers* [94]. The latter two are particularly relevant towards applications of the nice algorithmic properties which bounded-expansion and nowhere dense classes have been demonstrated to enjoy in several papers, e.g., [26, 44, 71]. As a matter of fact, subgraph closed nowhere dense classes are a natural limit for the efficient solvability of a wide class of problems [36, 44, 57].

It seems unlikely that bounded-expansion and nowhere dense classes admit global Robertson-Seymour style decompositions as they are available not only for classes excluding a fixed minor, but also classes excluding a topological minor [45], an immersion [93], or an odd minor [27]. However, Nešetřil and Ossona de Mendez showed [69] that bounded expansion and nowhere dense classes admit a 'local' decomposition, a so-called *low r-treedepth colouring*, in the following sense: for every integer r, every graph from a bounded expansion (nowhere dense) class can be coloured with $\chi_r(G) \leqslant f(r)$ (respectively $\chi_r(G) \leqslant \mathcal{O}(|G|^{o(1)})$) colours such that every union of $p < r$ colour classes induces a graph of treedepth at most p, where treedepth roughly measures how 'starlike' a graph is. These types of colourings generalise the star-colouring number introduced by Fertin, Raspaud, and Reed [39].

Among the many characterisations of bounded expansion and nowhere denseness, the notion of study in the first part of this dissertation is the so-called *generalised colouring numbers*. Roughly, the generalised colouring numbers describe how well the vertices of a graph can be linearly ordered such that for any vertex v, the number of vertices that can reach v via short paths of length at most a number r that use higher-order vertices is bounded. More specifically, they are the *weak r-colouring numbers* $\mathrm{wcol}_r(G)$, the *r-colouring numbers* $\mathrm{col}_r(G)$ and the *r-admissibility numbers* $\mathrm{adm}_r(G)$ of a graph G.

The two families of colouring numbers were introduced by Kierstead and Yang in [52], and the admissibility numbers go back to Kierstead and Trotter in [51] and were generalised by Dvořák in [35]. The name 'colouring numbers' reflects the fact that when we only allow paths of length at most one, the generalised colouring numbers correspond to the *degeneracy* of a graph, sometimes also called the *colouring number*, which is defined to be the minimum d such that there is a linear order of the vertices of G in which every vertex has at most d smaller neighbours.

Generalised colouring numbers are particularly relevant in the algorithmic context, especially with respect to *neighbourhood covers*. Neighbourhood covers play an important role in the study of distributed network algorithms and other application areas (see, for example, [76] and [1]). The neighbourhood covers for nowhere dense graph classes developed in [44] combine low radius and low degree making them interesting for the applications outlined above. Their existence is established through such colouring numbers—the *weak r-colouring numbers*, to be precise—and the value of these numbers is directly related to the degree of the neighbourhood covers.

Our contribution in the first part of the dissertation is twofold: first we provide various upper and lower bounds for these families of colouring numbers. In particular, we prove tight polynomial bounds for the colouring numbers for graphs of bounded treewidth which are contrasted with new, stronger exponential lower bounds that can already be achieved on graph classes of bounded degree. Moreover, in general the colouring numbers are also known to be bounded within each other through the following relations:

- $\mathrm{adm}_r(G) \leqslant \mathrm{col}_r(G) \leqslant \mathrm{wcol}_r(G) \leqslant \mathrm{adm}_r(G)^r$,
- $\mathrm{wcol}_r(G) \leqslant \mathrm{col}_r(G)^r$.

To our knowledge there are no examples in the literature showing optimality of the known gaps between the colouring numbers. We verify that the exponential dependency on r of these gaps in the inequalities above is unavoidable.

Secondly, the characterisation of bounded expansion through generalised colouring numbers in [94] was provided by relating low treedepth colourings to generalised colouring numbers. We believe that it is useful to highlight this interaction of the two notions, in the sense that when one can use one of the two notions as a direct proof tool, it might often be the case for the other as well (the most appropriate to be chosen depending on the occasion). This is also supported by the fact that the general known bounds relating low treedepth colourings and generalised colouring numbers seem to be very loose and most probably not optimal. For example, it is still unclear if one is always smaller than the other. Moreover, bounds for both parameters are not in general known for all kinds of specific graph glasses. It can then be the case that for different questions and different graph classes, generalised colouring numbers are more appropriate than low treedepth colourings or vice versa.

To this end, we point out some of the similarities and differences between the two approaches by proving two results utilising generalised colourings that can also be approached by low treedepth colourings. More specifically, we provide new bounds for the chromatic number of exact odd-distance powergraphs of bounded expansion graph classes, which are motivated by connections to graph homomorphisms. Moreover, we provide an upper bound for neighbourhood complexity—which reflects how many different subsets of a given vertex set in a graph can be the exact neighbourhood of another vertex of the graph— in terms of weak colouring numbers. This leads to an alternative characterisation of the property of bounded expansion.

1.3 ...to Highdimensionality and Denseness

The sparse classes discussed above are intimately related to trees, in the sense that through one treelike decomposition of bounded measure or another, they manage to lift characteristic properties of trees into their setting. But more generally, the usefulness of tree decompositions, for example, as a decomposition tool is highlighted not only in the theory of Graph Minors but also by other various, often very general, structural theorems ([19, 20, 21, 47, 84, 85]).

Since the concept of modelling a graph like a 'thick' tree allows several advantages of trees to be lifted onto more general graphs, it is tempting to investigate how to go beyond tree decompositions and try to model a graph on graphs other than trees (in the sense that the former has 'bounded width' in terms of the latter), maybe as a means to study how these more general decompositions can be used to form structural hierarchies of graph classes. Diestel and Kühn proposed a version of such general decompositions with interesting implications in [29], who also noted a disadvantage in their decompositions: all graphs, when modelled like a grid, have bounded 'gridwidth'.

A median graph is a connected graph, such that for any three vertices u, v, w there is exactly one vertex x that lies simultaneously on a shortest (u, v)-path, a shortest (v, w)-path and a shortest (w, u)-path. Examples of median graphs are grids and the i-dimensional hypercube Q_i, for every $i \geqslant 1$. One of the simplest examples of median graphs are trees themselves.

One might choose to see trees as the one-dimensional median graphs under a certain perspective: for example, the topological dimension of a tree continuum is one; or amalgamating one-dimensional cubes, namely edges, on a tree, will also produce a tree; or trees are the median graphs not containing a square (the two-dimensional cube) as an induced subgraph [62].

A subset S of vertices of a connected graph is *(geodesically) convex* if for every pair of vertices in S all shortest paths between them only contain vertices in S. The following is the core observation that inspired the second part of the dissertation:

Convexity degenerates to connectedness on trees!

In a tree decomposition, a vertex of the graph lives in a connected subgraph of the underlying tree. The properties of convex subsets of median graphs, one of them being the *Helly Property*, provide the means allowing the extension of the concept of tree decompositions into the setting of *median decompositions* in a rather natural way: when we use general median graphs as the underlying graph of the decomposition, a vertex of the original graph will live in a convex subgraph.

Median decompositions and their respective width parameters is the topic of the second part of the dissertation.

This generalisation of tree decompositions will, as a result, allow for finer decompositions of the decomposed graph. Median graphs are a high-dimensional generalisation of trees within the context of several notions of dimension, therefore median decompositions provide a rigorous framework that broadens substantially the perspective with which we can view graphs: they provide a means to see every graph as a high-dimensional object in a concrete geometric sense.

Having already discussed sparse graph classes, we shall see that medianwidth parameters are more suitable to classify rather the dense, than the sparse graph classes. Fittingly, this might perhaps add up more perspective to the intuition behind the dichotomy between sparse and dense: whenever the graphs of a graph class can be modelled in some sort of treelike—and hence one-dimensional—fashion of bounded measure, be it width or depth related, they can be seen to be sparse, but as soon as they have to be seen as high-dimensional objects they probably become dense.

It turns out that median decompositions preserve all the characteristic properties of tree decompositions. Surprisingly, we also prove that the corresponding width parameter $\mathrm{mw}(G)$ matches the *clique number* $\omega(G)$ of a graph G, the size of its largest complete subgraph. Moreover, we study a specific variation of median decompositions, which satisfy an additional axiom ensuring more regularity for them. Certain median decompositions, which we will call *chromatic median decompositions* and arise by making use of a proper colouring of the graph, enjoy this additional regularity by their definition. This allows us to show that the respective width parameter, to be called *smooth medianwidth*, is equivalent to the *chromatic number* $\chi(G)$ of G.

We also take a step even further and discuss a general framework of how to decompose a graph G in any fixed graphlike fashion, where the underlying graph of the decomposition is chosen from an arbitrary fixed graph class $\mathcal{H}$, and such that the most important properties of tree and median decompositions are preserved.

Now, every median graph can be isometrically embedded into the Cartesian product of a finite number of trees. We consider median decompositions whose underlying median graph must be isometrically embeddable into the Cartesian product of i trees, along with the respective medianwidth parameter, to be called i-medianwidth $\mathrm{mw}_i(G)$. By definition, the invariants mw_i will form a non-increasing sequence:

$$\mathrm{tw}(G) + 1 = \mathrm{mw}_1(G) \geqslant \mathrm{mw}_2(G) \geqslant \ldots \geqslant \mathrm{mw}(G)_\infty = \omega(G).$$

Since they are all bounded from below by the clique number of the graph, they do not share the same disadvantage as the decompositions of [29], where the 'gridwidth' of all graphs was bounded (note that in our setting a decomposition in a 'gridlike' fashion would only be a 2-median decomposition). Moreover, by considering complete multipartite graphs, we establish that this infinite hierarchy of parameters is proper in the strong sense that each of its levels is 'unbounded' in the previous ones: for $i < i'$, graphs classes of bounded i'-medianwidth can have unbounded i-medianwidth. This also provides a natural way to go beyond treewidth and obtain new 'bounded width' hierarchies of the class of all graphs, now in terms of bounded i-medianwidth, for different $i \geqslant 1$. Additionally, one of our main results is a characterisation of i-medianwidth in terms of tree decompositions: roughly, we prove that it corresponds to the largest 'intersection' of the 'best' choice of i tree decompositions of the graph.

Similarly, instead of considering median decompositions whose underlying median graph can be isometrically embedded into the Cartesian product of i trees, we study medianwidth parameters for which we consider median decompositions whose underlying median graph must be isometrically embeddable into the Cartesian product of i *paths*. For $i \geqslant 1$, the corresponding width parameters, to be called i-latticewidth $\mathrm{lw}_i(G)$, will give rise to a sequence converging to the clique number and starting from pathwidth:

$$\mathrm{pw}(G) + 1 = \mathrm{lw}_1(G) \geqslant \mathrm{lw}_2(G) \geqslant \ldots \geqslant \mathrm{lw}_\infty(G) = \omega(G).$$

Lastly, we provide a characterisation of i-latticewidth in terms of path decompositions: we prove that it corresponds to the largest 'intersection' of the best choice of i path decompositions of the graph.

A large variety of width parameters for graphs are characterised through so-called *search games*, introduced by Parsons and Petrov in [74, 75, 77]. A set of searchers and a fugitive move on a graph according to some rules specified by the game. The goal of the searchers is to capture the fugitive, whose goal is to avoid capture. Different variants of the rules according to which the searchers and the fugitive move, give rise to games that characterise related width parameters, often otherwise introduced and appearing in different contexts. These game characterisations provide a better understanding of the parameters. For a survey on search games, see [40].

Treewidth and pathwidth are known to be characterised by appropriate variations of the *Cops and Robber game*—sometimes seen as *helicopter Cops and Robber game* in the literature. The game is played on a finite, undirected graph G by the cop player, who controls k cops, and the robber player. The robber stands on a vertex of G and can run arbitrarily fast through a path of G to any other vertex, as long as there are no cops standing on the vertices of the path. Each of the k cops either stands on a vertex of G or is in a helicopter in the air. The cop player tries to capture the robber by landing a cop with a helicopter on the vertex where the robber stands and the robber tries never to be captured. The robber sees where each of the k cops stands or if they are going to land on a vertex of G and can move arbitrarily fast to another vertex to evade capture while some of the cops are still in the air.

While the robber can always see the cops at any point of the game, there are two forms of the game with respect to the information available to the cop player. In the first variation, the cop player can see the robber at all times and tries to surround her in some corner of the graph. This version of the game characterises the treewidth of G in the sense that the cop player has a winning strategy with at most k cops if and only if $\mathrm{tw}(G) \leqslant k-1$ [87]. In the second variation of the game, the robber is invisible to the cop player so he has to search the graph in a more methodical way. In this version of the game, the cop player can always win with at most k cops if and only if $\mathrm{pw}(G) \leqslant k-1$ [10, 53, 59].

We provide a game characterisation of medianwidth parameters by extending the above game to a setting where the robber now plays against not just one, but i teams of cops. More precisely, we show that i cop players can *monotonely* search a graph G for a visible robber with cooperation at most k if and only if $\mathrm{mw}_i(G) \leqslant k$. Similarly, we show that i cop players can *monotonely* search a graph G for an invisible robber with cooperation at most k if and only if $\mathrm{lw}_i(G) \leqslant k$. To our knowledge, this is also the first instance of a search game played between a single fugitive player against a team of many search players connected to a width parameter of graphs.

1.4 Structure of the Dissertation

This dissertation is organised as follows:

- In Chapter 2, we provide some general background from graph theory that is needed throughout all of the topics considered. Wherever needed, we discuss more specific background related to each topic at the beginning of each chapter.

Our own contributions are contained in the two parts that follow.

- In Part I, we study bounds on generalised colouring numbers and related methods or graph classes of bounded expansion.
  - In Chapter 3, we start by introducing the property of nowhere denseness and bounded expansion in Section 3.1 and we introduce the generalised colouring numbers along with their known properties in Section 3.2. We prove tight bounds for the generalised colouring numbers of graphs of bounded treewidth in Section 3.3 and provide exponential lower bounds for high-girth regular graphs in Section 3.4. In Sections 3.3 and 3.4, we show the tightness of the known inequalities between the generalised colouring numbers.
  - In Chapter 4, we utilise the weak colouring approach as proof method for results connected to graph classes of bounded expansion. In Section 4.1, we prove a singly exponential dependency between the chromatic numbers of exact odd-distance powergraphs obtained from a given graph and the weak colouring numbers of the graph, improving their doubly exponential dependency to the r-treedepth numbers of the graph. Part I is concluded by relating the neighbourhood complexity of a graph to its weak colouring numbers in Section 4.2.
- In Part II, we introduce and develop the theory behind median decompositions and their respective medianwidth parameters.
  - In Chapter 5, we start by summarising some relevant parts of the known theory on median graphs in Section 5.1. In Section 5.2, we formally introduce median decompositions, study their general properties, and prove that the corresponding width parameter $\mathrm{mw}(G)$ matches the *clique number* $\omega(G)$ of a graph G. Section 5.3 is devoted to smooth

median decompositions and we prove that *smooth medianwidth* is equivalent to the *chromatic number* $\chi(G)$ of a graph G. In Section 5.4, we discuss a general framework of $\mathcal{H}$-decompositions, where the underlying graph of the decomposition is chosen from an arbitrary fixed graph class $\mathcal{H}$.

– In Chapter 6, we study medianwidth parameters when seen through different notions of dimension. The i-medianwidth parameters, along with their characterisations in term of 'intersections' of tree decompositions are studied in Section 6.1. We discuss the i-latticewidth parameters in Section 6.2.

– In Chapter 7, we characterise medianwidth parameters in terms of the games. We characterise i-medianwidth in terms of the i-Cops and Robber game with vision in Section 7.1 and i-latticewidth in terms of the i-Cops and Robber game without vision in Section 7.2.

Finally, we conclude with Chapter 8 where we motivate some of the various questions that arise from the topics studied in this dissertation.

Nothing is built on stone; all is built on sand, but we must build as if the sand were stone.

Jorge Luis Borges

2

General Background

2.1 Sets, Functions and Inequalitites

Unless stated otherwise, we will usually denote numbers with lowercase letters and sets with capital letters. The natural numbers are denoted by $\mathbb{N}$. We denote the cardinality of a set X by $|X|$ and its power set by $\mathcal{P}(X)$. We will often refer to a set of sets as a *family* of sets.

Let k be a positive integer. We denote the k-element subsets of a set X by $[X]^k$. A *partition* of a non-empty set X into k sets is a family $\{X_1, \ldots, X_k\}$ of subsets of X such that $\bigcup_{i=1}^k X_i$ and $X_i \cap X_j =$ for $1 \leqslant i < j \leqslant s$. For sets $X_1, \ldots, X_k$, the set

$$X_1 \times \ldots \times X_k := \{(x_1, \ldots, x_k) : x_i \in X_i, 1 \leqslant i \leqslant k\}$$

is the *Cartesian product* of $X_1, \ldots, X_k$. We denote $\underbrace{X \times \ldots \times X}_{k \text{ times}}$ by X^k. A *transversal* of a family $X_1, \ldots, X_k$ is an element of $X_1 \times \ldots \times X_k$.

A *k-ary relation* R over $X_1, \ldots, X_k$ is a subset of $X_1 \times \ldots \times X_k$. When $X_i = X$ for every $1 \leqslant i \leqslant k$, we simply say that R is a relation *over* X. A binary relation R over X is *reflexive* if $(a,a) \in R$ for every $a \in X$, and it is *transitive* if for every $a, b, c \in X$, $(a,b) \in R$ and $(b,c) \in R$ imply $(a,c) \in R$. The relation R is *symmetric* if for every $a, b \in X$ we have $(a,b) \in R$ if and only if $(b,a) \in R$, and *antisymmetric* if $(a,b) \in R$

and $(b,a) \in R$ imply $a = b$. For binary relations represented by standard inequality symbols, we use infix notation, for example we write $x \leqslant y$ instead of $(x,y) \in \leqslant$.

A binary relation is a *partial order* if it is reflexive, antisymmetric and transitive. A partial order R over a set X is a *linear order* if for every two elements $x, y \in X$, it is either $(x,y) \in R$ or $(y,x) \in R$.

If a binary relation is reflexive, symmetric and transitive, it is an *equivalence relation*. For an equivalence relation R over a set X and an element $a \in X$, the *equivalence class of x under R*, denoted by $[x]_R$, is the set $\{y \in X \mid (x,y) \in R\}$. . An equivalence class of R over X is a set $[x]_R$ for some $x \in X$. When $(a,b) \in R$, we have $[x]_R = [y]_R$. An element of an equivalence class Y of R over X is a *representative* of Y.

A reflexive and transitive relation is called *quasi-ordering*. A quasi-ordering $\leqslant$ on a set A is a *well-quasi-ordering*, and the elements of A are well-quasi-ordered by $\leqslant$, if for every infinite sequence $a_0, a_1, \ldots$ in A there are indices $i < j$ such that $a_i \leqslant a_j$.

For a function $f : \mathbb{N} \to \mathbb{N}$, we denote

$$\mathcal{O}(f) = \left\{ g \colon \mathbb{N} \to \mathbb{N} \mid \limsup_{n \to \infty} \frac{g(n)}{f(n)} < \infty \right\}$$

and

$$o(f) = \left\{ g \colon \mathbb{N} \to \mathbb{N} \mid \limsup_{n \to \infty} \frac{g(n)}{f(n)} = 0 \right\}$$

When $f \in \mathcal{O}(g)$, we also write $g \in \Omega(f)$. When $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(f)$, we write $g \in \Theta(f)$.

Finally, the *Cauchy-Schwarz* inequality for real numbers states that for all positive reals $a_1, \ldots, a_k, b_1, \ldots, b_k$ the following holds:

$$\left(\sum_{i=1}^{k} a_i^2 \right) \left(\sum_{i=1}^{k} b_i^2 \right) \geqslant \left(\sum_{i=1}^{k} a_i b_i \right)^2 ,$$

with equality if and only if $\frac{a_1}{b_1} = \ldots = \frac{a_k}{b_k}$.

2.2 Graphs

A (simple, undirected) *graph* is a pair $G = (V(G), E(G))$ of sets, where $E(G) \subseteq [V(G)]^2$. The elements of $V(G)$ are the *vertices* (or *nodes*) of G and those of $E(G)$ are its *edges*. An edge $\{u, v\} \in E(G)$ will also be denoted as uv (or vu). We say that a vertex v is *incident* to an edge e if $v \in e$. The vertices u, v of an edge uv are the *endpoints* or *ends* of the edge. For the *empty graph* $(\emptyset, \emptyset)$ we write $\emptyset$. The *complement* G^c of G is the graph on $V(G^c) = V(G)$ with edge set $E(G^c) = [V(G)]^2 \setminus E(G)$. All graphs in this dissertation are finite.

We call two graphs G, G' *isomorphic*, and write $G \simeq G'$, if there exists a bijection $\varphi : V(G) \to V(G')$ with $uv \in E(G)$ if and only if $\varphi(u)\varphi(v) \in E(G')$. To simplify the notation, we will usually write $G = G'$ rather than $G \simeq G'$. A class of graphs closed under isomorphism is called a *graph property*.

For two graphs G, G', we define $G \cup G' := (V(G) \cup V(G'), E(G) \cup E(G'))$ and $G \cap G' := (V(G) \cap V(G'), E(G) \cap E(G'))$. When $G \cap G' = \emptyset$, we call them *disjoint*. G' is a *subgraph* of G, denoted by $G' \subseteq G$, if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. It is a *proper subgraph* when, additionally, $G' \neq G$. When $G' \subseteq G$, we say that G *contains* G' or that G' *is in* G.

Let $V' \subseteq V(G)$. The graph $G[V'] := (V', E(G) \cap (V' \times V'))$ is called the subgraph of G *induced* by V'; it is *spanning* if $V' = V(G)$. A subgraph G' of G is an *induced subgraph* of G if there is a vertex set $V' \subseteq V(G)$ such that $G' = G[V']$.

For a set of vertices U of G, we write $G \setminus U$ for $G[V(G) \setminus U]$. In that case, we say that $G \setminus U$ is obtained by *deleting* the vertices of U and their incident edges. Similarly, for $F \subseteq V(G) \times V(G)$, by $G \setminus F$ we mean the graph $(V(G), E(G) \setminus F)$. For a vertex v and an edge e, we will write $G - v$ and $G - e$ instead of $G \setminus \{v\}$ and $G \setminus \{e\}$.

Two vertices $u \neq v$ in G are *adjacent* or *neighbours* if $uv \in E(G)$, while two edges $e \neq f$ are *adjacent*, if they have an end in common. A subgraph G' of G is *complete* (or a *clique* in G) if its vertices are pairwise adjacent in G. We denote the complete graph on n vertices by K_n. Pairwise non-adjacent vertices in a graph are called *independent*, forming an *independent vertex set*. The *clique number* $\omega(G)$ of G is the size of the largest complete subgraph of G, while its *independence number* $\alpha(G)$ is the size of its largest independent set.

The Cartesian product $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$, in which vertices (a,x) and (b,y) are adjacent whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. The graphs G and H are called the *factors* of $G \square H$.

2.3 Degrees

The set of neighbours of a vertex v is called the *neighbourhood* of v and denoted by $N_G(v)$, or $N(v)$ if the referred graph is clear from the context.

The *degree* $d_G(v)$ of a vertex v is the size $|N_G(v)|$ of its neighbourhood. Again, when it is clear by the context, we drop the index and write $d(v)$. If all the vertices of the graph have the same degree k, we call it k-regular.

The *minimum degree* of G is the number $\delta(G) := \min\{d(v) \mid v \in V(G)\}$ and, similarly, its *maximum degree* the number $\Delta(G) := \max\{d(v) \mid v \in V(G)\}$. The *average degree* of G is

$$d(G) := \frac{1}{|V(G)|} \sum_{v \in V(G)} d(v).$$

By definition,

$$\delta(G) \leqslant d(G) \leqslant \Delta(G),$$

and a simple double-counting argument (also known as the *handshake lemma*) shows that

$$d(G) = 2\frac{|E(G)|}{|V(G)|}.$$

The right side of the above equation gives rise to one of the most fundamental measures of the sparsity of a graph G, the *edge density* of G, which is the ratio

$$\varepsilon(G) := \frac{|E(G)|}{|V(G)|}.$$

It follows that $d(G) = 2\varepsilon(G)$.

G is k-degenerate if every subgraph $H \subseteq G$ has a vertex of degree at most k. If G is k-degenerate, then a simple inductive argument on the number of vertices of G shows that $\varepsilon(G) < k$. In particular, iterative deletion of vertices of degree at most k in the resulting graph of each step implies the existence of a linear ordering of the vertices of G such that every vertex of G has at most k neighbours lower in the order.

2.4 Paths, Cycles, Distance and Connectivity

A *path* is a non-empty graph P with

$$V(P) = \{v_0, v_1, \dots, v_k\} \quad \text{and} \quad E(G) = \{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\}.$$

We will denote it by writing $P = (v_0, \dots, v_k)$. We say that P *links* or *connects* v_0 and v_k. The *length* of P is the number of its edges. The vertices v_0 and v_k are called the *ends* of P, while $v_1, \dots, v_{k-1}$ are its *inner* or *internal* vertices and we say that P is a (v_0, v_k)-path. The graph isomorphic to the path on m vertices is denoted by P_m.

A *cycle* is a graph obtained by connecting the ends of a path of length at least 2 with an edge. More specifically, a cycle is of the form $C = (V(P), E(P) \cup \{u_1u_2\})$, where P is a path with ends u_1, u_2 and $|V(P)| \geqslant 3$. Its *length* is the number of its edges (or vertices). We call the cycle of k edges a k-cycle and denote it by C_k. The minimum length of a cycle contained in a graph G is the *girth* $g(G)$ of G and the minimum length of an odd-cycle in G is the *odd-girth* of G. It G does not contain any cycle, we set $g(G)$ to ∞. A graph is called *chordal* if none of its cycles of length at least 4 is induced.

A pair (A, B) with $A, B \subseteq V(G)$ is a *separation* of G if $A \cup B = V(G)$ and G has no edge between $A \setminus B$ and $B \setminus A$. It is *proper* when $A \setminus B$ and $B \setminus A$ are both non-empty.

A non-empty graph G is *connected* if every pair of vertices is linked by a path in G. If $U \subseteq V(G)$ and $G[U]$ is a connected graph, we also call U *connected in* G. A maximal connected subgraph of G is a (connected) *component* of G. More generally, we say that G is *k-connected* if it has at least $k+1$ vertices and $G \setminus X$ is connected for every $X \subseteq V(G)$ with $|X| < k$. Then the definitions of connected and 1-connected graphs coincide. The greatest integer k such that G is k-connected is the *connectivity* $\kappa(G)$ of G.

The *distance* $d_G(u, v)$ in G is the length of a shortest (u, v)-path in G with the convention $d_G(u, v) = \infty$ if no such path exists. Whenever it is clear by the context, we drop the index and write $d(u, v)$. The *diameter* $\operatorname{diam}(G)$ of G is the largest distance between any two of its vertices, while its *radius* is the quantity $\mathrm{rad}(G) := \min_{u\in V(G)} \max_{v\in V(G)} d_G(u,v)$. It immediately follows from the definitions that

$$\mathrm{rad}(G) \leqslant \mathrm{diam}(G) \leqslant 2\cdot \mathrm{rad}(G).$$

2.5 Forests and Trees

A *forest* is an acyclic graph, one not containing any cycles. A connected forest is a *tree*. The *leaves* of a tree are its vertices of degree 1. Fix a vertex r of a tree T; we call r the *root* of the tree and T *rooted tree* with root r. We denote a tree T with root r by (T,r) or just T when the root of the tree is not necessary for the context. A rooted forest is a forest whose every component is a rooted tree. A rooted tree (T,r) naturally defines a partial order $\trianglelefteq^{T,r}$ (sometimes dropping the superscript accordinlgy) on $V(T)$: $u \trianglelefteq v$ if and only if u belongs to the (unique) path in T linking r and v. In that case, when $u \neq v$ we say that u is an *ancestor* of v.

The closure $\mathrm{clos}(T,r)$ of a rooted tree (T,r) is the graph with vertex set $V(T)$ and edge set $\{uv \mid u \trianglelefteq^{T,r} v\}$. In other words, $\mathrm{clos}(T,r)$ is the *comparability graph* of $\trianglelefteq^{T,r}$. The *height* of a vertex v of a tree T (or of a rooted tree (T,v)) is $\max_{u\in V(T)} d_G(v,u)$. The height of a tree T is the maximum of the heights of its vertices, hence it coincides with its radius $\mathrm{rad}(T)$. The closure of a rooted forest is the union of the closures of its components and its height is the maximum height over all of its components.

2.6 Colourings and Multipartite Graphs

A graph G is called r-partite if $V(G)$ can be partitioned into r sets that induce independent sets in G, called the parts of G. If for every pair of vertices u,v in G that belong to different parts we have $uv \in E(G)$, then G is a *complete r-partite graph*. The complete r-partite graph with parts of size $n_1,\ldots,n_r$ is denoted by $K_{n_1,\ldots,n_r}$.

For a graph G, a *proper vertex colouring* with k colours is a map $c : V(G) \to \{1,\ldots,k\}$ such that $c(v) \neq c(u)$ whenever $uv \in E(G)$. The *chromatic number* $\chi(G)$ is the smallest integer k such that G can be coloured with k colours. A graph with

$\chi(G) = k$ is called *k-chromatic*, while if $\chi(G) \leqslant k$, we call G *k-colourable*. Clearly, a graph is k-colourable if and only if it is k-partite.

Let k be the degeneracy of G. Recall from Section 2.3 that there is a linear ordering of the vertices of G such that every vertex has at most k neighbours which are lower in the order. By starting from the highest in the order vertex and iteratively using at most $k+1$ colours to colour each vertex along with its lower neighbours, we can directly see the well-known fact that the chromatic number of G is at most its degeneracy plus one.

2.7 Homomorphism and Minor Relations

In this section we take a look into the graph relations that we will need throughout this dissertation.

For graphs G, H, a *homomorphism* from G to H is a mapping $f\colon V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$. The existence of a homomorphism from G to H is denoted by $G \longrightarrow H$ (resp. $G \not\longrightarrow G'$). The $\longrightarrow$ relation induces a quasi-order on graphs: we write $G \leqslant_h H$ if $G \longrightarrow H$. It is well known that the relation $\leqslant_h$ is not a partial order as we may have non-isomorphic graphs G, H such that $G \leqslant_h H \leqslant_h G$. Such graphs are called *homomorphically equivalent*.

For any graph G, consider a graph G' homomorphically equivalent to G and such that G' has minimum number of vertices. It is then easy to prove that G' is unique (up to isomorphism). Such a graph G' is called the *core* of G, and it is isomorphic to an induced subgraph of G. A graph which is a core of some graph is called a *core graph*.

A (necessarily induced) subgraph H of a graph G is a *retract* of G, if there is a homomorphism $f\colon G \to H$ that fixes H, that is, $f(v) = v$ for every $v \in V(H)$. It is not difficult to observe that the core of G can be alternatively defined as the minimal retract of G. Actually, two graphs are homomorphically equivalent if and only if they have isomorphic cores.

Now, for an edge $e = xy$ of G, by G/e we denote the graph obtained by *contracting* the edge e into a new vertex v_e, adjacent to all former neighbours of x and y. We say that H is a *minor* of G and write $H \preceq^m G$, if H can be obtained from G by deleting edges and vertices and by contracting edges. Equivalently, H is a minor of G if for every $v \in V(H)$ there is a connected subgraph G_v of G, such that all the graphs in $(G_v)_{v \in V(H)}$ are pairwise vertex disjoint and for every edge in v_1v_2 of H, there is at least one edge u_1u_2 of G with $u_1 \in V(G_{v_1})$ and $u_2 \in V(G_{v_2})$. In other words, H arises from a subgraph of G after contracting connected subgraphs. We then say that the family $(G_v)_{v \in V(H)}$ is a *minor-model* of H in G and call its elements the *branch sets* of the model.

A *subdivision* H' of a graph H is obtained from H by replacing edges by pairwise internally disjoint paths. The vertices of H' corresponding to the vertices of H are the *principal vertices* of H'. We say that H is a *topological minor* of G (we write $H \preceq^t G$) if some subdivision of H is a subgraph of G.

2.8 Tree Decompositions and the Graph Minor Theorem

A *tree decomposition* $\mathcal{D}$ of a graph G is a pair $(T, \mathcal{Z})$, where T is a tree and $\mathcal{Z} = (Z_t)_{t \in V(T)}$ is a family of subsets of $V(G)$ (called *bags*) such that

(T1) for every edge $uv \in E(G)$ there exists $t \in V(T)$ with $u, v \in Z_t$,

(T2) for every $v \in V(G)$, the set $Z^{-1}(v) := \{t \in V(T) \mid v \in Z_t\}$ is a non-empty connected subgraph (a subtree) of T.

The *width* of a tree decomposition $\mathcal{D} = (T, \mathcal{Z})$ is the number

$$\max\{|Z_t| - 1 \mid t \in V(T)\}.$$

The *adhesion* of $\mathcal{D}$ is the number

$$\max\{|Z_t \cap Z_{t'}| \mid tt' \in E(T)\}.$$

Let $\mathcal{T}^G$ be the set of all tree decompositions of G. The *treewidth* $\operatorname{tw}(G)$ of G is the least width of any tree decomposition of G, namely

$$\operatorname{tw}(G) := \min_{\mathcal{D} \in \mathcal{T}^G} \max\{|Z_t| - 1 \mid t \in V(T)\}.$$

One can easily check that trees themselves have treewidth 1. Let us briefly summarise some of the most characteristic properties of treewidth (e.g., see [28, 78]).

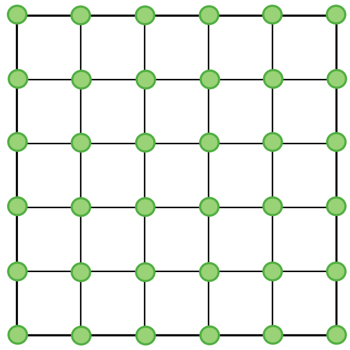


Figure 2.1: The 6*x*6-grid.

Lemma 2.1 *Let $\mathcal{D} = (T, \mathcal{Z}) \in \mathcal{T}^G$.*

(i) *For every $H \subseteq G$, the pair $(T, (Z_t \cap V(H))_{t \in T})$ is a tree decomposition of H, so that* $\mathrm{tw}(H) \leqslant \mathrm{tw}(G)$.

(ii) *Any complete subgraph of G is contained in some bag of $\mathcal{D}$, hence we have* $\omega(G) \leqslant \mathrm{tw}(G) + 1$.

(iii) *For every edge $t_1 t_2$ of T, the set $Z_{t_1} \cap Z_{t_2}$ separates the set $W_1 := \bigcup_{t \in T_1} Z_t$ from the set $W_2 := \bigcup_{t \in T_2} Z_t$, where T_1, T_2 are the components of $T - t_1 t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$.*

(iv) *If $H \preceq^m G$, then* $\mathrm{tw}(H) \leqslant \mathrm{tw}(G)$.

(v) $\chi(G) \leqslant \mathrm{tw}(G) + 1$.

(vi) $\mathrm{tw}(G) = \min\{\omega(G') - 1 \mid G \subseteq G' \text{ and } G' \text{ chordal}\}$.

It is well-known that a graph of treewidth k has a tree decomposition $(T, \mathcal{Z})$ of width k such that for every $st \in E(T)$ we have $|Z_s \setminus Z_t| \leq 1$. We call such decompositions *smooth*.

The $m \times n$-grid is the graph $P_m \square P_n$ (Fig. 2.1). Robertson and Seymour showed in [83] (and later greatly improved it together with Thomas [80]) that large grids serve as a canonical witness in a graph of high treewidth with respect to the minor relation.

Theorem 2.2 (Excluded Grid Theorem) *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph that does not contain the $k \times k$-grid as a minor has treewidth at most $f(k)$.*

Kruskal [58] proved the following for trees.

Theorem 2.3 *The trees are well-quasi-ordered by the topological minor relation $\preceq^t$.*

The above result does not hold for general graphs, but Wagner's conjecture stated that Theorem 2.3 can be extended to the class of all graphs if the corresponding graph relation is the minor relation $\preceq^t$ instead of the stronger topological minor relation $\preceq^t$. The deep theory on Graph Minors that Robertson and Seymour developed to prove Wagner's conjecture extensively revolves around the notion of tree decompositions (and surface embeddability of graphs). Starting from Theorem 2.3 as an inductive basis, they showed that the class of graphs of treewidth at most k are well-quasi-ordered by the minor relation. By combining it with Theorem 2.2, they were able to extend this to graphs excluding a planar minor.

Finally, they showed what is now known as 'Excluded Minor Theorem' [85], characterising the structure of all graphs that exclude a fixed minor: at a high level, every such graph has a tree decomposition of small adhesion into parts that can be 'almost' embedded into a surface of bounded genus. Using their structural theorem they were able to extend the well-quasi-ordering arguments for trees and bounded treewidth graphs and prove Wagner's Conjecture in its full generality.

Theorem 2.4 (Graph Minor Theorem[86]) *The class of all graphs is well-quasi-ordered by the minor relation $\preceq^m$.*

Part I

Colourings in Graph Classes of Bounded Expansion

When you look into an abyss, the abyss also looks into you.

Friedrich Nietzsche

3

Bounds on Generalised Colouring Numbers

3.1 Nowhere Denseness and Bounded Expansion

The *r-neighbourhood* $N_G^r(v)$ of v in G is the set of vertices of distance at most r from v in G. Note that $N_G^1(v) = N_G(v) \cup \{v\}$. For a set of vertices $X \subseteq V(G)$, we define $N_G^r(X) := \bigcup_{v \in X} N_G^r(v)$.

A set $X \subseteq V(G)$ is *r-independent* in G if $d_G(v, u) > r$ for every pair of distinct vertices of X. By definition, an independent set in G coincides with a 1-independent set.

Let $r \in \mathbb{N}$. We say that H is a *(shallow) minor at depth* r of G (denote $H \preceq_r^m G$) if there is a minor-model $(G_v)_{v \in V(H)}$ of H in G, whose branch sets G_v have radius at most r. The set of all depth-r shallow minors of G is denoted by $G \triangledown r$.

For $r \in \mathbb{N}$, an *r-subdivision* of a graph H is obtained from H by replacing edges by pairwise internally disjoint paths of length at most $r + 1$. If a graph G contains a $2r$-subdivision of H as a subgraph, then H is a *topological depth-r minor* of G, written $H \preceq_r^t G$ (Fig. 3.1).

More formally, let $\mathscr{P}_G = \{P \subseteq G \mid P \text{ path } \}$ be the set of subgraphs of a graph G isomorphic to a path. A *topological minor embedding* of a graph H into a graph G

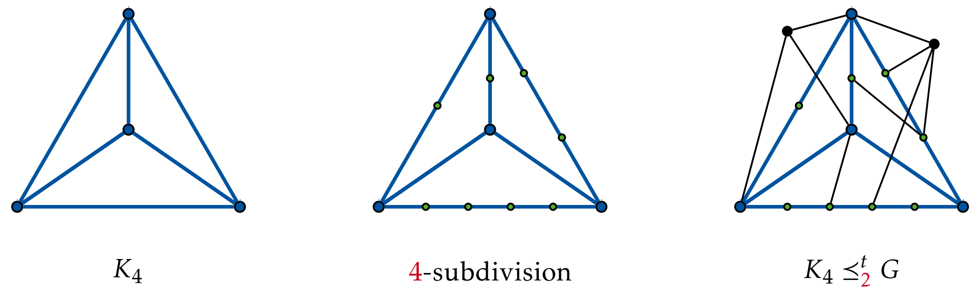


Figure 3.1: A graph G containing K_4 as a topological depth-2 minor

is a pair of functions $\varphi_V \colon V(H) \to V(G)$, $\varphi_E \colon E(H) \to \mathscr{P}_G$ where φ_V is injective and for every $uv \in E(H)$ we have that

1. $\varphi_E(uv)$ is a path in G with endpoints $\varphi_V(u), \varphi_V(v)$ and
2. for every $u'v' \in E(H)$ with $u'v' \neq uv$ the two paths $\varphi_E(uv)$, $\varphi_E(u'v')$ are internally vertex-disjoint.

We define the *depth* of the topological minor embedding φ_V, φ_E as the half-integer $(\max_{uv \in E(H)} |\varphi_E(uv)| - 1)/2$, that is, an embedding of depth r will map the edges of H onto paths in G of length at most $2r+1$. Clearly, if H has a topological minor embedding of depth r into G, then $H \preceq_r^t G$. Note that this relationship is monotone in the sense that a topological depth-r minor of G is also topological depth-$r+1$ minor of G. We denote the set of all topological minors of G at depth r by $G \widetilde{\triangledown} r$.

The maximum of the edge densities of all $H \preceq_r^m G$ is known as the *greatest reduced average density* (in short grad) $\nabla_r(G)$ of G with rank r, namely

$$\nabla_r(G) := \max\{\varepsilon(H) \mid H \in G \triangledown r\}.$$

Similarly, the maximum of the edge densities of all $H \preceq_r^t G$ is known as the *topological greatest reduced average density* (in short top-grad) $\widetilde{\nabla}_r(G)$ of G with rank r and defined through the formula

$$\widetilde{\nabla}_r(G) := \max\{\varepsilon(H) \mid H \in G \widetilde{\triangledown} r\}.$$

We extend the above notations to graph classes as $\nabla_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} \nabla_r(G)$ and $\widetilde{\nabla}_r(\mathcal{G}) = \sup_{G \in \mathcal{G}} \widetilde{\nabla}_r(G)$.

The following result provides polynomial dependency between grads and top-grads.

Theorem 3.1 (Dvořák[34]) *For every graph G and every integer $r \geqslant 0$,*

$$\widetilde{\nabla}_r(G) \leqslant \nabla_r(G) \leqslant 4\left(4\widetilde{\nabla}(G)\right)^{(r+1)^2}.$$

Large average degrees of graphs can force the existence of large complete graphs as minors or topological minors. Kostochka [56] and Thomason [88] show that every graph with average degree at least $c(t) \in \Omega(t\sqrt{\log t})$ contains K_t as a minor. Similarly for topological minors, Bollobás and Thomason [15], and independently Komlós and Szemerédi [55] showed that graphs which exclude a fixed graph as a topological minor are sparse: every graph G with average degree at least $c(t) \in \Omega(t^2)$ satisfies $K_t \preceq^t G$.

Dvořák [34] showed that for every real number $\varepsilon \geqslant 0$ there is a constant $c = c(\varepsilon)$ such that all sufficiently large graphs G (only the order of G depends on t and ε) with at least $|V(G)|^{1+\varepsilon}$ edges contain a c-subdivision of K_k. Jiang [50] improved the best bound for the constant $c(\varepsilon)$ known so far to $c(\varepsilon) \leqslant \lfloor \frac{10}{\varepsilon} \rfloor$.

Consider an infinite class $\mathcal{G}$ of graphs such that for some integer r_1, very intuitively speaking, every graph in $\mathcal{G}\widetilde{\nabla}r_1$ has 'asymptotically strictly more than linear' number of edges. More precisely, there exists an $\varepsilon > 0$ such that for every large enough graph $H \in \mathcal{G}\widetilde{\nabla}r_1$ we have $|E(H)| \geqslant |V(H)|^{1+\varepsilon}$. Then, using the result of Dvořák and Jiang, Nešetřil and Ossona de Mendez proved that there is an r_2 such that $\mathcal{G}\widetilde{\nabla}r_2$ contains all complete graphs. Switching back to a more high-level perspective again—and to highlight how surprising such a result is when seen purely intuitively—there is necessarily a depth $r_2 \in \mathbb{N}$ such that $\mathcal{G}\widetilde{\nabla}r_2$ contains an infinite (sub)sequence of graphs that have 'asymptotically quadratic number of edges' (in fact, all possible number of edges). Or in other words, 'somewhere' at some specific depth r_2, 'some' infinite part of $\mathcal{G}\widetilde{\nabla}r_2$ consists of 'dense' graphs.

This is captured precisely in the following trichotomy.

Theorem 3.2 (Nešetřil and Ossona de Mendez[70]) *For an infinite graph class $\mathcal{G}$ containing at least one non-edgeless graph,*

$$\lim_{r\to\infty} \limsup_{n\to\infty} \left\{ \frac{\log|E(H)|}{\log|V(H)|} \;\middle|\; H \in \mathcal{G}\widetilde{\nabla}r, |V(H)| \geqslant n \right\} \in \{0,1,2\}.$$

Based on the trichotomy above, they introduced nowhere dense graph classes. A class $\mathcal{G}$ is *nowhere dense* if the limit in Theorem 3.2 is at most 1 and *somewhere dense* if said limit is 2. Equivalently, by our above argumentation, $\mathcal{G}$ is nowhere dense if for every $r \in \mathbb{N}$ there is a $k(r) \in \mathbb{N}$ such that no graph in $\mathcal{G}$ has $K_{k(r)}$ as a topological r-minor.

There are already more than several characterisations of nowhere denseness in the literature, and it is not the scope of this dissertation to delve into all of them. As such, we will focus on the ones more relevant to this dissertation. Let us rephrase Theorem 3.2 into an equivalent definition of nowhere denseness, more suitable for our purposes: a class $\mathcal{G}$ of graphs is nowhere dense if for all $\varepsilon > 0$ and all $r \in \mathbb{N}$ there is an $n_0 \in \mathbb{N}$ such that all graphs $G \in \mathcal{G}$ with at least $|V(G)| \geqslant n_0$ satisfy $\widetilde{\nabla}_r(G) \leqslant |V(G)|^{\varepsilon}$. It is only natural to study the more specific case where $\widetilde{\nabla}_r(G)$ is bounded by a constant only depending on r, which was also the original motivation of Nešetřil and Ossona de Mendez.

To this end, a graph class $\mathcal{G}$ has *bounded expansion* if there exists a function $c\colon \mathbb{N} \to \mathbb{R}$ such that for all r we have that $\widetilde{\nabla}_r(\mathcal{G}) \leqslant c(r)$. It is easy to see that all classes of bounded expansion are nowhere dense; the converse does not hold. For example, it is well-known that the class $\mathcal{G} = \{G \mid \Delta(G) \leqslant g(G)\}$ of graphs with maximum degree at most their girth is nowhere dense, but does not have bounded expansion as it does not have bounded average degree ([71], Example 5.1).

Recall the polynomial dependency between grad and top-grad from Theorem 3.1. It can be easily seen that this allows us to freely switch between the minor and the topological minor relation for the notion of preference when studying nowhere dense or bounded expansion graph classes.

Let us note one exception to this rule of thumb. We say that $\mathcal{G}$ has *constant expansion* if there is a global constant c such that $\nabla_r(G) \leq c$ for all $r \in \mathbb{N}$ and $G \in \mathcal{G}$. As we have already stated every graph of average degree at least $c(k) \in \Omega(k\sqrt{\log k})$ contains a K_k as a minor. It easily follows that $\mathcal{G}$ has constant expansion if and only if $\mathcal{G}$ is a class excluding a minor.

Similarly, we say that $\mathcal{G}$ has *constant topological expansion* if there is a global constant c such that $\widetilde{\nabla}_r(G) \leq c$ for all $r \in \mathbb{N}$ and $G \in \mathcal{G}$. Recall that every graph of average degree at least $c(k) \in \Omega(k^2)$ contains a K_k as a topological minor. Then $\mathcal{G}$ has constant topological expansion if and only if it is a class excluding a topological minor. So, as also pointed out by the dependencies in Theorem 3.1, when we speak of constant expansion we must specify whether we speak about minors or topological minors. This will be relevant in Section 3.4.

We now switch to another notion for a characterisation of bounded expansion (and nowhere denseness). The *treedepth* $\mathrm{td}(G)$ of a graph G is the minimum height of a rooted forest F such that $G \subseteq \mathrm{clos}(F)$. Equivalently, treedepth can be alternatively defined according to the following recursive formula.

Lemma 3.3 (Nešetřil and Ossona de Mendez [66]) *Let $G_1, \dots, G_p$ be the connected components of G. Then*

$$\mathrm{td}(G) = \begin{cases} 1, & \textit{if } |V(G)| = 1; \\ 1 + \min_{v \in V(G)} \mathrm{td}(G - v), & \textit{if } G \textit{ is connected and } |V(G) > 1; \\ \max_{i=1}^{p} \mathrm{td}(G_i), & \textit{otherwise.} \end{cases}$$

A deep generalisation of the chromatic number of a graph are the so-called *low treedepth colourings*. An *r-treedepth colouring* of a graph G is a vertex colouring of G for which each $r' \leqslant r$ parts induce a subgraph with treedepth at most r'. The minimum number of colours of an r-treedepth colouring of G is denoted by $\chi_r(G)$. It then follows by definition that the invariants $\chi_r(G)$ form a non-increasing sequence starting from the chromatic number of the graph and stabilizing to the treedepth of the graph:

$$\chi(G) = \chi_1(G) \leqslant \chi_2(G) \leqslant \ldots \leqslant \chi_{\mathrm{td}(G)}(G) = \mathrm{td}(G).$$

Treedepth colourings were motivated by the fact that Nešetřil and Ossona de Mendez initially proved that graph classes excluding a minor— the starting point of sparse graph classes—enjoyed the property of having low treedepth colourings [66]. This was later extended to bounded expansion classes. Let us conclude this section by the characterisation of bounded expansion (an analogous characterisation can be derived for nowhere denseness) via χ_r.

Theorem 3.4 (Nešetřil and Ossona de Mendez [68]) *Let $\mathcal{G}$ be a graph class of bounded expansion. Then there exists a function f such that for every $r \in \mathbb{N}$ and every $G \in \mathcal{G}$ it holds that $\chi_r(G) \leqslant f(r)$.*

3.2 Generalised Colouring Numbers

Let $\Pi(G)$ be the set of linear orders on $V(G)$ and let $\leq \in \Pi(G)$. We represent $\leq$ as an injective function $L\colon V(G) \to \mathbb{N}$ with the property that $v \leq w$ if and only if $L(v) \leq L(w)$.

A vertex u is *weakly r-reachable* from v with respect to the order L, if there is a path P of length $\leqslant r$ from v to u such that $L(u) \leq L(w)$ for all $w \in V(P)$. If furthermore, all inner vertices w of P satisfy $L(v) < L(w)$, then u is *strongly r-reachable* from v. Let $\mathrm{WReach}_r[G,L,v]$ be the set of vertices that are weakly r-reachable from v with respect to L. The *weak r-colouring number* $\mathrm{wcol}_r(G)$ is now defined as

$$\mathrm{wcol}_r(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |\mathrm{WReach}_r[G,L,v]|.$$

Moreover, for a set of vertices $X \subseteq V(G)$, we let

$$\mathrm{WReach}_r[G,L,X] = \bigcup_{x\in X} \mathrm{WReach}_r[G,L,v].$$

Similarly, let $\mathrm{SReach}_r[G,L,v]$ be the set of vertices that are strogly r-reachable from v with respect to L. The *r-colouring number* $\mathrm{col}_r(G)$ is defined as

$$\mathrm{col}_r(G) = \min_{L\in\Pi(G)} \max_{v\in V(G)} |\mathrm{SReach}_r[G,L,v]|.$$

The *r-admissibility* $\mathrm{adm}_r[G,L,v]$ of v with respect to L is the maximum size k of a family $\{P_1,\dots,P_k\}$ of paths of length at most r in G that start in v, end at a vertex w with $L(w) \leq L(v)$, satisfy $V(P_i) \cap V(P_j) = \{v\}$ for $1 \leqslant i \neq j \leqslant k$ and the internal vertices of the paths are larger than v with respect to L. Note that $\mathrm{adm}_r[G,L,v]$ is an integer, whereas $\mathrm{WReach}_r[G,L,v]$ and $\mathrm{SReach}_r[G,L,v]$ are sets of vertices.

By their definition, it follows that $\mathrm{adm}_r(G) \leqslant \mathrm{col}_r(G) \leqslant \mathrm{wcol}_r G$ and that $\mathrm{adm}_1(G), \mathrm{col}_1(G), \mathrm{wcol}_1 G$ are equal to the degeneracy of G plus one. Moreover, it is not difficult to prove that $\mathrm{wcol}_{|V(G)|} = \mathrm{td}(G)$ [71]. Using elimination orderings (whose definition we omit), it was also noted in [42] that $\mathrm{col}_{|V(G)|} = \mathrm{tw}(G) + 1$. Hence, we obtain the following non-decreasing sequences.

$$\mathrm{col}_1(G) \leqslant \mathrm{col}_2(G) \leqslant \ldots \leqslant \mathrm{col}_{|V(G)|}(G) = \mathrm{tw}(G) + 1,$$

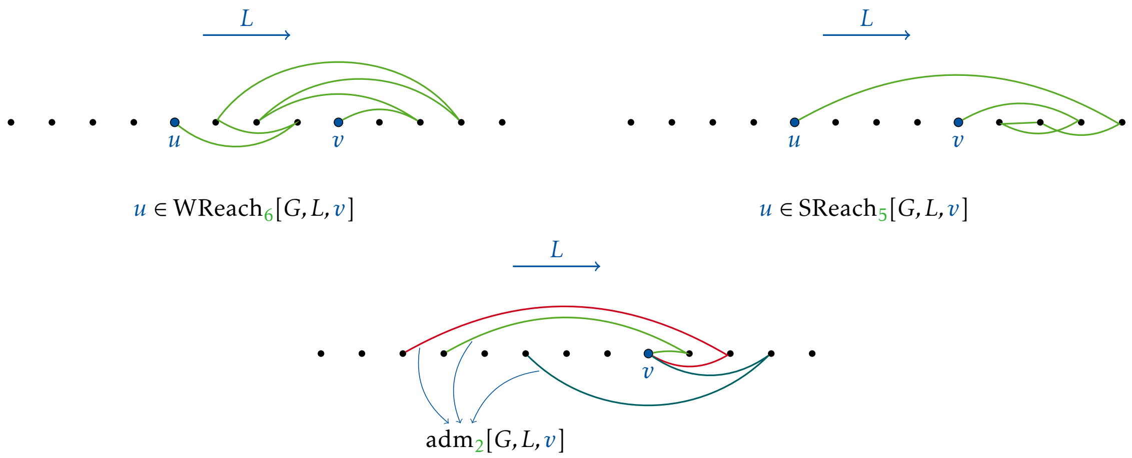


Figure 3.2: Visualising the generalised colouring numbers

$$\mathrm{wcol}_1(G) \leqslant \mathrm{wcol}_2(G) \leqslant \ldots \leqslant \mathrm{wcol}_{|V(G)|}(G) = \mathrm{td}(G).$$

Let us see a short independent proof that the r-colouring numbers stabilise to treewidth, which avoids elimination orderings.

Lemma 3.5 *For every graph G, $\mathrm{col}_{|V(G)|}(G) = \mathrm{tw}(G) + 1$.*

Proof. That $\mathrm{col}_{|V(G)|}(G) \leqslant \mathrm{tw}(G) + 1$ will immediately follow from the order of $V(G)$ defined in the proof of Theorem 3.13. Thus, we only prove the direction $\mathrm{tw}(G) \leqslant \mathrm{col}_{|V(G)|}(G)$.

Indeed, let L be an ordering of $V(G)$. We call a pair of vertices u, v to be L-*linked* if there is a (u, v)-path P in G, such that for every internal vertex w of P, $L(w) > \max\{L(v), L(u)\}$. Define the graph $G_L^+ \supseteq G$ such that

$$V(G_L^+) = V(G),\ E(G_L^+) = \{uv |\ u, v \text{ are } L\text{-linked}\}.$$

Let C be a minimum cycle of G_L^+ and let v_1, v_2, v_3 be its minimum vertices with respect to L. Then all three pairs v_i, v_j are L-linked, therefore

$$v_1v_2, v_2v_3, v_3v_1 \in E(G_L^+).$$

By its minimality, $C = v_1v_2v_3$. Since C was an arbitrary minimum cycle, it follows that G_L^+ is chordal.

Moreover, let K be a clique in G_L^+ and let $v \in V(K)$ with $L(v) = \max L(V(K))$. Then u, v are L-linked in G for every $u \in V(K) - v$, hence $|V(K)| \leqslant \left|\mathrm{SReach}_{|V(G)|}[G, L, v]\right|$. It follows that

$$\omega(G_L^+) \leqslant \max_{v \in V(G)} \left|\mathrm{SReach}_{|V(G)|}[G, L, v]\right|.$$

Hence, by Lemma 2.1 (vi) we have

$$\begin{aligned}
\mathrm{tw}(G) &\leqslant \min\{\omega(G') - 1 \mid G \subseteq G' \text{ and } G' \text{ chordal}\} \\
&\leqslant \min\left\{\omega(G_L^+) \mid L \in \Pi(G)\right\} - 1 \\
&\leqslant \min_{L \in \Pi(G)} \max_{v \in V(G)} \left|\mathrm{SReach}_{|V(G)|}[G, L, v]\right| - 1 \\
&= \mathrm{col}_{|V(G)|}(G) - 1.
\end{aligned}$$

■

The following lemma describes the known bounds between the generalised colouring numbers. Dvořák [35] proves the first one, while the last two ones were proven by Kierstead and Trotter [52, 71].

Lemma 3.6 *For every graph G and every $r \geqslant 1$, the following inequalities hold.*

(i) $\mathrm{adm}_r(G) \leqslant \mathrm{wcol}_r(G) \leqslant \mathrm{adm}_r(G)^r$

(ii) $\mathrm{adm}_r(G) \leqslant \mathrm{col}_r(G) \leqslant \mathrm{adm}_r(G)^r$

(iii) $\mathrm{col}_r(G) \leqslant \mathrm{wcol}_r(G) \leqslant \mathrm{col}_r(G)^r$.

Zhu [94] showed that a graph class has bounded expansion if and only if the generalised colouring numbers of every member is bounded by a function that only depends on the depth r. He established this in combination with Theorem 3.4 and by relating treedepth colourings and generalised colouring numbers, as the next theorem shows.

Theorem 3.7 *[94] For every graph G and $r \geqslant 1$, the following inequalities hold:*

(i) $\chi_r(G) \leqslant (col_{2^{r-2}}(G))^{2^{r-2}}$

(ii) $\mathrm{col}_r(G) \leqslant 1 + \left(2\binom{\chi_{r+2}(G)}{r+1}r\right)^2$.

As a corollary of the bounds of Theorem 3.7, Zhu showed the aforementioned characterisation of bounded expansion (and nowhere denseness) in the same paper.

Theorem 3.8 (Zhu [94]) *A class $\mathcal{G}$ is a graph class of bounded expansion if and only if there exists a function f such that for every $r \in \mathbb{N}$ and every $G \in \mathcal{G}$ it holds that $\mathrm{wcol}_r(G) \leqslant f(r)$.*

Let us mention one of the relations used as a step for the proof of Theorem 3.8 in the following lemma.

Lemma 3.9 *[94] For any graph G and any $r \geqslant 1$,*

$$\nabla_{\lfloor \frac{r-1}{2} \rfloor}(G) + 1 \leqslant \mathrm{wcol}_r(G).$$

Now, for $r \in \mathbb{N}$, an *r-neighbourhood cover* $\mathcal{X}$ of a graph G is a set of connected subgraphs of G called *clusters*, such that for every vertex $v \in V(G)$ there is some $X \in \mathcal{X}$ with $N_r(v) \subseteq X$. The *radius* $\mathrm{rad}(\mathcal{X})$ of a cover $\mathcal{X}$ is the maximum radius of any of its clusters. The *degree* $d^{\mathcal{X}}(v)$ of v in $\mathcal{X}$ is the number of clusters that contain v.

We say that a class $\mathcal{C}$ *admits sparse neighbourhood covers* if for every $\varepsilon > 0$ and every $r \in \mathbb{N}$, there is $n_0 \in \mathbb{N}$ and $c \in \mathbb{N}$ such that for all $G \in \mathcal{C}$ of order $n \geqslant n_0$ there exists an r-neighbourhood cover of radius at most $c \cdot r$ and degree at most n^{ε}. This is the case for monotone nowhere dense graph classes, as shown by Grohe et. al. [42].

Theorem 3.10 *Let $\mathcal{C}$ be a monotone graph class. Then $\mathcal{C}$ is nowhere dense if and only if it admits sparse neighbourhood covers.*

The forward direction of the above characterisation of nowhere denseness is a straight corollary of the nowhere dense version of Theorem 3.8 and the following result of Grohe, Kreutzer and Siebertz, which provides very low radius neighbourhood covers of maximum degree only depending on the weak r-colouring numbers. It is also one of the main motivations of studying the generalised colouring numbers of specific graph classes.

Theorem 3.11 *[44] For every graph G, there exists an r-neighbourhood cover of radius at most $2r$ and maximum degree at most $\mathrm{wcol}_{2r}(G)$.*

3.3 Bounded Treewidth Graphs

Several constructions of neighbourhood covers about graphs excluding a minor were known in the literature [1]. Apart from being of independent interest, graphs of bounded treewidth are one of the key building blocks in the Excluded Minor Theorem. Motivated by Theorem 3.11 and the very good neighbourhood covers it automatically provides in terms of radius, we are interested in the generalised colouring numbers of bounded treewidth graphs.

Let us start with an easy lemma about the generalised colouring numbers of a tree, which will be the basis of our intuition when dealing with graphs of bounded treewidth.

Lemma 3.12 *Let T be a tree. Then* $\mathrm{col}_r(G) = 2$ *and* $\mathrm{wcol}_r(T) \leqslant r+1$.

Proof. Choose a root s of T arbitrarily and consider a linear extension L of the respective partial order defined by T with root s. Clearly, this linear order shows that $\mathrm{col}_r(G) \leqslant 2$. Moreover, it is easy to see that from any given vertex v of T, the only weakly reachable vertices (without any distance constraint) lie on the path from v to s in T. Since that path is distance-preserving, it follows that $\mathrm{wcol}_r(T) \leqslant r+1$. ■

In what follows, let $L^{T,s}$ represent the standard tree order $\trianglelefteq^{T,s}$ of a rooted tree (T,s). The author thanks Sebastian Siebertz and Roman Rabinovich for their contribution to the presentation of the proof.

Theorem 3.13 *Let* $\mathrm{tw}(G) \leqslant k$. *Then* $\mathrm{wcol}_r(G) \leqslant \binom{r+k}{k}$.

Proof. Let (T,X) be a smooth tree decomposition of G of width at most k. Since if G' is a subgraph of G, then $\mathrm{wcol}_r(G') \leq \mathrm{wcol}_r(G)$, w.l.o.g we may assume that G is edge maximal of treewidth k, i.e. each bag induces a clique in G. We choose an arbitrary root s of T and let L' be some linear extension of $L^{T,s}$. For every $v \in V(G)$, let t_v be the unique node of T such that $L'(t_v) = \min\{L'(t) \mid v \in X_t\}$ and define a linear ordering $L := L_G^{T,s}$ of $V(G)$ such that:

(i) $L'(t_v) < L'(t_u) \Rightarrow L(v) < L(u)$,

(ii) if $L'(t_v) = L'(t_u)$ (which is possible in the root bag X_s), break ties arbitrarily.

Fix some $v \in V(G)$ and let $w \in \mathrm{WReach}_r[G,L,v]$. By Lemma 2.1(iii) and the definition of L, it is immediate that t_w lies on the path from t_v to s in T. Let $u \in X_{t_v}$ be such that $L(u) \leqslant L(u')$ for all $u' \in X_{t_v}$. If $t_v = s$, then $|\mathrm{WReach}_r[G,L,v]| \leq k+1$ and we are done. Otherwise, as the decomposition is smooth, $L'(t_u) < L'(t_v)$. We define two subgraphs G_1 and G_2 of G as follows. The graph G_1 is induced by the vertices from the bags between s and t_u, i.e. by the set $\bigcup\{X_t \in V(T) \mid L^{T,s}(t) \leq L^{T,s}(t_u)\}$. The graph G_2 is induced by $\bigcup\{X_t \in V(T) \mid L^{T,s}(t_u) \leq L^{T,s}(t) < L^{T,s}(t_v)\} \setminus V(G_1)$.

Let L_i be the restriction of L to $V(G_i)$, for $i = 1,2$, respectively. We claim that if $w \in \mathrm{WReach}_r[G,L,v]$, then $w \in \mathrm{WReach}_{r-1}[G_1,L_1,u] \cup \mathrm{WReach}_r[G_2,L_2,v]$. To see this, let $P = (v = v_1,\ldots,v_\ell = w)$ be a shortest path between v and w of length $\ell \leq r$ such that $L(w)$ is minimum among all vertices of $V(P)$.

We claim that $L(v_1) > \ldots > L(v_\ell)$ (and call P a decreasing path). This implies in particular that all t_{v_i} lie on the path from t_v to s and that $L^{T,s}(t_{v_1}) \geqslant \ldots \geqslant L^{T,s}(t_{v_\ell})$ (non-equality may only hold in the last step, if we take a step in the root bag).

Assume that the claim does not hold and let i be the first position with $L(v_i) < L(v_{i+1})$. It suffices to show that we can find a subsequence (which is also a path in G) $Q = v_i, v_j, \ldots, v$ of P with $j > i+1$. By definition of $t_{v_{i+1}} =: t$, X_t contains v_i. (Indeed, there is an edge between v_i and v_{i+1}, which must be contained in some bag, but v_{i+1} appears first in X_t counting from the root and each bag induces a clique in G). Let t' be the parent node of t. $X_{t'}$ also contains v_i, as the decomposition is smooth and v_{i+1} is the unique vertex that joins X_t. But by Lemma 2.1(iii), $X_{t'}$ is a separator that separates v_{i+1} from all vertices smaller than v_{i+1}. We hence must visit another vertex v_j from $X_{t'}$ in order to finally reach v. We can hence shorten the path as claimed.

If $L(w) \leq L(u)$, then P goes through X_{t_u} by Lemma 2.1(iii). Let u' be the first vertex of P that lies in X_{t_u}. We show that there is a shortest path from v to u' that uses u as the second vertex. By assumption, $v \neq u$. If $vu' \in E(G)$, then both v, u' must be contained in some common bag $X_{t'}$. By definition of t_v, we have $t' = t_v$, as t_v is the first node of T on the path from s to t_v containing v. By definition of u and because (T,X) is smooth, u is the only vertex from t_v that appears in t_u. Thus $u' = u$, so the shortest path from v to u' uses u. If the distance between v and u' is at least 2, a shortest path can be chosen as v, u, u'. Indeed $u \in X_{t_u} \cap X_{t_v}$ and every bag induces a clique by assumption.

It follows that if $L(w) \leq L(u)$ and $w \in \mathrm{WReach}_r[G,L,v]$, then there is a shortest path from v to w that uses u as the second vertex. Thus $w \in \mathrm{WReach}_{r-1}[G_1,L_1,u]$, as P is decreasing.

If $L(w) > L(u)$, then P never visits vertices of G_1. If P lies completely in G_2, we have $w \in \mathrm{WReach}_r[G_2,L_2,v]$. If P leaves G_2, it visits vertices of G that are contained only in bags strictly below t_v. However, this is impossible, as P is decreasing.

Hence

$$|\mathrm{WReach}_r[G,L,v]| \leqslant |\mathrm{WReach}_{r-1}[G_1,L_1,u]| + |\mathrm{WReach}_r[G_2,L_2,v]|. \qquad (\star)$$

The treewidth of G_2 is at most $k-1$, as we removed u from every bag. More precisely, the tree decomposition (T^2,X^2) of G_2 of width at most $k-1$ is the restriction of (T,X) to G_2, i.e. we take tree nodes t contained between t_u and t_v (including t_v and not including t_u) and define $X_t^2 = X_t \cap V(G_2)$.

Now, recall the definition of $L_G^{T,s}$ and let $w(r,k)$ be the maximum $|\mathrm{WReach}_r[H,L_H^{T,s},v]|$, ranging over all graphs G with $\mathrm{tw}(G) \leqslant k$, linear orders $L_H^{T,s}$ obtained by a $(T,\mathcal{X}) \in \mathcal{T}^H$ with $s \in V(H)$, and vertices $v \in V(H)$. By $(\star)$, we then have $|\mathrm{WReach}_r[G,L,v]| \leqslant w(r,k-1) + w(r-1,k)$. Since G, L and v where arbitrary, it follows that

$$w(k,r) \leqslant w(k,r-1) + w(k-1,r).$$

Recall that $\mathrm{wcol}_1(G)$ equals the degeneracy of G plus one and that every graph of treewidth $\leqslant k$ is k-degenerate, hence $w(k,1) \leqslant k+1$. Furthermore, by Lemma 3.12, $w(r,1) \leqslant r+1$. Since $\binom{r+k}{k} = \binom{r+k-1}{k} + \binom{r+k-1}{k-1}$, it follows by induction that $w(r,k) \leqslant \binom{r+k}{k}$ and the theorem follows. ■

Notice that the linear order defined in the proof of Theorem 3.13 clearly also shows that for every $r \geqslant 1$, $\mathrm{col}_r(G) \leqslant \mathrm{tw}(G)$. Moreover, the proof itself gives rise to a construction of a class of graphs that matches the upper bound proven there. We construct a graph of treewidth k and weak r-colouring number $\binom{k+r}{k}$ whose tree decomposition has a highly branching host tree. This enforces a path in the tree from the root to a leaf that realises the recursion $w(k,r) = w(k,r-1) + w(k-1,r)$ from the proof of Theorem 3.13.

Theorem 3.14 *There is a family of graphs G_r^k with $\mathrm{tw}(G_r^k) = k$, such that $\mathrm{wcol}_r(G_r^k) = \binom{r+k}{k}$. In fact, for all $r' \leq r$, $\mathrm{wcol}_{r'}(G_r^k) = \binom{r'+k}{k}$.*

Proof. Fix r, k and let $c = \binom{r+k}{k}$. We define graphs $G(k', r')$ for all $r' \leqslant r, k' \leqslant k$ and corresponding tree decompositions $\mathcal{T}(k', r') = (T(k', r'), X(k', r'))$ of $G(k', r')$ of width k' with a distinguished root $s(T(k', r'))$ by induction on k' and r'. We will show that $\mathrm{wcol}_{r'}(G(k', r')) \leqslant \binom{r'+k'}{k'}$. We guarantee several invariants for all values of k' and r' which will give us control over a sufficiently large part of any order that witnesses $\mathrm{wcol}_{r'}(G(k', r')) \leqslant \binom{r'+k'}{k'}$.

1. There is a bijection $f : V(T(k', r')) \to V(G(k', r'))$ such that $f(s(T(k', r')))$ is the unique vertex contained in $X_{s(T(k', r'))}$ and if t is a child of t' in $T(k', r')$, then $f(t)$ is the unique vertex of $X_t \setminus X_{t'}$. Hence any order defined on $V(T)$ directly translates to an order of $V(G)$ and vice versa.

2. In any order L of $V(G(k', r'))$ which satisfies $\mathrm{wcol}_r(G(k', r')) \leqslant c$, there is some root-leaf path $P = t_1, \ldots, t_m$ such that $L(f(t_1)) < \ldots < L(f(t_m))$.

3. Every bag of $T(k', r')$ contains at most $k' + 1$ vertices.

It will be convenient to define the tree decompositions first and to define the corresponding graphs as the unique graphs induced by the decomposition in the following sense. For a tree T and a family of finite and non-empty sets $(X_t)_{t \in V(T)}$ such that if $z, s, t \in V(T)$ and s is on the path of T between z and t, then $X_z \cap X_t \subseteq X_s$, we define the graph *induced* by $(T, (X_t)_{t \in V(T)})$ as the graph G with $V(G) = \bigcup_{t \in V(T)} X_t$ and $\{u, v\} \in E(G)$ if and only if $u, v \in X_t$ for some $t \in V(T)$. Then $(T, (X_t)_{t \in V(T)})$ is a tree decomposition of G.

For $k' \geqslant 1, r' = 1$, let $T(k', r') =: T$ be a tree of depth $k' + 1$ and branching degree c with root s. Recall that $L^{T,s}$ represents the natural partial tree order $\trianglelefteq^{T,s}$. Let $f \colon V(T) \to V$ be a bijection to some new set V. We define $X_t := \{f(t) \mid L^{T,s}(t') \leqslant L^{T,s}(t)\}$. Let $G(k', r')$ be the graph induced by the decomposition. The first and the third invariants clearly hold. For the second invariant, consider a simple pigeonhole argument. For every non-leaf node t, the vertex $f(t)$ has c neighbours $f(t')$ in the child bags $X_{t'}$ of t. Hence some $f(t')$ must be larger in the order. This guarantees the existence of a path as required.

For $k' = 1, r' \geqslant 1$, let $T(k', r') =: T$ be a tree of depth $r' + 1$ and branching degree c with root s and let f be as before. Let $X_s := \{f(s)\}$ and for each $t' \in V(T)$ with parent $t \in V(T)$ let $X_{t'} := \{f(t), f(t')\}$. Let $G(k', r')$ be the graph induced by the decomposition. All invariants hold by the same arguments as above. Note that G_1^1 is the same graph in both constructions and is hence well defined.

Now assume that $G(k', r'-1)$ and $G(k'-1, r')$ and their respective tree decompositions have been defined. Let $T(k', r')$ be the tree which is obtained by attaching c copies of $T(k'-1, r')$ as children to each leaf of $T(k', r'-1)$. We define the bags that belong to the copy of $T(k', r'-1)$, exactly as those of $T(k', r'-1)$. To every bag of a copy of $T(k'-1, r')$ which is attached to a leaf z, we add $f'(z)$ (where f' is the bijection from $T(k', r'-1)$). Let $G(k', r')$ be the graph induced by the decomposition.

It is easy to see how to obtain the new bijection f on the whole graph such that it satisfies the invariant. It is also not hard to see that each bag contains at most $k'+1$ vertices. For the second invariant, let $P_1 = t_1, \ldots, t_m$ be some root-leaf path in $T(k', r'-1)$ which is ordered such that $L(f(t_1)) < \ldots < L(f(t_m))$. Let $v = f(t_m)$ be the unique vertex in the leaf bag in which P_1 ends. By the same argument as above, this vertex has many neighbours s' such that $f^{-1}(s')$ is a root of a copy of $T(k'-1, r')$. One of them must be larger than v. In an appropriate copy we find a path P_2 with the above property by assumption. We attach the paths to find the path $P = t_1 \ldots t_\ell$ in $T(k', r')$.

We finally show that $\mathrm{WReach}_{r'}[G(k', r'), L, f(t_\ell)] = c$. This is again shown by an easy induction. Using the notation of the proof of Theorem 3.13, we observe that the graph G_1 is isomorphic to $G(k', r'-1)$ in $G(k', r')$ and G_2 is isomorphic to $G(k'-1, r')$. Furthermore we observe that the number of vertices reached in these graphs are exactly $w(k', r'-1)$ and $w(k'-1, r')$, so that the upper bound is matched. Similarly one shows that $\mathrm{wcol}_{r'}(G(k, r)) = \binom{r'+k}{k}$. The theorem follows by letting $G_r^k := G(k, r)$. ■

Recall that by lemma 3.6, for every graph G it holds $\mathrm{wcol}_r(G) \leqslant (\mathrm{col}_r(G))^r$. To our knowledge, there is no example in the literature that verifies the exponential gap between wcol_r and col_r. As $\mathrm{col}_r(G) \leq \mathrm{tw}(G)$ and G_r^k contains a $k+1$-clique, we have $\mathrm{col}_{r'}(G_r^k) = k+1$. Theorem 3.14 provides an example that is close to an affirmative answer for arbitrarily large generalised colouring numbers, in a rather uniform manner.

Corollary 3.15 *For every $k \geqslant 1$, $r \geqslant 1$, there is a graph G_r^k such that for all $1 \leq r' \leq r$ we have $\mathrm{col}_{r'}(G_r^k) = k+1$ and $\mathrm{wcol}_{r'}(G_r^k) \geqslant \left(\frac{\mathrm{col}_{r'}(G_r^k)}{r'}\right)^{r'}$.*

Proof. Since $\mathrm{col}_{r'}(G_r^k) = k+1$, we have

$$\mathrm{wcol}_{r'}(G_r^k) = \binom{r'+k}{k} = \binom{k+r'}{r'} \geqslant \left(\frac{k+r'}{r'}\right)^{r'} \geqslant \left(\frac{\mathrm{col}_{r'}(G_r^k)}{r'}\right)^{r'}.$$

■

3.4 High-Girth Regular Graphs

In light of Theorem 3.11 again, we want to explore if assuming constant topological expansion for a graph class results to polynomial colouring numbers.

Surprisingly, we prove that, in fact, even classes of bounded degree (which are of the simplest classes that can exclude a topological minor) cannot have polynomial r-colouring numbers. This is established by considering regular graphs of large girth, making use of the property that such graphs are locally acyclic, namely r-neighbourhoods up to a certain radius are trees.

Several results for the existence of d-regular graphs of girth at least g are known in the literature. Using Cayley graphs, a simple construction of 3-regular graphs of arbitrarily high girth—which can be easily transposed into d-regular graphs for every $d \geqslant 3$—can be seen in [12], albeit without optimality in the size of the constructed graphs. Notice that for a vertex v of a d-regular graph with girth $g > 2r+1$, $N^r(v)$ induces a d-regular tree of radius r, hence we have

$$|N^r(v)| = d(1 + (d-1) + \ldots + (d-1)^{r-1}).$$

It follows that a d-regular graph on n vertices has girth

$$g \leqslant 2 + 2\frac{\log n}{\log(d-1)}.$$

Erdős, Sachs, Sauer and Walther (see [14], pp. 103-110) proved the existence of d-regular graphs on n vertices whose girth asymptotically matches this upper bound. Margulis gave an explicit construction of such graphs in [61].

Having discussed the existence of high-girth regular graphs, recall from Section 3.1 that the class of graphs that satisfy $\Delta(G) \leqslant g(G)$ is a nowhere dense class, but does not have bounded expansion. Intuitively, d-regular graphs of high girth are the densest graphs of 'constant expansion' fragments of the above nowhere dense class. Taking into account their innate vertex expansion properties as well, it is hence no coincidence that we look at high-girth regular graphs to prove lower bounds for the generalised colouring numbers. For this section, we let $n := |V(G)|$.

Theorem 3.16 *Let G be a d-regular graph of girth at least $4g+1$, where $d \geqslant 7$. Then for every $r \leqslant g$,*

$$\mathrm{col}_r(G) \geqslant \frac{d}{2}\left(\frac{d-2}{4}\right)^{2^{\lfloor log r \rfloor}-1}.$$

Proof. For an ordering L of G, let $R_r(v) = \mathrm{SReach}_r[G,L,v] \setminus \mathrm{SReach}_{r-1}[G,L,v]$ and $U_r = \sum_{v \in V(G)} |R_r(v)|$.

Suppose that $r \leqslant g$ and notice that for $u, w \in R_r(v)$, we have that either $u \in R_{2r}(w)$ or $w \in R_{2r}(u)$. Therefore, every vertex $v \in V(G)$ contributes at least $\binom{|R_r(v)|}{2}$ times to U_{2r}. Moreover, since $r \leqslant g$, for every u, w with $u \in R_{2r}(w)$ there is at most one vertex $v \in V(G)$ such that $u, w \in R_r(v)$ (namely the middle vertex of the unique (u,v)-path of length $2r$ in G). It follows that for every $r \leqslant g$,

$$\begin{aligned} U_{2r} &\geqslant \sum_{v \in V(G)} \binom{|R_r(v)|}{2} = \frac{1}{2}\sum_{v \in V(G)} |R_r(v)|^2 - \frac{1}{2}\sum_{v \in V(G)} |R_r(v)| \\ &\geqslant \frac{1}{2n}\Big(\sum_{v \in V(G)} |R_r(v)|\Big)^2 - \frac{1}{2}U_r = \frac{1}{2n}U_r^2 - \frac{1}{2}U_r \end{aligned}$$

where for the second inequality we have used the Cauchy-Schwarz inequality.

Let $c_r = \frac{U_r}{n}$. Then for every $r \leqslant g$, we obtain $c_{2r} \geqslant \frac{1}{2}c_r(c_r - 1)$. But,

$$U_1 = \sum_{v \in V(G)} |\mathrm{SReach}_1[G,L,v] \setminus \{v\}| = \frac{1}{2}dn,$$

so that $c_1 = \frac{d}{2} > 3$, since $d \geqslant 7$. By induction and because $c_{2r} \geqslant \frac{1}{2}c_r(c_r - 1)$, for every $r = 2^{r'} \leqslant g$ we have $c_{2r} \geqslant c_r \geqslant 3$. Therefore $c_r \geqslant c_1 = \frac{d}{2}$. Again because $c_{2r} \geqslant \frac{1}{2}c_r(c_r-1)$, for every $r = 2^{r'} \leqslant g$ we have

$$c_{2r} \geqslant \frac{1}{2}c_r^2 - \frac{1}{2}c_r \geqslant \frac{1}{2}c_r^2 - \frac{1}{d}c_r^2 = \frac{d-2}{2d}c_r^2.$$

Then for every $r = 2^{r'} \leqslant g$, it easily follows by induction that $c_r \geqslant \frac{d}{2}\left(\frac{d-2}{4}\right)^{r-1}$.

Finally, let $C_r = \frac{1}{n}\sum_{v\in V(G)}|\mathrm{SReach}_r[G,L,v]|$. Then, $C_r = \sum_{i=1}^{r} c_i$. In particular, it is $C_r \geqslant c_{2^{\lfloor log r\rfloor}} \geqslant \frac{d}{2}\left(\frac{d-2}{4}\right)^{2^{\lfloor log r\rfloor}-1}$, and hence for every $r \leqslant g$ there exists a vertex $v_r \in V(G)$ such that $|\mathrm{SReach}_r[G,L,v_r]| \geqslant \frac{d}{2}\left(\frac{d-2}{4}\right)^{2^{\lfloor log r\rfloor}-1}$. Since L was arbitrary, the theorem follows. ■

Notice that our proof above makes sense only if $d \geqslant 7$, which is also best possible with this approach, since for $d \leqslant 6$, we have $c_1 \leqslant 3$. Then from the recurrence relation $c_{2r} = \frac{1}{2}c_r^2 - \frac{1}{2}c_r$, we get $c_{2^i} \leqslant c_{2^{i-1}}$ for every i and we clearly need c_{2^i} to increase in the proof above.

Actually, by combining a known result for the ∇_r of high-girth regular graphs ([30]) and Lemma 3.9 we get exponential lower bounds for the weak colouring number of high-girth d-regular graphs, already for $d \geqslant 3$. In particular, for a 3-regular graph G of high enough girth, $\mathrm{wcol}_r(G) \geqslant 3\cdot 2^{\lfloor r/4\rfloor-1}$. These methods can be extended to get corresponding bounds in terms of their degree for regular graphs of higher degree, but by adopting a more straightforward approach, we get better bounds for high-girth d-regular graphs for $d \geqslant 4$. The author thanks Martin Grohe for providing the initial observation that inspired the next proof.

Theorem 3.17 *Let G be a d-regular graph of girth at least $2g+1$, where $d \geqslant 4$. Then for every $r \leqslant g$,*

$$\mathrm{wcol}_r(G) \geqslant \frac{d}{d-3}\left(\left(\frac{d-1}{2}\right)^r - 1\right).$$

Proof. Let L be an ordering of G. For $u,v \in V(G)$ with $d(u,v) \leqslant r$, let P_{uv} be the unique (u,v)-path of length at most r, due to the girth of G. Let

$$Q_r(v) = \mathrm{WReach}_r[G,L,v] \setminus \mathrm{WReach}_{r-1}[G,L,v],$$

and define $S_r = \sum_{v\in V(G)}|Q_r(v)|$. For $r \leqslant g-1$, a vertex $u \in Q_r(v)$ and $w \in N(v)\setminus V(P_{uv})$, it holds that either $w \in Q_{r+1}(u)$ or $u \in Q_{r+1}(w)$. Notice that $|N(v)\setminus V(P_{uv})| = d-1$ and that P_{vu} and P_{uw} are unique. Therefore, every pair of vertices v,u with $u \in Q_r(v)$ corresponds to at least $d-1$ pairs of vertices u,w with either $u \in Q_{r+1}(w)$ or $w \in Q_{r+1}(u)$ and hence contributes at least $d-1$ times to S_{r+1}. Since every path of length $r+1$ contains exactly two subpaths of length r, we have for every $r \leqslant g-1$ that

$$2S_{r+1} \geqslant (d-1)S_r.$$

Let $w_r = \frac{S_r}{n}$. Then, for every $r \leqslant g-1$ we have $w_{r+1} \geqslant \frac{d-1}{2} w_r$.

But,

$$\sum_{v \in V(G)} |\mathrm{WReach}_1[G, L, v] \setminus \{v\}| = \frac{1}{2} dn,$$

so that $w_1 = \frac{d}{2}$.

It easily follows by induction that for every $r \leqslant g$, we have $w_r \geqslant \frac{d}{2}\left(\frac{d-1}{2}\right)^{r-1}$. Finally, let $W_r = \frac{1}{n}\sum_{v \in V(G)} |\mathrm{WReach}_r[G, L, v]|$. Then,

$$W_r = \sum_{i=1}^{r} w_i \geqslant \sum_{i=1}^{r} \frac{d}{2}\left(\frac{d-1}{2}\right)^{i-1} = \frac{d}{d-3}\left(\left(\frac{d-1}{2}\right)^{r} - 1\right),$$

and hence for every $r \leqslant g$ there exists a vertex $v_r \in V(G)$ such that $\mathrm{WReach}_r[G, L, v_r]| \geqslant \frac{d}{d-3}\left(\left(\frac{d-1}{2}\right)^{r} - 1\right)$. Since L was arbitrary, the theorem follows. ■

Let us remark that for every d-regular graph G and every radius r, we have $\mathrm{adm}_r(G) \leqslant \Delta(G) + 1 = d + 1$, so by Theorem 3.16 for every $d \geqslant 7$ and every $r \leqslant g$, the d-regular graphs of girth at least $4g + 1$ verify the exponential dependency on r of the gap between $\mathrm{adm}_r, \Delta(G)$ and $\mathrm{col}_r, \mathrm{wcol}_r$ in the known relations from Lemma 3.6.

Lupus dentis, taurus cornis.

4

The Weak-Colouring Approach

4.1 Exact Odd-Distance Powergraphs

For a proper vertex colouring of a graph we ask for every pair of vertices at distance 1 to have different colours. A generalisation of this concept is to study colourings where we ask for every pair of vertices at higher distance (for example, at most or exactly p) to receive different colours.

Consider the variation of the above colourings where we ask for different colours between every pair of vertices of a graph G that are connected in G by a path (not necessarily shortest) of length exactly a fixed odd p. Nešetřil and Ossona de Mendez proved in [67] that for a graph G with odd-girth at least p from a bounded expansion class $\mathcal{G}$ the number of colours needed for such colourings depends only on $\mathcal{G}$ and the distance p, but not on the size of the graph G considered.

This is a consequence of the deep fact that every graph class of bounded expansion has *all restricted homomorphism dualities* [67], which means the following. Let $\mathcal{F}$ be a finite set of connected core graphs and let $\mathrm{Forb}_h(\mathcal{F})$ be the set of graphs G such that no graph in $\mathcal{F}$ has a homomorphism to G. Then for every bounded expansion graph class $\mathcal{G}$ there exists a graph (a *dual* of $\mathcal{F}$ with respect to $\mathcal{G}$) $D(\mathcal{F},\mathcal{G}) \in \mathrm{Forb}_h(\mathcal{F})$ such that every graph in $\mathcal{G} \cap \mathrm{Forb}_h(\mathcal{F})$ has a homomorphism to $D(\mathcal{F},\mathcal{G})$.

Now, consider $\mathcal{F} = \{C_p\}$ with p odd. Then it is easy to see that for every graph G with odd-girth more than p, every homomorphism $c : G \to D(\mathcal{F},\mathcal{G})$ gives a desired colouring of G with $|V(D(\mathcal{F},\mathcal{G}))|$ colours.

On the other hand, it is not generally known what is the graph $D(\mathcal{F},\mathcal{G})$ with the minimum number of vertices. Knowledge on the number of colours needed for such graph colourings described above, which by the above argumentation provides an upper bound for the size of a minimum such $D(\mathcal{F},\mathcal{G})$, motivates the study of such colourings. More can be said when we study colourings of graphs asking for different colours for pairs of vertices whose distance is *exactly* p.

For a graph G and positive integer p, denote by $G^{[\natural p]}$ the graph $(V(G), E^{[\natural p]})$, where uv is an edge in $E^{[\natural p]}$ if and only if $d_G(u,v) = p$. Clearly, if p is even, $G^{[\natural p]}$ can have unbounded chromatic number (and even clique number) even when G is a tree, simply by considering subdivisions of stars. But for p odd, Nešetřil and Ossona de Mendez proved in [71](Theorem 11.8) the following surprising result.

Theorem 4.1 *For every graph class and every odd integer $p \geqslant 1$,*

$$\chi(G^{[\natural p]}) \leqslant \chi_p(G) \cdot 2^{\chi_p(G) \cdot 2^{\chi_p(G)}}.$$

For classes of bounded expansion, an immediate corollary Theorems 3.4 and 4.1 is the following.

Corollary 4.2 *For every graph class $\mathcal{G}$ with bounded expansion and every odd integer $p \geqslant 3$, there exists an integer $c = c(\mathcal{G}, p)$ such that all graphs $G \in \mathcal{G}$,*

$$\chi(G^{[\natural p]}) \leqslant c.$$

Even though the growth of $c(\mathcal{G}, p)$ is very fast, it is not known if it is optimal or even unbounded (with respect to p) for very sparse graph classes. More specifically, van den Heuvel and Naserasr posed the problem whether there exists a constant C such that for every odd integer p and every planar graph G it holds $\chi(G^{[\natural p]}) \leqslant C$ ([71], Problem 11.1).

In pursue of an answer of the problem above and the improvement of the bounds of $\chi(G^{[\natural p]})$, we present the following theorem utilising the weak-colouring approach, whose proof is inspired by that of Theorem 4.1.

Theorem 4.3 *For every graph G and every odd $p \geqslant 3$,*

$$\chi(G^{[\natural p]}) \leqslant \mathrm{wcol}_{2p-3}(G) \cdot 2^{\mathrm{wcol}_{2p-3}(G)}.$$

Proof. Note that since $p \geqslant 3$, it is $2p - 3 \geqslant p$. Let $L \in \Pi(G)$ be such that

$$d := \mathrm{wcol}_{2p-3}(G) = \max_{v \in V(G)} |\mathrm{WReach}_p[G, L, v]|.$$

Let G' be the graph with vertex set $V(G') = V(G)$ and edge set

$$E(G') = \{uv \mid u \in \mathrm{WReach}_{2p-3}[G, L, v]\}.$$

Now, we always have that $v \in \mathrm{WReach}_{2p-3}[G, L, v]$, so, clearly, G' has degeneracy at most $d - 1$. It follows that there exists a proper colouring φ of G' using a set X of at most d colours. Then, the same colouring φ on G corresponds to a colouring where for every two vertices $u, v \in V(G)$ with $u \in \mathrm{WReach}_{2p-3}[G, L, v]$, it must be $\varphi(u) \neq \varphi(v)$.

Let c be a colour in X. Let us define the mapping $\pi_v : X \to \{0, 1\}$ by:

$$\pi_v(c) = \begin{cases} 1, & \text{if there is a vertex } w \text{ of colour } c \\ & \text{with } w \in \mathrm{WReach}_{p-2}[G, L, v] \\ & \text{and } d(v, w) \equiv 1 \mod 2; \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the number of different π_v's is at most the number of different mappings from X to $\{0, 1\}$, which is 2^d. Let ψ be the colouring of G defined by $\psi(v) = (\varphi(v), \pi_v)$. The number of colours in ψ is at most $d \cdot 2^d$.

Let us prove that ψ is a proper colouring of $G^{[\natural p]}$. Assume that there are two vertices of u, v at distance p in G, such that $\psi(u) = \psi(v)$. Then $\varphi(u) = \varphi(v)$ and $\pi_u = \pi_v$. Consider a shortest (u, v)-path P of length p and let w be the minimum vertex of P with respect to L. Since $\varphi(u) = \varphi(v)$, by the definition of φ neither $u \in \mathrm{WReach}_{2p-3}[G, L, v]$, nor $v \in \mathrm{WReach}_{2p-3}[G, L, u]$, hence w is an internal vertex of P.

Now, $d(u, w) + d(v, w) = p \equiv 1 \mod 2$. Therefore, one of the two distances is odd and the other even. W.l.o.g. let $d(u, w) = r \equiv 1 \mod 2$ and $d(v, w) = p - r \equiv 0 \mod 2$. Then $1 \leqslant r \leqslant p - 2$, because r is odd and w is internal in P, and by the definition of π_u, we have $\pi_u(\varphi(w)) = 1$. But $\pi_u = \pi_v$, therefore we have that $\pi_v(\varphi(w)) = \pi_u(\varphi(w)) = 1$, so there exists a vertex $w' \in \mathrm{WReach}_{p-2}[G,L,v]$ with $\varphi(w') = \varphi(w)$. Finally, since $w \in \mathrm{WReach}_{p-1}[G,L,v]$ and $w' \in \mathrm{WReach}_{p-2}[G,L,v]$, it follows that either $w \in \mathrm{WReach}_{2p-3}[G,L,w']$ or $w' \in \mathrm{WReach}_{2p-3}[G,L,w]$, a contradiction to the definition of φ. ■

Notice that combined with Theorem 3.8, Theorem 4.3 reproves Corollary 4.2. Expanding on the results of Section 3.13, van den Heuvel et al. showed in [91] polynomial bounds for the weak colouring numbers of graphs excluding a fixed minor.

Theorem 4.4 *Let $t \geqslant 4$. For every graph G that excludes K_t as a minor, we have*

$$\mathrm{wcol}_r(G) \leqslant \binom{r+t-2}{t-2} \cdot (t-3) \cdot (2r+1) \in \mathcal{O}(r^{t-1}).$$

An immediate corollary of Theorems 4.3 and 4.4 is a bound on the chromatic number of $G^{[\natural p]}$ when G excludes a fixed minor, greatly improving the doubly exponential bounds obtained by Theorem 4.1.

Corollary 4.5 *Let $t \geqslant 4, p \geqslant 3$. For every graph G that excludes K_t as a minor, we have*

$$\chi(G^{[\natural p]}) \in 2^{\mathcal{O}((2p-3)^{t-1})}.$$

4.2 Characterising Bounded Expansion by Neighbourhood Complexity

In this section, we further highlight the interplay between low treedepth colourings and generalised colouring numbers. The following notion, due to Felix Reidl, is the central notion of this section.

For a graph G the *r-neighbourhood complexity* is a function ν_r defined via

$$\nu_r(G) := \max_{H \subseteq G, X \subseteq V(H)} \frac{|\{N^r(v) \cap X\}_{v \in V(H)}|}{|X|}.$$

We extend this definition to graph classes $\mathcal{G}$ via $\nu_r(\mathcal{G}) := \sup_{G \in \mathcal{G}} \nu_r(G)$.

Note that we define the value over all possible subgraphs: otherwise complete graphs would yield very low values, which is undesirable for a measure for sparse graph classes. Alternatively, we can define the neighbourhood complexity via the index of an equivalence relation. This turns out to be a useful notational perspective in the main proof of this section. For $r \in N$ and $X \subseteq V(G)$, we define the (r, X)-*twin equivalence* over $V(G)$ as

$$u \simeq_r^{G,X} v \iff N^r(u) \cap X = N^r(v) \cap X$$

which gives rise to the alternative definition

$$\nu_r(G) = \max_{H \subseteq G, X \subseteq V(G)} \frac{|V(H)/\simeq_r^{H,X}|}{|X|}.$$

Whenever we fix a graph, we will omit the superscript G of this relation.

We say that a graph class $\mathcal{G}$ has *bounded neighbourhood complexity* if there exists a function f such that for every r it holds that $\nu_r(\mathcal{G}) \leqslant f(r)$.

The main result of this section is the following characterisation of bounded expansion through neighbourhood complexity for graph classes by utilising the weak-colouring approach.

Theorem 4.6 *A graph class $\mathcal{G}$ has bounded expansion if and only if it has bounded neighbourhood complexity.*

Felix Reidl and Fernando Sánchez Villaamil establish in [79] the forward direction of the above characterisation of bounded expansion by bounding the neighbourhood complexity of a graph in terms of their r-treedepth colouring numbers. In the same paper, the author of this dissertation provides a second perspective to the above characterisation of bounded expansion by bounding the neighbourhood complexity of a graph in terms of their weak r-colouring numbers.

4.2.1 Neighbourhood Complexity and Weak Colouring Numbers

In the following theorem we derive a bound of the r-neighbourhood complexity in terms of the weak r-colouring number. From this, it directly follows from

Theorem 3.8 that for a graph class $\mathcal{G}$, bounded expansion implies bounded neighbourhood complexity, which is the one direction of Theorem 4.6. For the next proof, we say that two vertices $u, v \in V(G)$ have *the same distances* to $Z \subseteq V(G)$ if for every $z \in Z$ we have $d_G(u,z) = d_G(v,z)$.

Theorem 4.7 *For every graph G it holds that*

$$\nu_r(G) \leqslant \frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)}\mathrm{wcol}_{2r}(G) + 1.$$

Proof. Fix a graph G and choose any subset $\emptyset \neq X \subseteq V(G)$. We will show in the following that

$$|V(G)/\simeq_r^X| \leqslant \left(\frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)}\mathrm{wcol}_{2r}(G) + 1\right)|X|,$$

from which the claim immediately follows.

Let $\alpha_0 \in V(G)/\simeq_r^X$ be the equivalence class of $\simeq_r^X$ corresponding to the vertices of G with an empty r-neighbourhood in X and let $\mathcal{W} = \left(V(G)/\simeq_r^X\right) \setminus \{\alpha_0\}$. Moreover, let $L \in \Pi(G)$ be such that $\mathrm{wcol}_{2r}(G) = \max_{v \in V(G)} |\mathrm{WReach}_{2r}[G, L, v]|$. We will estimate the neighbourhood complexity of X via the neighbourhood complexity of a certain *good* subset of $\mathrm{WReach}_r[G, L, X]$.

For a vertex $v \in N^r(X)$ and a vertex $x \in N^r(v) \cap X$, let $\mathcal{P}_v^x$ be the set of all shortest (v,x)-paths (of length at most r). We define $G^r[v]$ as the graph induced by the union of the paths of all $\mathcal{P}_v^x$. Formally,

$$G^r[v] = G\Big[\bigcup_{x \in N^r(v) \cap X} \bigcup_{P \in \mathcal{P}_v^x} V(P)\Big].$$

By its construction, $G^r[v]$ contains, for every $x \in N^r(v) \cap X$, all shortest paths of length at most r that connect v to x.

Now, for every equivalence class $\kappa \in \mathcal{W}$, choose a representative vertex $v_\kappa \in \kappa$. Let $C = \{v_\kappa\}_{\kappa \in \mathcal{W}}$ be the set of representative vertices for all classes in $\mathcal{W}$. Using the representatives from C, we define for every class $\kappa \in \mathcal{W}$ the set (see Fig. 4.1)

$$Y_\kappa = \mathrm{WReach}_r[G^r[v_\kappa], L, v_\kappa] \cap \mathrm{WReach}_r[G, L, N^r(v_\kappa) \cap X]$$

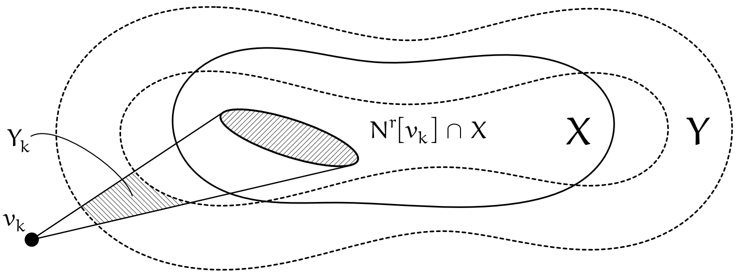


Figure 4.1: A set Y_κ and the set Y.

and join all such sets into $Y = \bigcup_{\kappa \in \mathcal{W}} Y_\kappa$. Then,

$$Y \subseteq \bigcup_{\kappa \in \mathcal{W}} \mathrm{WReach}_r[G, L, N^r(v_\kappa) \cap X] \subseteq \mathrm{WReach}_r[G, L, X].$$

Moreover, by definition and the fact that L is an ordering achieving $\mathrm{wcol}_{2r}(G)$ (and not necessarily one achieving $\mathrm{wcol}_r(G)$), we have

$$|Y_\kappa| \leqslant |\mathrm{WReach}_r[G, L, v_\kappa]| \leqslant |\mathrm{WReach}_{2r}[G, L, v_\kappa]| \leqslant \mathrm{wcol}_{2r}(G).$$

Notice that for every $x \in N^r(v) \cap X$, the minimum vertex (according to L) of a path in $\mathcal{P}_v^x$ will always belong to Y_κ, therefore the set Y_κ intersects every path of the sets $\mathcal{P}_{v_\kappa}^x$ forming $G^r[v_\kappa]$. We want to see how many different equivalence classes of $\mathcal{W}$ produce the same Y_κ set. This will allow us to bound the neighbourhood complexity of X by relating it to the number of different Y_κ's.

Suppose that $\kappa \neq \lambda$ with $Y_\kappa = Y_\lambda = Z$. Recall that Y_κ intersects all the shortest paths from v_κ to the vertices of $N^r(v_\kappa) \cap X$ and that $G^r[v_\kappa]$ is formed by all such shortest paths. Hence, if v_κ and v_λ have the same distances to Z, then we get $N^r(v_\kappa) \cap X = N^r(v_\lambda) \cap X$, a contradiction. This means that if $Y_\kappa = Y_\lambda = Z$, the vertices v_κ and v_λ cannot have the same distances to Z. But there are at most $(r+1)^{|Z|}$ possible configurations of distances of the vertices of a set Z to a vertex v that has distance at most r to every vertex of Z. It follows that the number of equivalence classes of $\mathcal{W}$ that produce the same set Y_κ through their representative v_κ from C is at most $(r+1)^{|Y_\kappa|} \leqslant (r+1)^{\mathrm{wcol}_{2r}(G)}$.

Let $\mathcal{Y} := \{Y_\kappa \mid \kappa \in \mathcal{W}\}$ be the set of all (*different*) Y_κ's, and define $\gamma\colon \mathcal{Y} \to Y$ by $\gamma(Y_\kappa) = \arg\max_{y \in Y_\kappa} L(y)$. That is, $\gamma(Y_\kappa)$ is that vertex in Y_κ that comes last according to L. Observe that—by definition—every vertex in Y_κ is weakly r-reachable from v_κ. It follows that every vertex in Y_κ is weakly $2r$-reachable from $\gamma(Y_\kappa)$ via v_κ. In other words, $Y_\kappa \subseteq \mathrm{WReach}_{2r}[G, L, \gamma(Y_\kappa)]$. Consequently, for every vertex $y \in \gamma(\mathcal{Y})$, it holds that[1]

$$\bigcup \gamma^{-1}(y) \subseteq \mathrm{WReach}_{2r}[G, L, y],$$

i.e. the union $\bigcup \gamma^{-1}(y)$ of all Y_κ's that choose the same vertex y via γ has size at most $\mathrm{wcol}_{2r}(G)$. But every set in the family $\gamma^{-1}(y)$ is a subset of $\bigcup \gamma^{-1}(y)$ that contains y. Since there are at most $2^{|\bigcup \gamma^{-1}(y)|-1}$ different such subsets of $\bigcup \gamma^{-1}(y)$, the number of *different* Y_κ's for which the same vertex is chosen via γ is bounded by $2^{\mathrm{wcol}_{2r}(G)-1}$, i.e.

$$|\gamma^{-1}(y)| \leqslant 2^{\mathrm{wcol}_{2r}(G)-1}.$$

Recalling that one Y_κ corresponds to at most $(r+1)^{\mathrm{wcol}_{2r}(G)}$ equivalence classes of $\mathcal{W}$ and that $Y \subseteq \mathrm{WReach}_r[G, L, X]$, we can now bound the size of $\mathcal{W}$ as follows:

$$\begin{aligned}
|\mathcal{W}| &\leqslant (r+1)^{\mathrm{wcol}_{2r}(G)} \cdot |\mathcal{Y}| = (r+1)^{\mathrm{wcol}_{2r}(G)} \cdot \sum_{y \in \gamma(\mathcal{Y})} |\gamma^{-1}(y)| \\
&\leqslant (r+1)^{\mathrm{wcol}_{2r}(G)} \cdot \sum_{y \in \gamma(\mathcal{Y})} 2^{\mathrm{wcol}_{2r}(G)-1} \\
&= \frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)} \cdot |\gamma(\mathcal{Y})|
\end{aligned}$$

from which we obtain that

$$\begin{aligned}
|V(G)/{\simeq_r^X}| \leqslant |\mathcal{W}| + 1 &\leqslant \frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)} \cdot |\gamma(\mathcal{Y})| + 1 \\
&\leqslant \frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)} \cdot |Y| + 1 \\
&\leqslant \frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)} \cdot |\mathrm{WReach}_r[G, L, X]| + 1 \\
&\leqslant \frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)} \mathrm{wcol}_{2r}(G) \cdot |X| + 1 \\
&\leqslant \left(\frac{1}{2}(2r+2)^{\mathrm{wcol}_{2r}(G)} \mathrm{wcol}_{2r}(G) + 1\right)|X|,
\end{aligned}$$

as claimed. ∎

[1] We remind the reader that this union expresses the union of a set in the set theoretical sense, i.e. the union of a set is the union of all of its elements (as sets).

4.2.2 Completing the Characterisation

We have seen in the previous subsection that bounded expansion implies bounded neighbourhood complexity. Let us now prove the other direction to arrive at the full characterisation. The results of this subsection have been obtained in close collaboration with Felix Reidl and Fernando Sánchez Villaamil. We begin by proving that every bipartite graph with low neighbourhood complexity must have low minimum degree. To that end, we will need the following Lemma.

Lemma 4.8 (Nešetřil & Ossona de Mendez [72]) *Let $G = (A, B, E)$ be a bipartite graph and let $1 \leqslant r \leqslant s \leqslant |A|$. Assume each vertex in B has degree at least r. Then there exists a subset $A' \subseteq A$ and a subset $B' \subseteq B$ such that $|A'| = s$ and $|B'| \geqslant |B|/2$ and every vertex in B' has at least $r\frac{|A'|}{|A|}$ neighbours in A'.*

The minimum degree and depth-one neighbourhood complexity ν_1 of a bipartite graph can now be related to each other as follows:

Lemma 4.9 *Let $G = (A, B, E)$ be a non-empty bipartite graph. Then*

$$\delta(G) < 4\nu_1(G)\Big(2\lceil\log\nu_1(G)\rceil + 1\Big)\Big(64\nu_1(G)^3\lceil\log\nu_1(G)\rceil + 16\nu_1(G)^2 + 1\Big).$$

Proof. Let

$$\alpha = 4\nu_1(G)\Big(2\lceil\log\nu_1(G)\rceil + 1\Big)\Big(64\nu_1(G)^3\lceil\log\nu_1(G)\rceil + 16\nu_1(G)^2 + 1\Big)$$

and suppose that $\delta(G) \geqslant \alpha$. Assume without loss of generality that $|B| \geqslant |A|$ and let $\nu = 2^{\lceil\log\nu_1(G)\rceil}$. Observe that both ν and $\log\nu$ are integers and that $\nu_1(G) \leqslant \nu < 2\nu_1(G)$. Therefore,

$$|B| \geqslant |A| \geqslant \delta(G) > 2\nu(2\log\nu + 1)\Big(8\nu^3\log\nu + 4\nu^2 + 1\Big).$$

Let us apply Lemma 4.8 on G with $r = 8\nu^3\log\nu + 4\nu^2 + 1$ and $s = \lfloor\frac{|A|}{2\nu(2\log\nu+1)}\rfloor$. Notice that this is indeed possible, because $|A| > 2\nu(2\log\nu + 1)\cdot r$ and therefore $s \geqslant r$. We obtain a subgraph $G' = (A', B', E')$ with

1. $\frac{|A|}{2\nu(2\log\nu+1)} - 1 < |A'| = s \leqslant \frac{|A|}{2\nu(2\log\nu+1)}$,
2. $|B'| \geqslant \frac{|B|}{2}$, and thus $|B'| \geqslant \frac{|A|}{2} \geqslant \nu(2\log\nu + 1)|A'|$,

3. and such that for every $v \in B'$ we have that $\deg_{G'}(v) \geqslant r \cdot \frac{|A'|}{|A|}$.

Combining the first and third property with $|A| > 2\nu(2\log\nu + 1)\cdot r$, we obtain

$$\begin{aligned}\deg_{G'}(v) \geqslant r\cdot\frac{|A'|}{|A|} &> r\Big(\frac{1}{2\nu(2\log\nu+1)} - \frac{1}{|A|}\Big)\\ &> r\Big(\frac{1}{2\nu(2\log\nu+1)} - \frac{1}{2\nu(2\log\nu+1)\cdot r}\Big)\\ &= \frac{r-1}{2\nu(2\log\nu+1)} = 2\nu^2.\end{aligned}$$

Now, note that any graph H with at least two vertices trivially has $\nu_1(H) \geqslant 2$ by taking X to be a single vertex of H. Hence, if $K_{2\nu^2, 2\log\nu+1}$ is a subgraph of G', we have that

$$\nu_1(G) \geqslant \nu_1(G') \geqslant \nu_1(K_{2\nu^2,2\log\nu+1}) \geqslant \frac{2\nu^2}{2\log\nu+1} > \nu,$$

where the last inequality follows by the fact that $\nu \geqslant 2$, a contradiction.

So, let us partition B' into twin-classes $B'_1, \ldots B'_\ell$. Since each twin-class has at least $2\nu^2$ neighbours, the size of each twin-class must be bounded by $|B'_i| < 2\log\nu + 1$. Hence, the number of twin-classes is at least $\ell > \frac{|B'|}{2\log\nu+1}$. Since each twin-class has, by definition, a unique neighbourhood in A', we conclude that

$$\nu_1(G') \geqslant \frac{\ell}{|A'|} > \frac{|B'|}{2\log\nu+1}\,\frac{\nu(2\log\nu+1)}{|B'|} = \nu \geqslant \nu_1(G),$$

a contradiction. ■

It easily follows that every graph with low neighbourhood complexity must have low average degree.

Corollary 4.10 *Let G be a graph. Then $\widetilde{\nabla}_0(G) < 5445\cdot\nu_1(G)^4\log^2\nu_1(G)$.*

Proof. We assume that $\widetilde{\nabla}_0(G) = |E(G)|/|V(G)|$, otherwise we restrict ourselves to a suitable subgraph of G with that property. The case where $|V(G)| = 1$ is trivial, therefore we may assume that $|V(G)| \geqslant 2$. It is folklore that G contains a bipartite graph H such that $|E(H)| \geqslant |E(G)|/2$. We can further ensure that $\delta(H) \geqslant |E(H)|/|V(H)|$ by excluding vertices of lower degree (this operation cannot decrease the density of H). Applying Lemma 4.9 to H, we obtain that

$$\widetilde{\nabla}_0(G) = \frac{|E(G)|}{|V(G)|} \leqslant 2\frac{|E(H)|}{|V(H)|} \leqslant 2\delta(H).$$

We apply the bound provided by Lemma 4.9 and relax it to the more concise polynomial $(5445/2) \cdot \nu_1(G)^4 \log^2 \nu_1(G)$, using the fact that $\nu_1(G) \geqslant 2$. ■

The next theorem now leads to the full characterisation as stated in Theorem 4.6.

Theorem 4.11 *For every graph G and every half-integer r it holds that*

$$\widetilde{\nabla}_r(G) \leqslant (2r+1)\max\Big\{5445\nu_1(G)^4\log^2\nu_1(G), \nu_2(G), \dots, \nu_{\lceil r+1/2\rceil}(G)\Big\}.$$

Proof. Fix r and let $H \preceq_r^t G$ be an topological depth-r minor of maximal density, i.e. $\widetilde{\nabla}_0(H) = \widetilde{\nabla}_r(G)$. Let further (φ_V, φ_E) be a topological minor embedding of H into G of depth r.

Let us label the edges of H by the respective path-length in the embedding φ_V, φ_E: an edge $uv \in H$ receives the label $|E(\varphi_E(uv))|$. Let r' be the label of highest frequency and let $H' \subseteq H$ be the graph obtained from H by only keeping edges labelled with r'. Since there were up to $2r+1$ labels in H, we have that $(2r+1)|E(H')| \geqslant |E(H)|$ and therefore

$$\widetilde{\nabla}_r(G) = \widetilde{\nabla}_0(H) \leqslant (2r+1)\frac{|E(H')|}{|V(H')|} \leqslant (2r+1)\widetilde{\nabla}_0(H').$$

First, consider the case that $r' = 1$, i.e. H' is a subgraph of G. Combining (4.2.2) with Corollary 4.10, we obtain

$$\begin{aligned}\widetilde{\nabla}_r(G) &\leqslant (2r+1)\widetilde{\nabla}_0(H') \leqslant (2r+1)\widetilde{\nabla}_0(G) \\ &\leqslant (2r+1) \cdot 5445\,\nu_1(G)^4\log^2\nu_1(G).\end{aligned}$$

Otherwise, assume that $r' \geqslant 2$, i.e. every edge of H' is embedded into a path of length at least 2 in G by φ_V, φ_E. Construct the subgraph $G' \subseteq G$ that contains all edges and vertices involved in the embedding of H' into G, that is, G' has vertices $\bigcup_{v \in H'} V(\varphi_V(v)) \cup \bigcup_{e \in H'} V(\varphi_E(e))$ and edges $\bigcup_{e \in H'} E(\varphi_E(e))$.

Let $X = \bigcup_{v \in H'} V(\varphi_V(v))$ and let $S \subseteq V(G')$ be a set constructed as follows: for every edge $e \in H'$ we add the middle vertex of the path $\varphi_E(e)$ to S—in case r' is odd, we pick one of the two vertices that lie in the middle of $\varphi_E(e)$ arbitrarily. Because X is an independent set in G' and $r' > 1$, every vertex in S has exactly two neighbours in X at distance at most $\lceil r'/2 \rceil$ in the graph G'. By construction, there is a one-to-one correspondence between these $\lceil r'/2 \rceil$-neighbourhoods and the edges of H'. Accordingly,

$$|E(H')| = |\{N_{G'}^{\lceil r'/2 \rceil}(v) \cap X\}_{v \in S}|$$

and therefore, using also the fact that G' is a subgraph of G,

$$\frac{|E(H')|}{|V(H')|} = \frac{|\{N_{G'}^{\lceil r'/2 \rceil}(v) \cap X\}_{v \in S}|}{|X|} \leqslant \nu_{\lceil r'/2 \rceil}(G') \leqslant \nu_{\lceil r'/2 \rceil}(G).$$

This, taken together with (4.2.2) and the fact that G' is a subgraph of G, yields

$$\widetilde{\nabla}_r(G) \leqslant (2r+1)\widetilde{\nabla}_0(H') \leqslant (2r+1)\nu_{\lceil r'/2 \rceil}(G).$$

Putting everything together, we finally arrive at

$$\widetilde{\nabla}_r(G) = (2r+1)\max\Big\{5445\,\nu_1(G)^4 \log^2 \nu_1(G),\, \nu_2(G), \ldots, \nu_{\lceil r+1/2 \rceil}(G)\Big\},$$

proving the theorem. ∎

We conclude that graph classes with bounded neighbourhood complexity have bounded expansion. Theorem 4.6 follows by Theorems 3.8, 4.7 and 4.11.

Part II

Generalized Graph Decompositions

Ουκ εν τω πολλω το ευ, αλλ' εν τω ευ το πολυ.

Αριστοτελης

5

Median Decompositions and Medianwidth

Median graphs will be the generalisation of trees, whose good geometric properties will allow us to extend in a consistent way the concept of tree decompositions in order to be able to model graphs not only after trees, but also more complicated graphs. We start by summarising some of the relevant parts around the theory of median graphs. For a detailed view on median graphs, the reader can refer to books [38, 48, 90] and papers [5, 54], or a general survey on metric graph theory and geometry [2].

5.1 Median Graphs

For $u, v \in V(G)$, a (u, v)-*geodesic* is a shortest (u, v)-path. A path P in G is a *geodesic* if there are vertices u, v such that P is a (u, v)-geodesic.

The *interval* $I(u, v)$ consists of all vertices lying on a (u, v)-geodesic, namely

$$I(u,v) = \{x \in V(G) \mid d(u,v) = d(u,x) + d(x,v)\}.$$

A graph G is called *median* if it is connected and for any three vertices $u, v, w \in V(G)$ there is a unique vertex x, called the *median* of u, v, w, that lies simultaneously on a (u, v)-geodesic, (v, w)-geodesic and a (w, u)-geodesic. In other words, G is median if $|I(u,v) \cap I(v,w) \cap I(w,u)| = 1$, for every three vertices u, v, w.

A set $S \subseteq V(G)$ of a connected graph G is called *geodesically convex* or just *convex* if for every $u, v \in S$, $I(u, v) \subseteq S$ (we will only talk about geodesic convexity and not other graph convexities, so it is safe to refer to geodesically convex sets as just convex, without confusion). By definition, convex sets are connected. As with convex sets in Euclidean spaces (or more generally, as a prerequisite of abstract convexities), it is easy to see that the intersection of convex sets is again convex. Note that the induced subgraphs corresponding to convex sets of median graphs are also median graphs.

For $S \subseteq V(G)$, its *convex hull* $< S >$ is the minimum convex set of G containing S.

For the rest of the section, we present without proofs some well-known basic theory on median graphs and summarize some of their most important properties, that will be important for our needs throughout this part of the dissertation.

Let us present some examples. Let C_k be the cycle graph on k vertices. Notice that the cycles C_3 and C_k, where $k \geqslant 5$, are not median, simply because there are always 3 vertices with no median. As we will later see, every median graph is bipartite. On the other hand, apart from the even cycles of length at least six, examples of bipartite graphs that aren't median are the complete bipartite graphs $K_{n,m}$ with $n \geqslant 2$ and $m \geqslant 3$, since all n vertices of one part are medians of every three vertices of the other part.

The *i-dimensional hypercube* or *i-cube* Q_i, $i \geqslant 1$, is the graph with vertex set $\{0, 1\}^i$, two vertices being adjacent if the corresponding tuples differ in precisely one position. Note that the Cartesian product of i-copies of $K_2 = Q_1$ is an equivalent definition of the i-cube Q_i. The hypecubes are also the only regular median graphs [65].

In the Cartesian products of median graphs, medians of vertices can be seen to correspond to the tuple of the medians in every factor of the product. The following Lemma is folklore.

Lemma 5.1 *Let $G = \square_{i=1}^{k} G_i$, where G_i is median for every $i = 1, \ldots, k$. Then G is also median, whose convex sets are precisely the sets $C = \square_{i=1}^{k} C_i$, where C_i is a convex subset of G_i.*

There are several characterizations of median graphs: they are exactly the retracts of hypercubes; they can be obtained by successive applications of convex amalgamations of proper median subgraphs; they can also be obtained by K_1 after a sequence of *convex* or *peripheral expansions*. We see them in more detail in what follows.

A graph G is a *convex amalgam* of two graphs G_1 and G_2 (along $G_1 \cap G_2$) if G_1 and G_2 constitute two intersecting induced convex subgraphs of G whose union is all of G.

Median graphs are easily seen to be closed under retraction, and since they include the i-cubes, every retract of a hypercube is a median graph. Actually, the inverse is also true, one of whose corollaries is that median graphs are bipartite graphs.

Theorem 5.2 *[3, 49, 89] A graph G is median if and only if it is the retract of a hypercube. Every median graph with more than two vertices is either a Cartesian product or a convex amalgam of proper median subgraphs.*

A graph H is *isometrically embeddable* into a graph G if there is a mapping $\varphi : V(H) \to V(G)$ such that $d_G(\varphi(u), \varphi(v)) = d_H(u,v)$ for any vertices $u, v \in H$. Isometric subgraphs of hypercubes are called *partial cubes*. Retracts of graphs are isometric subgraphs, hence median graphs are partial cubes, but not every partial cube is a median graph: C_6 is an isometric subgraph of Q_3, but not a median graph.

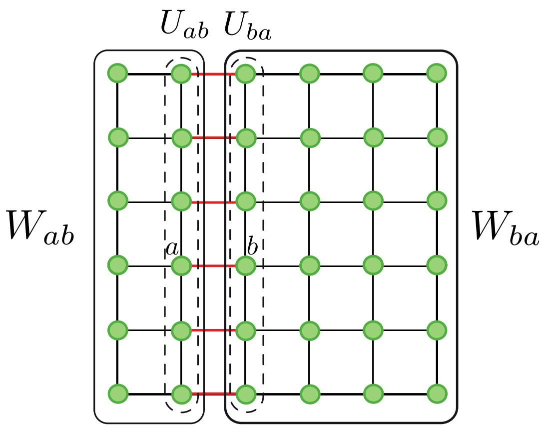


Figure 5.1: Θ-class, W- and U-sets of an edge ab.

Suppose that (A,B) is a separation of G, where $A \cap B \neq \emptyset$ and $G[A], G[B]$ are isometric subgraphs of G. An *expansion* of G with respect to (A,B) is a graph H obtained from G by the following steps:

(i) Replace each $v \in A \cap B$ by vertices v_1, v_2 and insert the edge v_1v_2.

(ii) Insert edges between v_1 and all neighbours of v in $A \setminus B$. Insert edges between v_2 and all neighbours of v in $B \setminus A$.

(iii) Insert the edges v_1u_1 and v_2u_2 if $v, u \in A \cap B$ and $vu \in E(G)$.

An expansion is *convex* if $A \cap B$ is convex in G. We can now state Mulder's Convex Expansion Theorem on median graphs.

Theorem 5.3 *[64, 62] A graph is median if and only if it can be obtained from K_1 by a sequence of convex expansions.*

For a connected graph and an edge ab of G we denote

- $W_{ab} = \{v \in V(G) \mid d(v,a) < d(v,b)\}$,
- $U_{ab} = W_{ab} \cap N_G(W_{ba})$.

Sets of the graph that are W_{ab} for some edge ab will be called W-*sets* and similarly we define U-*sets*. If $U_{ab} = W_{ab}$ for some edge ab, we call the set U_{ab} a peripheral set of the graph. Note that if G is a bipartite graph, then $V(G) = W_{ab} \cup W_{ba}$ and $W_{ab} \cap W_{ba} = \emptyset$ is true for any edge ab. If G is a median graph, it is easy to see that

W-sets and U-sets are convex sets of G. Moreover, the W-sets of G play a similar role to that of the halfspaces of the Euclidean spaces, which is highlighted by the following lemma:

Lemma 5.4 *For a median graph, every convex set is an intersection of W-sets.*

Edges $e = xy$ and $f = uv$ of a graph G are in the *Djokovic-Winkler* relation Θ [31, 92] if $d_G(x,u)+d_G(y,v) \neq d_G(x,v)+d_G(y,u)$. Relation Θ is reflexive and symmetric. If G is bipartite, then Θ can be defined as follows: $e = xy$ and $f = uv$ are in relation Θ if $d(x,u) = d(y,v)$ and $d(x,v) = d(y,u)$. Winkler [92] proved that on bipartite graphs relation Θ is transitive if and only if it is a partial cube and so, by Theorem 5.2 it is an equivalence relation on the edge set of every median graph, whose classes we call Θ-classes.

The following lemma summarizes some properties of the Θ-classes of a median graph:

Lemma 5.5 *[48] Let G be a median graph and for an edge ab, let $F_{ab} = F_{ba}$ denote the set of edges between W_{ab} and W_{ba}. Then the following are true:*

1. *F_{ab} is a matching of G.*
2. *F_{ab} is a minimal cut of G.*
3. *A set $F \subseteq E(G)$ is a Θ-class of G if and only if $F = F_{ab}$ for some edge $ab \in E(G)$.*

An expansion with respect to a separation (A,B) of G is called *peripheral*, if $A \subseteq B$ and $A = A \cap B$ is a convex set of G. In other words, if A is a convex set, the peripheral expansion along A is the graph H obtained by taking the disjoint union of a copy of G and A and joining each vertex in the copy of A to its corresponding vertex of the subgraph A of G in the copy of G. Note that in the new graph H, the new copy of A is a peripheral set of H, hence the name of the expansion. Moreover, during a peripheral expansion of a median graph, exactly one new Θ-class appears. Peripheral expansions are enough to get all median graphs.

Theorem 5.6 *[63] A graph G is a median graph if and only if it can be obtained from K_1 by a sequence of peripheral expansions.*

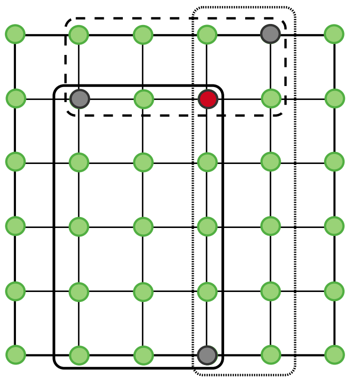


Figure 5.2: The Helly property for three intervals on the Grid.

Finally, a family of sets $\mathcal{F}$ on a universe U has the *Helly property*, if every finite subfamily of $\mathcal{F}$ with pairwise-intersecting sets, has a non-empty total intersection. A crucial property for our purposes is the following well-known lemma for the convex sets of a median graph.

Lemma 5.7 *[48] The convex sets of a median graph G have the Helly property.*

5.2 Median Decompositions and General Properties

Let us recall some of the most important properties of tree decompositions as in Lemma 2.1.

Lemma 2.1 *Let $\mathcal{D} = (T, \mathcal{Z}) \in \mathcal{T}^G$.*

(i) For every $H \subseteq G$, the pair $(T, (Z_t \cap V(H))_{t \in T})$ is a tree decomposition of H, so that $\mathrm{tw}(H) \leqslant \mathrm{tw}(G)$.

(ii) Any complete subgraph of G is contained in some bag of $\mathcal{D}$, hence we have $\omega(G) \leqslant \mathrm{tw}(G) + 1$.

(iii) For every edge $t_1 t_2$ of T, the set $Z_{t_1} \cap Z_{t_2}$ separates the set $W_1 := \bigcup_{t \in T_1} Z_t$ from the set $W_2 := \bigcup_{t \in T_2} Z_t$, where T_1, T_2 are the components of $T - t_1 t_2$, with $t_1 \in T_1$ and $t_2 \in T_2$.

(iv) If $H \preceq^m G$, then $\mathrm{tw}(H) \leqslant \mathrm{tw}(G)$.

(v) $\chi(G) \leqslant \mathrm{tw}(G)+1$.

(vi) $\mathrm{tw}(G) = \min\{\omega(G') - 1 \mid G \subseteq G' \text{ and } G' \text{ chordal}\}$.

In a tree decomposition, every vertex of the graph lives in a connected subtree of the tree. Recall that trees are median graphs. As we already foreshadowed in Chapter 1, the crucial observation, which (together with the Helly property of the convex sets of median graphs) is actually the reason that enables the development of the whole theory in this part, is the following:

A subgraph of a tree is convex if and only if it is connected.

Inspired by this observation and the general theory on tree decompositions, it is only natural to define this concept of decomposition of a graph, not only on trees such that every vertex of the graph lives in a connected subtree, but generally on median graphs such that every vertex lives in a convex subgraph of the median graph.

A *median decomposition* $\mathcal{D}$ of a graph G is a pair $(M, \mathcal{X})$, where M is a median graph and $\mathcal{X} = (X_a)_{a \in V(M)}$ is a family of subsets of $V(G)$ (called bags) such that

(M1) for every edge $uv \in E(G)$ there exists $a \in V(M)$ with $u, v \in X_a$,

(M2) for every $v \in V(G)$, the set $X^{-1}(v) := \{a \in V(M) \mid v \in X_a\}$ is a non-empty convex subgraph of M.

The *width* of a median decomposition $\mathcal{D} = (T, \mathcal{X})$ is the number

$$\max\{|X_a| \mid a \in V(M)\}.^{2}$$

Let $\mathcal{M}^G$ be the set of all median decompositions of G. The *medianwidth* $\mathrm{mw}(G)$ of G is the least width of any median decomposition of G:

$$\mathrm{mw}(G) := \min_{\mathcal{D} \in \mathcal{M}^G} \max\{|X_a| \mid a \in V(M)\}.$$

[2] While the definition of the width of tree decompositions is adjusted so that trees are exactly the graphs of treewidth 1, by Theorem 5.12 all trianglefree graphs have minimum medianwidth. Since there wouldn't be a similar exact correspondence of graphs of minimum medianwidth to the underlying graph class of median decompositions as in the case of treewidth, we felt that such an adjustment is not meaningful for medianwidth.

Since $\mathcal{T}^G \subseteq \mathcal{M}^G$, by definition of $\mathrm{mw}(G)$ we have $\mathrm{mw}(G) \leqslant tw(G)+1$. Let us find out which of the properties of tree decompositions in Lemma 2.1 can be translated in any sense to properties of median decompositions. For the Lemmas that follow, $\mathcal{D} = (T, \mathcal{X}) \in \mathcal{M}^G$ is a median decomposition of a graph G. It is straightforward that median decompositions are passed on to subgraphs.

Lemma 5.8 *For every $H \subseteq G$, $(M, (X_a \cap V(H))_{a \in M})$ is a median decomposition of H, hence* $\mathrm{mw}(H) \leqslant \mathrm{mw}(G)$. ■

The Helly property of the convex sets of median graphs was the secondary reason that indicated that median decompositions seem to be a natural notion. It is what allows us to prove the direct analogue of Lemma 2.1 (ii).

Lemma 5.9 *Any complete subgraph of G is contained in some bag of $\mathcal{D}$. In particular,* $\omega(G) \leqslant \mathrm{mw}(G)$.

Proof. Let K be a complete subgraph of G. By (M1), for every $u, v \in V(K)$, there exists a bag of M that contains both u and v, so that $X^{-1}(u) \cap X^{-1}(v) \neq \emptyset$. By (M2), the family $\mathcal{F} = \{X^{-1}(v) \mid v \in V(K)\}$ is a family of pairwise-intersecting convex sets of the median graph M. By Lemma 5.7,

$$\bigcap \mathcal{F} = \bigcap_{v \in V(K)} X^{-1}(v) \neq \emptyset$$

and hence, there is a bag of M that contains all vertices of K. ■

For a median decomposition $(M, \mathcal{X})$ and a minimal cut $F \subseteq E(M)$ of M that separates $V(M)$ into W_1 and W_2, let U_i be the vertices of W_i adjacent to edges of F, and let $Y_i := \bigcup_{x \in W_i} X_x$, $Z_i := \bigcup_{x \in U_i} X_x$, where $i = 1, 2$. Observe that minimal cuts on a tree are just single edges by themselves. This leads us to an analogue of Lemma 2.1(iii), which says that minimal cuts of M correspond to separations of G.

Lemma 5.10 *For every minimal cut F of M and Y_i, Z_i, $i = 1, 2$, defined as above, $Z_1 \cap Z_2$ separates Y_1 from Y_2.*

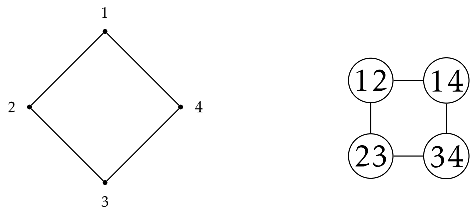


Figure 5.3: A median decomposition of C_4 of width 2.

Proof. Let $v \in Y_1 \cap Y_2$. Then there are $a \in W_1, b \in W_2$, such that $v \in X_a \cap X_b$, i.e. $a, b \in X^{-1}(v)$. By the convexity of $X^{-1}(v)$, it must be $I(a,b) \subseteq X^{-1}(v)$. But F is a minimal cut between W_1 and W_2, therefore there is an $xy \in F$ with $x \in W_1, y \in W_2$, such that $x, y \in X^{-1}(v)$, so that $v \in X_x \cap X_y \subseteq Z_1 \cap Z_2$. This proves that $Y_1 \cap Y_2 \subseteq Z_1 \cap Z_2$.

It remains to show, that there is no edge u_1u_2 of G with $u_1 \in Y_1 \setminus Y_2$ and $u_2 \in Y_2 \setminus Y_1$. If u_1u_2 was such an edge, then by (M1) there is an $x \in V(M)$ with $u_1, u_2 \in X_x$, hence $x \in X^{-1}(u_1) \cap X^{-1}(u_2) \subseteq (Y_1 \setminus Y_2) \cap (Y_2 \setminus Y_1) = \emptyset$, a contradiction. ■

Recall that by Lemma 5.5, for an edge ab of M, the Θ-class F_{ab} is a minimal cut of M. Denote $Y_{ab} := \bigcup_{x \in W_{ab}} X_x$ and $Z_{ab} := \bigcup_{x \in U_{ab}} X_x$. We will refer to them as the *Y-sets* and *Z-sets* of a median decomposition $\mathcal{D}$. Note that the Y-sets and Z-sets are subsets of the decomposed graph G, while the W-sets and U-sets are subsets of the median graph M of the decomposition. Observe that a more special way to look at the edges of a tree is that each edge of a tree forms a degenerated Θ-class by itself and its two corresponding U-sets are the ends of the edge. As a special case of Lemma 5.10, we obtain a more specific analogue of Lemma 2.1(iii), which says that intersections of unions of bags across opposite sides of a whole Θ-class of M also correspond to separations of G.

Lemma 5.11 *For every edge ab of M, $Z_{ab} \cap Z_{ba}$ separates Y_{ab} from Y_{ba}.* ■

While the first three properties of Lemma 2.1 can be translated into the setting of median decompositions, it is not the case that $\operatorname{mw}(H) \leqslant \operatorname{mw}(G)$, whenever $H \preceq^m G$. The median decomposition of C_4 in Fig. 5.3, shows that $\operatorname{mw}(C_4) \leqslant 2$, while (by Lemma 5.9) $\operatorname{mw}(C_3) \geqslant \omega(C_3) = 3$ and $C_3 \preceq^m C_4$. An insight to why medianwidth is not a minor-closed parameter, is that while the union of two intersecting connected subsets of a tree is again a connected subset (which allows you to safely replace in the bags of a tree decomposition both vertices of a contracted edge

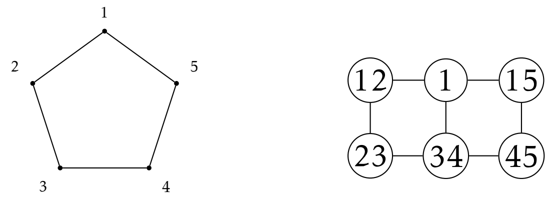


Figure 5.4: $\mathrm{mw}(C_5) = 2$, while $\chi(C_5) = 3$.

of the original graph with the new vertex obtained by the contraction without hurting (T2) and get a tree decomposition of the contracted graph with at most the same width), it is not true in general that the union of two intersecting convex sets of a median graph is again convex.

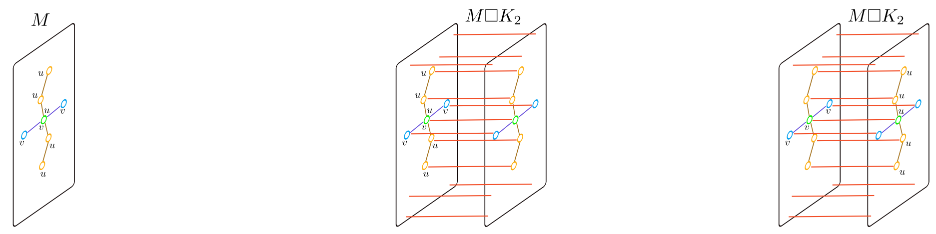
The *simplex graph* $\kappa(G)$ of G, is the graph with vertex set the set of complete subgraphs of G, where two vertices of $\kappa(G)$ are adjacent if the corresponding cliques differ by exactly one vertex of G. It is well-known that $\kappa(G)$ is a median graph [8, 9].

We have seen that $\omega(G) \leqslant \mathrm{mw}(G) \leqslant \mathrm{tw}(G) + 1$ and that medianwidth is not a minor-closed parameter. It is natural to ask if medianwidth is related to other non-minorclosed graph parameters between the clique number and the treewidth. Treewidth is not a minor-closed parameter and in general, $\mathrm{mw}(G) < \mathrm{tw}(G) + 1$, so one immediate candidate is the clique number itself. By Lemma 2.1(v), and for reasons that will become apparent in Section 5.3, the chromatic number $\chi(G)$ is the other candidate that we thought of. C_5 and Fig. 5.4 show that the medianwidth and the chromatic number are not equivalent, but in Section 5.3 we will still attempt to compare the two parameters.

As indicated by the simplex graph, it turns out that clique number is indeed the correct answer. While one might be able to argue by considering $\kappa(G)$, we will adopt a different approach for the proof, which we believe that highlights the fact that the directions we consider in Section 6.1 are natural for the development of this theory.

Note that in the proof below the underlying median graph of the median decomposition differs in general from the simplex graph of the decomposed graph.

Theorem 5.12 *For any graph G,* $\mathrm{mw}(G) = \omega(G)$.



$\mathcal{D}_0 = (M, \mathcal{X})$ Duplicated $\mathcal{D}_0$ $\mathcal{D}' = (M', \mathcal{X}')$

Figure 5.5: Proof Idea of Theorem 5.12.

Proof. By Lemma 5.9, it is enough to show $\mathrm{mw}(G) \leqslant \omega(G)$. For a median decomposition $\mathcal{D} = (M, \mathcal{X})$, let $\beta(\mathcal{D})$ be the number of non-edges of G contained in a bag of $\mathcal{D}$, namely

$$\beta(\mathcal{D}) := \left|\left\{\{v, u\} \mid vu \notin E(G) \text{ and } X^{-1}(v) \cap X^{-1}(u) \neq \emptyset\right\}\right|.$$

Let $\mathcal{D}_0 = (M, \mathcal{X}) \in \mathcal{M}^G$ with $\beta(\mathcal{D}_0)$ minimum. We will prove that $\beta(\mathcal{D}_0) = 0$ and therefore, every bag of $\mathcal{D}_0$ will induce a clique in G. Then by Lemma 5.9 the Theorem will follow.

Suppose that $\beta(\mathcal{D}_0) > 0$. Then there exists a node $a_0 \in V(M)$ and two vertices in $v, u \in X_{a_0}$, such that $vu \notin E(G)$. Consider the decomposition $\mathcal{D}' = (M', \mathcal{X}')$ of G (Fig. 5.5), where:

- $M' = M \square K_2$ is the median graph obtained by the peripheral expansion of M on itself, where $V(M') = M_1 \cup M_2$ and M_1, M_2 induce isomorphic copies of M. Let a_1, a_2, be the copies of $a \in V(M)$ in M_1, M_2 respectively.
- For every $a \in V(M)$, $X'_{a_1} := X_a \setminus \{v\}$, $X'_{a_2} := X_a \setminus \{u\}$.

It is straightforward to check that $\mathcal{D}'$ is a valid median decomposition of G, where every bag of $\mathcal{D}_0$ has been duplicated, but u lives only in M_1 and v only in M_2. Clearly, in $\mathcal{D}'$ we have that $X'^{-1}(v) \cap X'^{-1}(u) = \emptyset$, hence $\beta(\mathcal{D}') = \beta(\mathcal{D}_0) - 1$, a contradiction. ∎

5.3 Medianwidth vs Chromatic Number

As we discussed in the previous section, the chromatic number was another promising candidate, which we thought we could compare to medianwidth. Even though the standard medianwidth is equivalent to the clique number of a graph, the following construction gives us an indication that suitable variations of medianwidth can become equivalent to the chromatic number.

A k-dimensional lattice graph L is a graph obtained by the Cartesian Product of k paths. By Lemma 5.1, lattice graphs are median graphs. For a k-colourable graph G, let $c : V(G) \to \{1,\dots,k\}$ be a proper colouring of G and for $i = 1,\dots,k$, let P_i be a path with $|c^{-1}(i)|$ vertices, whose vertices are labeled by the vertices of $c^{-1}(i)$ with arbitrary order. Consider the k-dimensional lattice graph $L = \square_{i=1}^{k} P_i$, whose vertices $\mathbf{a} = (v_1,\dots,v_k) \in V(L)$ are labelled by the k-tuple of labels of $v_1,\dots,v_k$. For a vertex $\mathbf{a} \in V(L)$, define $X_{\mathbf{a}}$ to be the set of vertices that constitute the k-tuple of labels of $\mathbf{a}$. Let $\mathcal{X} = (X_{\mathbf{a}})_{\mathbf{a}\in V(L)}$.

Lemma 5.13 *The pair $\mathcal{D} = (L,\mathcal{X})$ is a median decomposition of G of width k.*

Proof. Since every colour class $c^{-1}(i)$ is an independent set and since the bags of $\mathcal{X}$ are all the transversals of the colour classes, every edge of G is contained in a bag, so that (M1) holds. To see (M2), as c defines a partition of $V(G)$, every vertex of G will be a label in some k-tuple labelling a vertex of L, which means that there is a bag in $\mathcal{X}$ containing it. Let $v \in V(G)$ be the label of $x_v \in P_i$. Then

$$X^{-1}(v) = \left(\square_{j\neq i} P_j\right) \square \{x_v\},$$

which, by Lemma 5.1, is a convex subgraph of L. ■

We will refer to median decompositions obtained from a colouring of $V(G)$ as in Lemma 5.13 as *chromatic median decompositions*. Fig. 5.6 shows a chromatic decomposition of a bipartite graph. In an attempt to add some intuition to chromatic median decompositions (if needed), borrowing terminology from geometry, in a chromatic median decomposition we make every vertex $v \in V(G)$ live in its own hyperplane of the lattice, a maximal sublattice of the lattice of codimension 1, which is of course convex.

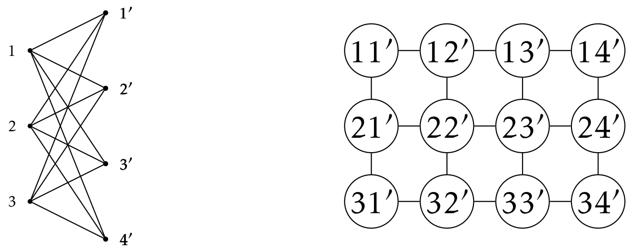


Figure 5.6: A chromatic median decomposition of $K_{3,4}$.

As one can observe, chromatic median decompositions enjoy more regularity than general ones. One would hope that by adding in the definition of median decompositions a suitable third axiom to exploit this regularity, and which axiom would automatically hold for chromatic median decompositions, we would be able to make the respective variation of medianwidth equivalent to the chromatic number.

As we mentioned in Section 2.8, a graph of treewidth k has a *smooth* tree decomposition $(T, \mathcal{Z})$ of width k, i.e. one such that for every $st \in E(T)$ we have $|Z_s \setminus Z_t| = |Z_t \setminus Z_s| = 1$. Recall the definition of the Z-sets of a median decomposition. Similarly to tree decompositions, we define a median decomposition $(M, \mathcal{X})$ to be Θ-*smooth*, if for every $ab \in E(M)$, we have $|Z_{ab} \setminus Z_{ba}| = |Z_{ba} \setminus Z_{ab}| = 1$ and additionally, $X^{-1}(v_a) \cup X^{-1}(v_b)$ is convex in M, where $\{v_a\} = Z_{ab} \setminus Z_{ba}$, $\{v_b\} = Z_{ba} \setminus Z_{ab}$. Notice that since the Θ-classes of a tree are single edges, smoothness and Θ-smoothness coincide on tree decompositions.

We consider the following third axiom in the definition of median decompositions:

(M3) $\mathcal{D}$ is Θ-smooth.

The *smooth-medianwidth* $\text{s-mw}(G)$ of G is the minimum width over all median decompositions of G that additionally satisfy (M3).

Lemma 5.14 *For any graph G, $\chi(G) \leqslant \text{s-mw}(G)$.*

Proof. Let $\text{s-mw}(G) \leqslant k$. Consider a Θ-smooth median decomposition $(M, \mathcal{X})$ of G of width at most k and $P = U_{ab}$ a peripheral set of M. Like in the definition of Θ-smoothness, let v_a the single element of $Z_{ab} \setminus Z_{ba}$ and v_b the single element of $Z_{ba} \setminus Z_{ab}$. By Lemma 5.11, $Z_{ab} \cap Z_{ba}$ separates v_a and v_b, hence they are not adjacent in G. Notice that since P is peripheral, all neighbours of v_a in G are contained in $Z_{ab} \cap Z_{ba}$. Let G' be the graph obtained by G by identifying v_a and

v_b into one new vertex v. Then, by letting $M' = M \setminus P = M[W_{ba}]$ and replacing v_b with v in every bag of $X^{-1}(v_b)$ (which remains convex in M'), we obtain a decomposition $(M', \mathcal{X}')$ of G' of width as most k, for which (M2) is immediately passed onto.

To see (M1), notice that $N_{G'}(v) = N_G(v_a) \cup N_G(v_b)$. By the convexity of $X^{-1}(v_a) \cup X^{-1}(v_b)$ in M, $X^{-1}(v_a) \cap U_{ab}$ and $X^{-1}(v_b) \cap U_{ba}$ are joined by a perfect matching $F \subseteq F_{ab}$ in M, hence v_b is also contained in a common bag with every neighbour of v_a. This means that v is contained in a common bag with every one of its neighbours in G', hence $(M', \mathcal{X}')$ is a valid smooth median decomposition of G'.

By induction on the number of vertices of a graph with smooth-medianwidth at most k, G' is k-colourable. Let c' be a k-colouring of G'. Since v_a, v_b are not adjacent in G, by letting $c(v_a) = c(v_b) = c'(v)$ and $c(u) = c'(u)$ for every $u \in V(G) \setminus \{v_a, v_b\}$, we obtain a proper k-colouring c of G. The Lemma follows. ■

Lastly, by the way they are defined, chromatic median decompositions are Θ-smooth, hence $\text{s-mw}(G) \leqslant \chi(G)$. An immediate corollary of this observation and Lemma 5.14 is the following characterisation of the chromatic number.

Theorem 5.15 *For any graph G, $\text{s-mw}(G) = \chi(G)$.* ■

5.4 More General Decompositions

Let K be a subset of vertices in a graph G, and let $u \in V(G)$. A *gate* for $u \in K$ is a vertex $x \in K$ such that x lies in $I(u, w)$, for each vertex $w \in K$. Trivially, a vertex in K is its own gate. Moreover, if u has a gate in K, then it must be unique and it is the vertex in K closest to u. A subset K of $V(G)$ is called *gated*, if every vertex v of G has the gate $p_K(v)$ in K.

Some general properties of gated sets are that every gated set is also geodesically convex (see [33]), that a map which maps a vertex to its gate in a gated set is a retraction (see Lemma 16.2 in [48]), that the intersection of two gated sets yields a gated set again (see Lemma 16.3 in [48]) and, very importantly, that the family of gated sets has the Helly property (see Corollary 16.3 in [48]). In the case of median graphs, gated sets are exactly the convex sets (see Lemma 12.5 in [48]).

Lemma 5.9, which essentially says that cliques behave as a compact, inseparable object of the decomposed graph, can be also seen in the following way: when the decomposition is seen as a hypergraph on the vertex set of the decomposed graph with hyperedges the bags of the decomposition, a tree or median decomposition becomes a *conformal hypergraph*[3] that covers the edges of the decomposed graph.

If we want to decompose a graph modelling it after any certain kind of graphs and in a way that the most characteristic properties of tree and median decompositions are preserved, like the one described above, then gated sets seem to provide a natural tool for such decompositions, exactly like convex sets do for median decompositions.

Let $\mathcal{H}$ be a class of graphs. An *$\mathcal{H}$-decomposition* $\mathcal{D}$ of a graph G is a pair $(H, \mathcal{X})$, where $H \in \mathcal{H}$ and $\mathcal{X} = (X_h)_{h \in V(H)}$ is a family of subsets of $V(G)$, such that

($\mathcal{H}$1) for every edge $uv \in E(G)$ there exists $h \in V(H)$ with $u, v \in X_h$,

($\mathcal{H}$2) for every $v \in V(G)$, the set $X^{-1}(v) := \{h \in V(H) \mid v \in X_h\}$ is a non-empty gated set of H.

The *width* of an $\mathcal{H}$-decomposition $\mathcal{D} = (H, \mathcal{X})$ is the number

$$\max\{|X_h| \mid h \in V(H)\}.$$

The *$\mathcal{H}$-width* $\mathcal{H}\mathrm{w}(G)$ of G is the least width of any $\mathcal{H}$-decomposition of G.

Since the Helly property holds for the gated sets of any graph, a direct imitation of the proof of Lemma 5.9 shows that every clique of a graph has to be fully contained in some bag of any $\mathcal{H}$-decomposition, so that $\omega(G) \leqslant \mathcal{H}\mathrm{w}(G)$ (and hence $\mathcal{H}\mathrm{w}$ is an unbounded parameter when considered on all graphs). Moreover, the convexity of gated sets ensures that the analogue of Lemma 5.10 holds for general $\mathcal{H}$-decompositions as well. Lastly, general laminar cuts in the decomposition graph H correspond to laminar separations in the decomposed graph G, exactly as in Lemma 6.5.

In the case that the structure of the gated sets of the graphs of a class $\mathcal{H}$ is relatively poor, the corresponding decompositions are not very flexible. For example, the gated sets of a clique are only the singletons and the whole clique itself. For a vertex set $S \subseteq V(G)$, let $\mathcal{C}^G(S)$ be the set of components of $G \setminus S$. Let us see then in

[3] A hypergraph H is conformal if the hyperedges of its dual hypergraph H* satisfy the Helly Property.

the next lemma what the corresponding parameter $\mathcal{K}$w represents, when $\mathcal{K}$ is the graph class of all cliques.

Lemma 5.16 *For every graph G,*

$$\mathcal{K}\mathrm{w}(G) = \min_{S \subseteq V(G)} \max\{|S \cup C| \mid C \in \mathcal{C}^G(S)\}.$$

Proof. For the rest of the proof, let

$$k := \min_{S \subseteq V(G)} \max\{|S \cup C| \mid C \in \mathcal{C}^G(S)\}.$$

Now, let $\mathcal{D} = (K, \mathcal{X})$ be a $\mathcal{K}$-decomposition of width $\mathcal{K}\mathrm{w}(G)$. Consider the set W of the vertices of G that appear in all bags of $\mathcal{D}$. Then every vertex of $V(G) \setminus W$ appears in at most one bag of $\mathcal{D}$. Thus, for every node $x \in V(K)$, either $X_x = W$ or the set $X_x \setminus W$ is a connected component of $G \setminus W$. In any case, we have $\max\{|W \cup C| \mid C \in \mathcal{C}^G(S)\} = \max\{|X_x| \mid x \in V(G)\}$, which implies that $k \leqslant \mathcal{K}\mathrm{w}(G)$.

For the opposite implication, let $W := \arg\min\{\max\{|S \cup C| \mid C \in \mathcal{C}^G(S)\}\}$, i.e. W is a set of vertices of G whose combined size together with each of the components of $G \setminus W$ is at most k. Let $l = |\mathcal{C}^G(W)|$ and consider the $\mathcal{K}$-decomposition $\mathcal{D} = (K_l, \mathcal{X})$ with $\mathcal{X} = \{W \cup C \mid C \in \mathcal{C}^G(W)\}$, which is straightforward to see that is well-defined. The width of $\mathcal{D}$ implies that $\mathcal{K}\mathrm{w}(G) \leqslant k$. ■

On the other hand, letting $\mathcal{H}$ be the class of cliques doesn't seem to be the natural direction one would want to take, when trying to decompose a graph. In general, one would want to decompose a graph in a sparser graphlike structure than the graph itself, not in denser ones like the cliques, so in such cases a richer structure of gated sets than the trivial ones of the cliques might then be expected.

For example, there is a wide variety of generalisations of median graphs, whose structure is closely related to gated sets. A bipartite generalization of median graphs are the modular graphs. Most of other generalizations of median graphs connected with gated sets are non-bipartite. These include quasi-median graphs [7, 62], pseudo-median graphs [6], weakly median graphs [4], pre-median graphs [22], fiber-complemented graphs [22], weakly modular graphs [16, 24], cage-amalgamation graphs [18], absolute C-median graphs [16] and bucolic graphs [17].

We do not grow absolutely, chronologically. We grow sometimes in one dimension, and not in another; unevenly. We grow partially. We are relative. We are mature in one realm, childish in another. The past, present, and future mingle and pull us backward, forward, or fix us in the present. We are made up of layers, cells, constellations.

Anais Nin

6 Median Decompositions of Bounded Dimension

6.1 The i-Medianwidth of Graphs

For the proof of Theorem 5.12, we promised an approach that indicates which directions we can consider to develop this theory. We believe this is the case, because the proof makes apparent the fact that in order to find a median decomposition of width equal to the clique number, our underying median graph of the decomposition might need to contain hypercubes of arbitrarily large dimension as induced subgraphs or, more generally, it might need to contain Cartesian products of arbitrarily many factors.

There are many notions of dimension for median graphs (or, more generally, partial cubes) in the literature [73, 37]. The one most suitable for our purposes is the *tree dimension* of a graph G, the minimum k such that G has an isometric embedding into a Cartesian product of k trees. The graphs with finite tree dimension are just the partial cubes [73], hence every median graph has finite tree dimension. Since trees are exactly the median graphs of tree dimension 1, we are led to the following definition.

Fon an $i \geqslant 1$, an *i-median decomposition* of G is a median decomposition $\mathcal{D} = (M, \mathcal{X})$ satisfying (M1),(M2), where M is a median graph of tree dimension at most i. We denote the set of i-median decompositions of G as $\mathcal{M}_i^G$. The *i-medianwidth* $\mathrm{mw}_i(G)$ of G is the least width of any i-median decomposition of G:

$$\mathrm{mw}_i(G) := \min_{\mathcal{D} \in \mathcal{M}_i^G} \max\{|X_a| \mid a \in V(M)\}.$$

The 1-median decompositions are the tree decompositions of G, therefore $\mathrm{mw}_1(G) = \mathrm{tw}(G) + 1$. By definition, the invariants mw_i form a non-increasing sequence:

$$\mathrm{tw}(G) + 1 = \mathrm{mw}_1(G) \geqslant \mathrm{mw}_2(G) \geqslant \ldots \geqslant \mathrm{mw}(G) = \omega(G).$$

An immediate observation is that i-medianwidth is not a bounded parameter on all graphs. Furthermore, we would like that i-medianwidth and i'-medianwidth for different $i, i' \geqslant 1$ do not constitute the same parameters, so that the hierarchy above is one that makes sense. In fact, we will see that complete multipartite graphs establish this in a notably strong fashion: for $i < i'$, a class of graphs of bounded i'-medianwidth can have unbounded i-medianwidth.

For a Cartesian product of trees $H = \square_{j=1}^k T^j$, let $\pi_j : H \to T^j$ be the j-th projection of H to its j-th factor T^j. We can always embed a median graph into a Cartesian product of trees that isn't unnecessarily large.

Lemma 6.1 *Let k be the tree dimension of a median graph M. Then there is an isometric embedding φ of M into the Cartesian product of k trees $\square_{j=1}^k T^j$ such that for every $j = 1, \ldots, k$ and every $t^j \in E(T^j)$,*

$$\pi_j^{-1}(t^j) \cap \varphi(V(M)) \neq \emptyset.$$

Proof. Let $\varphi : M \to H = \square_{j=1}^k T^j$ be an isometric embedding into the Cartesian product of k trees H with $V(H)$ minimal. Then, for every $j = 1, \ldots, k$ and every leaf $l^j \in V(T^j)$ it must be $\pi_j^{-1}(l^j) \cap \varphi(V(M)) \neq \emptyset$, otherwise we can embed M into $(\square_{h \neq j} T^h) \square (T^j - l^j)$, a contradiction to the choice of H. Since $\varphi(M)$ is a connected subgraph of H, the Lemma follows. ■

We say that two Θ-classes $F_{x_1x_2}, F_{x'_1x'_2}$ of a median graph M *cross* if $W_{x_ix_{3-i}} \cap W_{x'_jx'_{3-j}} \neq \emptyset$ for any $i, j \in \{1,2\}$. Otherwise, if there is a choice $i, j \in \{1,2\}$ such that $W_{x_ix_{3-i}} \subseteq W_{x'_jx'_{3-j}}$ and $W_{x_{3-i}x_i} \subseteq W_{x'_{3-j}x'_j}$, we call $F_{x_1x_2}, F_{x'_1x'_2}$ *laminar*. Two U-sets are laminar if their adjacent Θ-classes are laminar.

For a median graph M, let Θ^M be the set of its Θ-classes, $\mathcal{U}^M$ the family of its U-sets and $\mathcal{P}^M$ the family of its peripheral sets.

A Θ-*system* of M is a set of Θ-classes of M. We call a Θ-system of M a *direction* in M if all of its members are pairwise laminar. In [8], Bandelt and Van De Vel show that a median graph is isometrically embeddable into the Cartesian product of k trees if and only if Θ^M can be 'covered' with k directions. We will extensively use the one implication of the above result, which we reformulate (together with some facts obtained from its proof) in a more convenient way for what follows. For a mapping $\psi : G \to H$ and an edge $e \in E(H)$, by $\psi^{-1}(e)$ we mean $\{uv \in E(G) \mid \psi_j(u)\psi_j(v) = e\}$.

Lemma 6.2 *[8] Let $\varphi : M \to H$ be an isometric embedding of a median graph M into the Cartesian product of k trees $H = \square_{j=1}^k T^j$ as in Lemma 6.1. Then for every $j = 1, \ldots, k$ the following are true:*

(i) for every $e^j \in E(T^j)$, $\varphi^{-1}(\pi_j^{-1}(e^j))$ is a Θ-class of M

(ii) the family $\Delta_j = \{\varphi^{-1}(\pi_j^{-1}(e^j)) \mid e^j \in E(T^j)\}$ is a direction of M

(iii) for every node t^j adjacent to an edge e^j in T^j, one of the two U-sets of M adjacent to $\varphi^{-1}(\pi_j^{-1}(e^j))$ is a subset of $\varphi^{-1}(\pi_j^{-1}(t^j))$.

We say that a set of vertices $S \subseteq V(G)$ *intersects* a subgraph H of a graph G if it contains a vertex of H. We need the following lemma:

Lemma 6.3 *[63] Let S be a set of vertices intersecting every peripheral set of a median graph M. Then $< S >= V(M)$.*

As promised, let us now show that complete $i+1$-partite graphs have unbounded i-medianwidth and thus strongly distinguish mw_{i+1} from mw_i.

Lemma 6.4 *For every $i \geqslant 1$, $\mathrm{mw}_i(K_{n_1,\ldots,n_{i+1}}) \geqslant \min_{j=1}^{i+1}\{n_j\} + 1$, while* $\mathrm{mw}_{i+1}(K_{n_1,\ldots,n_{i+1}}) = i+1$.

Proof. Let $K = K_{n_1,\dots,n_{i+1}}$. Since complete $i+1$-partite graphs are $i+1$-colourable, its clique number and a chromatic median decomposition of it establish that $\mathrm{mw}_{i+1}(K) = i+1$.

Let $(M,\mathcal{X})$ be an i-median decomposition of K. We can assume that $|V(M)| \geqslant 2$ (since K is not a clique) and that for every peripheral set U_{ab}, it must be $Z_{ab} \backslash Z_{ba} \neq \emptyset$ (otherwise we just remove the peripheral set and its bags and obtain a median decomposition of K with fewer bags). We call the vertices in $Z_{ab} \setminus Z_{ba}$ and the sets Z_{ab} for some peripheral set U_{ab}, *the peripheral vertices* and *the peripheral Z-sets* (of K), respectively, with respect to the decomposition. The *peripheral bags* of $(M,\mathcal{X})$ are the bags corresponding to nodes belonging to peripheral sets of M.

Let $k \leqslant i$ be the tree dimension of M and let $\varphi : M \to \square_{j=1}^{k} T^j$ be an isometric embedding into the Cartesian product of k trees H as in Lemma 6.1. Since the peripheral sets of H correspond to the leaves of the factors of H, it clearly follows that

$$\mathcal{P}^M = \{\varphi^{-1}(\pi_j^{-1}(l^j)) \mid j = 1,\dots,k \text{ and } l^j \text{ is a leaf of } T^j\}.$$

We partition the peripheral sets of M as inherited by the natural partition of $\mathcal{P}^H$ into the families corresponding to the leaves of each tree factor of H, namely we partition $\mathcal{P}^M$ into the sets $\mathcal{P}_1^M,\dots,\mathcal{P}_k^M$, where for $j = 1,\dots,k$,

$$\mathcal{P}_j^M = \{\varphi^{-1}(\pi_j^{-1}(l^j)) \mid l^j \text{ is a leaf of } T^j\}.$$

By Lemma 6.14, the sets of every $\mathcal{P}_j^M$ are adjacent to Θ-classes which belong to the same direction. Hence, $\mathcal{P}_j^M$ consists of pairwise laminar peripheral sets of M, so, by Lemma 5.11, two peripheral vertices of Z-sets corresponding to different peripheral sets of the same $\mathcal{P}_j^M$ are always non-adjacent in K. It follows that every transversal of peripheral vertices chosen from different Z-sets corresponding to peripheral sets from the same family $\mathcal{P}_j^M$ is an independent set in K. Recall that $|V(M)| \geqslant 2$, and therefore each $\mathcal{P}_j^M$ has at least two elements. Moreover, since K is complete multipartite, if $uv, vw \notin E(K)$, then also $uw \notin E(K)$. It follows that all the peripheral vertices belonging to Z-sets corresponding to the same $\mathcal{P}_j^M$ belong to the same part of K, for all $j = 1,\dots,k$.

But $k \leqslant i$ and thus, there is a part A_{j_0} of K that contains no peripheral vertices with respect to $(M,\mathcal{X})$. As the neighbourhood of a peripheral vertex must lie completely in the corresponding Z-set, every vertex of A_{j_0} is contained in every peripheral

Z-set. Namely, for every vertex v in A_{j_0}, $X^{-1}(v)$ intersects every peripheral set of M. By the convexity of $X^{-1}(v)$ and Lemma 6.3, v must belong to every bag of $(M,\mathcal{X})$. Hence, there are peripheral bags that contain the whole A_{j_0} plus a peripheral vertex of G, so that the width of $(M,\mathcal{X})$ is at least $|A_{j_0}|+1$. As $(M,\mathcal{X})$ was arbitrary, the lemma follows. ∎

We call two separations $(U_1,U_2),(W_1,W_2)$ of a graph G *laminar* if there is a choice $i,j \in \{1,2\}$ such that $U_i \subseteq W_j$ and $U_{3-i} \supseteq W_{3-j}$, otherwise we say they *cross*. A set of separations is called laminar if all of its members are pairwise laminar separations of G.

Lemma 6.5 *Let $(M,\mathcal{X})$ a median decomposition of G. If the Θ-classes F_{ab}, F_{cd} are laminar in M, then the corresponding separations (Y_{ab},Y_{ba}) and (Y_{cd},Y_{dc}) are laminar in G.*

Proof. Let F_{ab}, F_{cd} be laminar in M. Then, $F_{cd} \subseteq E(M[W_{ab}])$ or $F_{cd} \subseteq E(M[W_{ab}])$, otherwise F_{ab}, F_{cd} cross. W.l.o.g we can assume $F_{cd} \subseteq E(M[W_{ab}])$. Then $W_{cd} \subseteq W_{ab}$ and $W_{dc} \supseteq W_{ba}$. It follows that $Y_{cd} \subseteq Y_{ab}$ and $Y_{dc} \supseteq Y_{ba}$, therefore (Y_{ab},Y_{ba}), (Y_{cd},Y_{dc}) are laminar in G. ∎

Note that the converse is in general not true. If F_{ab}, F_{cd} cross in M, but at least one of the four sets $(Y_{ab}\setminus Y_{ba})\cap(Y_{cd}\setminus Y_{dc})$, $(Y_{ab}\setminus Y_{ba})\cap(Y_{dc}\setminus Y_{cd})$, $(Y_{ba}\setminus Y_{ab})\cap(Y_{cd}\setminus Y_{dc})$, $(Y_{ba}\setminus Y_{ab})\cap(Y_{dc}\setminus Y_{cd})$ is empty, then (Y_{ab},Y_{ba}), (Y_{cd},Y_{dc}) are still laminar in G. Moreover, one can see that the proof of Lemma 6.5 also works if one defines laminarity not only for Θ-classes, but also for general minimal cuts of the median graph M in the natural way. In that case, Lemma 6.5 holds for general laminar minimal cuts of M accordingly.

In [84], Robertson and Seymour construct the so-called *standard tree decomposition* of a graph into its *tangles* (the definition of which we omit, since we don't need it for this paper). To do that, they make use of the following lemma, also used by Carmesin *et al.* in [21] (where laminar separations stand under the name *nested separations*), which we will also need.

Lemma 6.6 *For a tree decomposition $(T,\mathcal{Z})$ of G, the set of all separations of G that correspond to the edges of T as in Lemma 2.1(iii) is laminar. Conversely, if $\{(A_i,B_i)\mid 1\leqslant i\leqslant k\}$ is a laminar set of separations of G, there is a tree decomposition $(T,\mathcal{Z})$ of G such that* *(i) for $1 \leqslant i \leqslant k$, (A_i, B_i) corresponds to a unique edge of T*

(ii) for each edge e of T, at least one of the separations of the two separations that corresponds to e equals (A_i, B_i) for some $i \in \{1, \ldots, k\}$.

We are ready to present the main result of this section, which roughly says that the i-medianwidth of a graph corresponds to the largest 'intersection' of the best choice of i many tree decompositions of the graph. In the following theorem, when we denote tree decompositions with $\mathcal{D}^j$, we mean $\mathcal{D}^j = (T^j, \mathcal{Z}^j)$.

Theorem 6.7 *For any graph G and any integer $i \geqslant 1$,*

$$\mathrm{mw}_i(G) = \min_{\mathcal{D}^1,\ldots,\mathcal{D}^i \in \mathcal{T}^G} \max\Big\{\Big|\bigcap_{j=1}^{i} Z^j_{t_j}\Big| \mid t_j \in V(T^j)\Big\}.$$

Proof. Let

$$\mu := \min_{\mathcal{D}^1,\ldots,\mathcal{D}^i \in \mathcal{T}^G} \max\Big\{\Big|\bigcap_{j=1}^{i} Z^j_{t_j}\Big| \mid t_j \in V(T^j)\Big\}.$$

For $\mathcal{D}^1, \ldots, \mathcal{D}^i \in \mathcal{T}^G$, consider the pair $(M, \mathcal{X})$, where $M = \square_{j=1}^i T^j$ and $X_{(t_1,\ldots,t_i)} = \bigcap_{j=1}^i Z^j_{t_j}$. Observe that (M1) follows directly by (T1) for $\mathcal{D}^1, \ldots, \mathcal{D}^i$. Moreover, for every $v \in V(G)$, we have

$$X^{-1}(v) = \square_{j=1}^i Z^{j^{-1}}(v),$$

which, by Lemma 5.1, is a convex subset of M, so (M2) also holds. Then $(M, \mathcal{X})$ is a valid i-median decomposition of G, therefore

$$\mathrm{mw}_i(G) \leqslant \max\Big\{\bigcap_{j=1}^{i} Z^j_{t_j} \mid t^j \in V(T^j)\Big\}.$$

Since $\mathcal{D}^1, \ldots, \mathcal{D}^i$ were arbitrary, it follows that $\mathrm{mw}_i(G) \leqslant \mu$.

For the opposite implication, consider an i-median decomposition $(M, \mathcal{X})$ of G of width $\mathrm{mw}_i(G)$. Let $k \leqslant i$ be the tree dimension of M and let $\varphi : M \to H = \square_{j=1}^k T^j$ be an isometric embedding as per Lemma 6.1. By Lemma 6.14(i),(ii), each

$$\Delta_j = \{\varphi^{-1}(\pi_j^{-1}(e^j)) \mid e^j \in E(T^j)\}$$

is a direction in M. By the definition of a direction, Lemma 6.5 and Lemma 6.6, there are tree decompositions $\mathcal{D}^j = (T^j, \mathcal{Z}^j)$ of G obtained by each Δ_j and by Lemma 6.14(iii), for each $t^j \in V(T^j)$ we have

$$Z^j_{t^j} = \bigcup_{\pi_j(\varphi(a))=t^j} X_a.$$

Observe that for each $a \in V(M)$, it is

$$\{a\} = \bigcap_{\substack{\pi_j(\varphi(a))=t^j \\ j=1,\dots,k}} \varphi^{-1}(\pi_j^{-1}(t^j)).$$

It follows that

$$X_a = \bigcap_{\substack{\pi_j(\varphi(a))=t^j \\ j=1,\dots,k}} Z^j_{t^j}.$$

Clearly, the maximal intersections of bags, one taken from each of $\mathcal{D}^1, \dots, \mathcal{D}^k$, correspond to the elements of $\mathcal{X}$. Therefore, by considering for μ the decompositions $\mathcal{D}^1, \dots, \mathcal{D}^k$ together with the trivial decomposition of G consisting of one bag being the whole $V(G)$ and repeated $i-k$ times, we obtain

$$\mu \leqslant \max\Big\{\Big|\bigcap_{j=1}^{k} Z^j_{t_j}\Big| \mid t_j \in V(T^j)\Big\} = \max\{|X_a| \mid a \in V(M)\} = \mathrm{mw}_i(G).$$

■

The combination of Theorems 5.12 and 6.7, imply the following, rather unnatural characterisation of the clique number.

Theorem 6.8 *Let $\overline{m} = |E(G^c)|$. Then*

$$\omega(G) = \min_{\mathcal{D}^1,\dots,\mathcal{D}^{\overline{m}} \in \mathcal{T}^G} \max\Big\{\Big|\bigcap_{j=1}^{i} Z^j_{t_j}\Big| \mid t_j \in V(T^j)\Big\}.$$

■

Recall that for a k-colourable graph a corresponding chromatic median decomposition is, by Lemma 5.13, a k-median decomposition of width k. This immediately implies the following.

Lemma 6.9 *For any graph G, $\mathrm{mw}_{\chi(G)} \leqslant \chi(G)$.* ■

Moreover, to obtain Theorem 5.15 we can clearly choose to restrict to Θ-smooth median decompositions where the underlying median graph is always a Cartesian product of trees. In such a case, by Θ-smoothness all the tree decompositions obtained following the directions in the Cartesian product as in Lemma 6.14 are smooth. Let $\mathcal{T}^G_{smooth}$ be the set of smooth tree decompositions of G. A direct adaptation of the proof of Theorem 6.7 combined with Theorem 5.15 provide an alternative (and seemingly unintuitive) characterization of the chromatic number with respect to smooth tree decompositions.

Theorem 6.10 *A graph G is k-chromatic if and only if*

$$\min_{\mathcal{D}^1,\ldots,\mathcal{D}^k \in \mathcal{T}^G_{smooth}} \max\left\{\left|\bigcap_{j=1}^{k} Z^j_{t_j}\right| \mid t_j \in V(T^j)\right\} = k.$$

■

6.2 The i-Latticewidth of Graphs

In this section, we turn into another notion of dimension for median graphs, the *lattice dimension*, namely the minimum k such that a graph can be isometrically embedded into a k-lattice graph. As with tree dimension, median graphs have finite lattice dimension [73, 37]. Paths are exactly the median graphs of lattice dimension equal to 1. We are led to the following definition.

For an $i \geqslant 1$, an *i-lattice decomposition* of G is a median decomposition $\mathcal{D} = (M, \mathcal{X})$ satisfying (M1),(M2), where M is a median graph of lattice dimension at most i. We denote the set of i-lattice decompositions of G as $\mathcal{L}^G_i$. The *i-latticewidth* $\mathrm{lw}_i(G)$ of G is the least width of any i-lattice decomposition of G:

$$\mathrm{lw}_i(G) := \min_{\mathcal{D} \in \mathcal{L}^G_i} \max\{|X_a| \mid a \in V(M)\}.$$

Since $\mathcal{L}^G_i \subseteq \mathcal{M}^G_i$, we have $\mathrm{mw}_i(G) \leqslant \mathrm{lw}_i(G)$. The 1-lattice decompositions are the path decompositions of G, therefore $\mathrm{lw}_1(G) = \mathrm{pw}(G) + 1$. Similarly to the case of mw_i, the parameters lw_i form a hierarchy starting from pathwidth and converging to the clique number:

$$\mathrm{pw}(G)+1=\mathrm{lw}_1(G)\geqslant \mathrm{lw}_2(G)\geqslant \ldots \geqslant \mathrm{lw}_\infty(G)=\mathrm{mw}(G)=\omega(G).$$

An immediate corollary of Lemma 5.13 is the following bound.

Lemma 6.11 *For any graph G, $\mathrm{lw}_{\chi(G)}\leqslant\chi(G)$.* ■

The results and proofs in the rest of this section are in the spirit of Section 6.1. Recall that by Lemma 6.4, complete $i+1$-partite graphs strongly distinguish mw_{i+1} from mw_i. Consequently, this fact directly translates to the case of latticewidth parameters: complete $i+1$-partite graphs have unbounded i-latticewidth, but bounded $i+1$-latticewidth.

Lemma 6.12 *For every $i\geqslant 1$, $\mathrm{lw}_i(K_{n_1,\ldots,n_{i+1}})\geqslant \min_{j=1}^{i+1}\{n_j\}+1$, while $\mathrm{lw}_{i+1}(K_{n_1,\ldots,n_{i+1}})=i+1$.*

Proof. Let $K=K_{n_1,\ldots,n_{i+1}}$. Since $\chi(K)=\omega(K)=i+1$, Lemmas 5.9 and 6.11 show that $\mathrm{lw}_{i+1}(K)=i+1$.

On the other hand, recall that $\mathrm{lw}_i(K)\geqslant\mathrm{mw}_i(K)$. Lemma 6.4 completes the proof. ■

The lattice dimension of partial cubes has been studied in [37] and [23], but we shall only need simpler versions of some of the machinery used there.

For a k-lattice $L=\square_{j=1}^k P^j$, let $\pi_j : L\to P^j$ be the j-th projection of L to its j-th factor P^j. Let us prove the analogue of Lemma 6.1, i.e. that we can always embed a median graph into a lattice in an irredundant way.

Lemma 6.13 *Let k be the lattice dimension of a median graph M. Then there is an isometric embedding φ of M into a k-lattice $\square_{j=1}^k P^j$ such that for every $j=1,\ldots,k$ and every $u_j\in V(P^j)$,*

$$\pi_j^{-1}(u_j)\cap\varphi(V(M))\neq\emptyset.$$

Proof. Let $\varphi : M \to L = \square_{j=1}^{k} P^j$ be an isometric embedding into a k-lattice L with $V(H)$ minimal. Then, for every $j = 1, \dots, k$ and each of the two ends $l_j \in V(P^j)$ it must be $\pi_j^{-1}(l_j) \cap \varphi(V(M)) \neq \emptyset$, otherwise we can embed M into $(\square_{h \neq j} P^h) \square (P^j - l_j)$, a contradiction to the choice of L. The Lemma follows by the fact that $\varphi(M)$ is a connected subgraph of L. ■

Recall that a Θ-*system* of M is a set of Θ-classes of it. We call a Θ-system $\mathcal{F}$ of M a *strong direction* in M if all of its members are pairwise laminar and for every $F_{a_1b_1}, F_{a_2b_2}, F_{a_3b_3} \in \mathcal{F}$, if $W_{a_1b_1} \subseteq W_{a_2b_2}$ and $W_{a_1b_1} \subseteq W_{a_3b_3}$, then $W_{a_2b_2} \subseteq W_{a_3b_3}$ or $W_{a_3b_3} \subseteq W_{a_2b_2}$ (or in other words, if there is a $\subseteq$-chain containing a W-set from every pair of complementary W-sets corresponding to the Θ-classes of $\mathcal{F}$, with their complementary W-sets forming a $\supseteq$-chain). For a mapping $\psi : G \to H$ and an edge $e \in E(H)$, by $\psi^{-1}(e)$ we mean $\{uv \in E(G) \mid \psi_j(u)\psi_j(v) = e\}$. Notice that for a k-lattice $H = \square_{j=1}^{k} P^j$, the family $\{\pi_j^{-1}(e_j) \mid e_j \in E(P^j)\}$ is a strong direction in H. When embedded into a lattice, a median graph inherits in a natural way the lattice's strong directions.

Lemma 6.14 *Let $\varphi : M \to L$ be an isometric embedding of a median graph M into a k-lattice $L = \square_{j=1}^{k} P^j$ as in Lemma 6.13. Then for every $j = 1, \dots, k$ the following are true:*

(i) for every $e_j \in E(P^j)$, $\varphi^{-1}(\pi_j^{-1}(e_j))$ is a Θ-class of M

(ii) the family $\Sigma_j = \{\varphi^{-1}(\pi_j^{-1}(e_j)) \mid e_j \in E(P^j)\}$ is a strong direction of M $\varphi^{-1}(\pi_j^{-1}(e_j))$ is a subset of $\varphi^{-1}(\pi_j^{-1}(u_j))$.

Proof. (i) Let $e_j = u_j v_j$. Since $\pi_j^{-1}(u_j), \pi_j^{-1}(v_j)$ are complementary W-sets of L and φ is an isometry, we have that $\varphi^{-1}(\pi_j^{-1}(u_j)), \varphi^{-1}(\pi_j^{-1}(v_j))$ are complementary W-sets of M. Since $\varphi^{-1}(\pi_j^{-1}(e_j))$ is the set of edges between them, they constitute a Θ-class of M.

(ii) Follows from (i) and the fact that $\{\pi_j^{-1}(e_j) \mid e_j \in E(P^j)\}$ is a strong direction in L.

■

We are ready to present the analogue of Theorem 6.7, which roughly says that the i-latticewidth of a graph corresponds to the largest 'intersection' of the best choice of i path decompositions of the graph. More specifically, in the following theorem let us denote path decompositions $(P^j, \mathcal{Z}^j)$ with $\mathcal{D}^j = (P^j, \mathcal{Z}^j)$.

Theorem 6.15 *For any graph G and any integer $i \geqslant 1$,*

$$\mathrm{lw}_i(G) = \min_{\mathcal{D}^1,\ldots,\mathcal{D}^i \in \mathcal{P}^G} \max\Big\{\Big|\bigcap_{j=1}^{i} Z^j_{u_j}\Big| \mid u_j \in V(P^j)\Big\}.$$

Proof. Let

$$\lambda := \min_{\mathcal{D}^1,\ldots,\mathcal{D}^i \in \mathcal{P}^G} \max\Big\{\Big|\bigcap_{j=1}^{i} Z^j_{u_j}\Big| \mid u_j \in V(P^j)\Big\}.$$

For $\mathcal{D}^1,\ldots,\mathcal{D}^i \in \mathcal{P}^G$, consider the pair $(L, \mathcal{X})$, where $L = \square_{j=1}^{i} P^j$ and $X_{(u_1,\ldots,u_i)} = \bigcap_{j=1}^{i} Z^j_{u_j}$. Note that (T1) for $\mathcal{D}^1,\ldots,\mathcal{D}^i$ implies (M1) for $(L, \mathcal{X})$. Clearly, for every $v \in V(G)$, we have

$$X^{-1}(v) = \square_{j=1}^{i} Z^{j^{-1}}(v),$$

so by Lemma 5.1, (M2) also holds. Then $(L, \mathcal{X})$ is a valid i-lattice decomposition of G, therefore

$$\mathrm{lw}_i(G) \leqslant \max\Big\{\Big|\bigcap_{j=1}^{i} Z^j_{u_j}\Big| \mid u_j \in V(P^j)\Big\}.$$

Since $\mathcal{D}^1,\ldots,\mathcal{D}^i$ were arbitrary, it follows that $\mathrm{lw}_i(G) \leqslant \lambda$.

For the opposite implication, consider an i-lattice decomposition $(M, \mathcal{X})$ of G of width $\mathrm{lw}_i(G)$. Let $k \leqslant i$ be the lattice dimension of M and let $\varphi : M \to L = \square_{j=1}^{k} P^j$ be an isometric embedding as per Lemma 6.13. By Lemma 6.14 (i),(ii), each

$$\Sigma_j = \{\varphi^{-1}(\pi_j^{-1}(e_j)) \mid e_j \in E(P^j)\}$$

is a strong direction in M. By the definition of a strong direction and Lemma 6.5, there are path decompositions $\mathcal{D}^j = (P^j, \mathcal{Z}^j)$ of G obtained by each Σ_j where for each $u_j \in V(P^j)$ we have

$$Z^j_{u_j} = \bigcup_{\pi_j(\varphi(a)) = u_j} X_a.$$

Every vertex of L is exactly the intersection of all the sublattices of L of codimension 1 that contain it. In other words, for each $a \in V(M)$, we have

$$\{a\} = \bigcap_{\substack{\pi_j(\varphi(a))=u_j \\ j=1,\dots,k}} \varphi^{-1}(\pi_j^{-1}(u_j)).$$

It follows that

$$X_a = \bigcap_{\substack{\pi_j(\varphi(a))=u_j \\ j=1,\dots,k}} Z^j_{u_j}.$$

The trasversals from $\mathcal{Z}^1,\dots,\mathcal{Z}^i$ that can achieve maximal size for the intersection of its elements, clearly correspond to the elements of $\mathcal{X}$. For λ, consider the decompositions $\mathcal{D}^1,\dots,\mathcal{D}^k$ together with the trivial decomposition of G consisting of one bag being the whole $V(G)$ and repeated $i-k$ times. Then

$$\lambda \leqslant \max\left\{\left|\bigcap_{j=1}^{k} Z^j_{u_j}\right| \mid u_j \in V(P^j)\right\} = \max\{|X_a| \mid a \in V(M)\} = \mathrm{lw}_i(G).$$

■

Naturally, the analogue of Theorem 6.8 also immediately follows.

Theorem 6.16 *Let* $\overline{m} = |E(G^c)|$. *Then*

$$\omega(G) = \min_{\mathcal{D}^1,\dots,\mathcal{D}^{\overline{m}} \in \mathcal{P}^G} \max\left\{\left|\bigcap_{j=1}^{i} Z^j_{t_j}\right| \mid t_j \in V(T^j)\right\}.$$

■

In every real man a child is hidden that wants to play.

Friedrich Nietzsche

7

Medianwidth Parameters and Games

We already mentioned in Chapter 1 that treewidth and pathwidth are known to be characterised by the classical Cops and Robber game. In light of the interplay of i-medianwith with intersections of bags of tree decompositions and that of i-latticewidth with intersections of bags of path decompositions of the graph from Chapter 6, we introduce the *i-Cops and Robber game* which turns out to be closely connected to the respective medianwidth and latticewidth parameters. The robber player now plays against i cop players which need to cooperate in order to capture the robber with the least 'cooperation' possible (to be explained later). Every cop player has at his disposal a team of $|V(G)|$ cops, each of which can stand on a vertex or move with a helicopter in the air. Cop teams are 'undercover' though, meaning that they are invisible to the other cop teams.

But the robber is very powerful: she can see all i cop teams and how they move at all times and additionally, the only way that the robber can be fully caught is by having a cop of every team on the vertex currently occupied by the robber. Moreover, she has a way to restrict their movement by selecting each time a cop team that she allows to move, while forcing all the other cop teams to remain still during said move. In other words, the cop teams move one at a time, with any order the robber prefers.

On the other hand, each cop team can also restrict the robber. If one cop player manages to somehow catch the robber (notice that the robber would still not be completely captured), then the cops of that team lock down on the vertices they currently occupy and they trap the robber in the following sense: from then on, she is allowed to only move to vertices occupied by said cop team and disabled from choosing that particular cop team to move again for the rest of the game. Moreover, when the robber chooses a cop team to move (if allowed), during the time the respective cop player moves some of his cops with helicopters to some other vertices, the robber can then move through a path of G to any other vertex, as long as there are no cops of a team that has not trapped her yet standing on the vertices of the path.

The *cooperation* of the cop players is the maximum number of vertices simultaneously occupied by a cop of every team at any point of the game. In case the cop players have a winning strategy to always catch the robber with cooperation at most k, we say that *i cop players can search the graph with cooperation at most k*. Moreover, we say that the cop players can capture the robber *monotonely* if with each of their moves the robber space of available escape options always shrinks.

As with the classical game, we study two variations of this game as well: one where the robber is *visible* and one where the robber is *invisible* to each cop player. When the robber is visible (respectively invisible), we say that the cop players search the graph *with vision* (respectively *without vision*).

Note that for $i = 1$ the game described above becomes the classical Cops and Robber game where only one cop player searches the graph, because in that case the cooperation degenerates to just being the size of the only cop team. Recall that 1-medianwidth corresponds to treewidth and 1-latticewidth corresponds to pathwidth, which are both characterised by the classical Cops and Robber game depending on the visibility of the robber. In this chapter we extend this connection between Cops and Robber games and width parameters, and we show that i cop players *with vision* can *monotonely* search a graph G with cooperation at most k if and only if $\mathrm{mw}_i(G) \leqslant k$. Similarly, we show that i cop players *without vision* can *monotonely* search a graph G with cooperation at most k if and only if $\mathrm{lw}_i(G) \leqslant k$. To our knowledge, this is also the first instance of a search game played between a single fugitive player against a team of many search players connected to a width parameter of graphs.

In the following sections, we describe in detail the game we sketched above, where the robber player plays against i cop players. As in the case of treewidth and pathwidth, we will examine two variations of the game: one where the robber is *visible* to each cop player and one where the robber is *invisible* to them.

7.1 i Cop Players vs a Visible Robber

Let us precisely describe the *'i-Cops and visible Robber'* game on a graph G with *cooperation at most k*, played by the i *Cop players* and a visible *Robber player*. Let $X \subseteq V(G)$. An *X-flap* is the vertex set of a component of $G \setminus X$.

A *position* of the game is an i-tuple of pairs $((Z^1,R^1),\ldots,(Z^i,R^i))$ where $Z^1,\ldots,Z^i,R^1,\ldots,R^i \subseteq V(G)$. A *move* is a triple (j,Z,R), where $j \in \{1,\ldots,i\}$ and $Z,R \subseteq V(G)$. The initial position of the game is always of the form $((Z_0^1,R_0^1),\ldots,(Z_0^i,R_0^i)) = ((\emptyset,C),\ldots,(\emptyset,C))$, where C is a connected component of G. A *play* is a sequence of moves and their corresponding positions. The l-th *round* of a play starts from a position $((Z_{l-1}^1,R_{l-1}^1),\ldots,(Z_{l-1}^i,R_{l-1}^i))$. The players then make a move and a new position $((Z_l^1,R_l^1),\ldots,(Z_l^i,R_l^i))$ is obtained according to the following steps:

(i) the *robber player* chooses a cop team $j \in \{1,\ldots,i\}$ with $R_{l-1}^j \not\subseteq Z_{l-1}^j$, to be the next to move (if $R_{l-1}^j \subseteq Z_{l-1}^j$, the j-th team can't be chosen by the robber),

(ii) the *j-th cop player* then chooses a new vertex set Z_l^j for the j-th cop team, based only on knowledge of sets $Z = Z_{l-1}^j$ and $R = R_{l-1}^j$, (each cop player can't see though in which round the overall game is at any point, how many times they were chosen or what the other cop players have played so far, but they remember what was their last choice Z and the choice R of the robber in the same round),

(iii) if $R_{l-1}^j \subseteq Z_l^j$, the *robber player* keeps the same $R_l^j = R_{l-1}^j$, otherwise the robber chooses a Z_l^j-flap R_l^j such that R_{l-1}^j, R_l^j are subsets of a common $(Z_{l-1}^j \cap Z_l^j)$-flap and such that R_l^j *intersects* $\bigcap_{j' \neq j} R_{l-1}^{j'}$,

(iv) the move (j,Z_l^j,R_l^j) *updates* the position $((Z_{l-1}^1,R_{l-1}^1),\ldots,(Z_{l-1}^i,R_{l-1}^i))$ of the game into $((Z_l^1,R_l^1),\ldots,(Z_l^i,R_l^i))$ by replacing the j-th pair (Z_{l-1}^j,R_{l-1}^j) with (Z_l^j,R_l^j) and leaving the rest of the position as is, namely for every $j' \neq j$, $(Z_l^{j'},R_l^{j'}) = (Z_{l-1}^{j'},R_{l-1}^{j'})$.

Observe that by (iii), the set $\bigcap_{j=1}^{i} R_l^j$, which contains the vertices that the robber can occupy, is non-empty for every round l of the game. The play ends when it arrives at a position $((Z^1,R^1),\dots,(Z^i,R^i))$ with $|\bigcap_{j=1}^{i} Z^j| > k$, in which case the robber wins, or when it arrives at a position with $R^j \subseteq Z^j$ for every $j = 1,\dots,i$ and $|\bigcap_{j=1}^{i} Z^j| \leqslant k$, in which case the cop players win. Otherwise, when the game never ends, the robber player also wins.

If the cop players have a winning strategy in the above game, we say that 'i teams of cops *with vision* can search the graph with cooperation at most k'. If the cop players can always win in such a way that $R_l^j \subseteq R_{l-1}^j$ for every $j = 1,\dots,i$ and every round l, we say that 'i teams of cops *with vision* can *monotonely* search the graph with cooperation at most k'.

Note that for $i = 1$, the game clearly becomes the classical 'Cops and Robber' game with one cop player and a visible robber, which characterises treewidth (recall that treewidth is the 1-medianwidth). Analogously, monotone winning strategies for the i cop players characterise i-medianwidth.

Theorem 7.1 *A graph G can be monotonely searched with cooperation at most k by i teams of cops* with vision *if and only if* $\mathrm{mw}_i(G) \leqslant k$.

Proof. Let $\mathrm{mw}_i(G) \leqslant k$. By Theorem 6.7, there are tree decompositions $(T^1,\mathcal{Z}^1),\dots,(T^i,\mathcal{Z}^i) \in \mathcal{T}^G$ with

$$\max\{|\bigcap_{j=1}^{i} Z_{t_j}^j| \mid t_j \in V(T^j)\} \leqslant k.$$

For each T^j, choose an arbitrary root r_j and consider the respective partial order $\trianglelefteq^j$ obtained by the rooted tree (T^j,r_j) with r_j its $\trianglelefteq^j$-minimal element. For every $t_j \in V(T^j)$, let

$$V_{t_j}^j := \bigcup_{\substack{s_j \in V(T^j) \\ t_j \trianglelefteq^j s_j}} Z_{s_j}^j.$$

Then the cop players have the following winning strategy, which is easily seen to be well-defined:

- For every $j \in \{1,\dots,i\}$, the j-th player always chooses bags of $(T^j,\mathcal{Z}^j)$ when he is selected to move.

- The first time the j-th cop player is chosen by the robber to move, he chooses $Z^j_{r_j}$.
- Suppose that at the last time the j-th cop moved, he chose $Z^j_{t_j}$ and the robber chose the $Z^j_{t_j}$-flap that is a subset of $V^j_{s_j}$ for a unique child s_j of t_j in T^j. Then, the next time he is selected by the robber to move, he chooses $Z^j_{s_j}$.

Clearly, by the properties of tree decompositions, the above strategy is monotone. The strategy is winning, because for every position $((Z^1,R^1),\ldots,(Z^i,R^i))$ of a play we have $Z^j \subseteq \mathcal{Z}^j$ and hence, $|\bigcap_{j=1}^{i} Z^j| \leqslant k$.

Conversely, suppose that the cop players have a monotone winning strategy σ. Since the cop players are invisible to each other, we can view σ as $\sigma = (\sigma^1,\ldots,\sigma^i)$, where projection σ^j of σ corresponds to the individual strategy of the j-th player. Each σ^j can be represented by a directed rooted tree $(\vec{T}^j, r_j)$ as follows: r_j is labeled with $Z_{r_j} = \emptyset$ and its outgoing arcs are labeled with the vertex sets of the connected components of G. The rest of the nodes t_j are labeled with $Z^j_{t_j} \subseteq V(G)$ corresponding to subsets of $V(G)$ occupied by cops of the j-th cop player and the rest of the arcs (t_j,s_j) are labeled with $R^j_{(t_j,s_j)} \subseteq V(G)$ corresponding to possible (legal) moves of the robber player. That is, for every pair of arcs $(t_j,s_j),(s_j,u_j)$ of $\vec{T}^j$, we have that $R^j_{(t_j,s_j)}$ is a $Z^j_{t_j}$-flap, $R^j_{(s_j,u_j)}$ is a $Z^j_{s_j}$-flap, and $R^j_{(s_j,u_j)}, R^j_{(t_j,s_j)}$ are subsets of a common $(Z^j_{t_j} \cap Z^j_{s_j})$-flap.

Moreover, since σ is monotone, for every pair of arcs $(t_j,s_j),(s_j,u_j)$ of $\vec{T}^j$, it must hold that $R^j_{(s_j,u_j)} \subseteq R^j_{(t_j,s_j)}$. Hence, for every arc (t_j,s_j), $R^j_{(t_j,s_j)}$ is a $(Z^j_{t_j} \cap Z^j_{s_j})$-flap, too (otherwise the robber can break the monotonicity condition). In other words, for every arc (t_j,s_j), $(Z^j_{t_j} \cap Z^j_{s_j})$ separates $R^j_{(t_j,s_j)} \cup Z^j_{s_j}$ from $V(G) \setminus R^j_{(t_j,s_j)}$. It is easy to see that the satisfaction of the statement of Lemma 2.1 (iii), combined with the fact that every vertex of the graph is in a $Z^j_{t_j}$ set, is a sufficient condition for the pair $(T^j, \mathcal{Z}^j = (Z^j_{t_j})_{t_j \in V(T^j)})$ to be a tree decomposition of G, where T^j is the underlying undirected tree of $\vec{T}^j$ (the fact that $Z^j_{r_j} = \emptyset$ does not hurt (T1),(T2), and this can even be easily avoided by contracting r_j to one of its children and removing $Z^j_{r_j}$ from $\mathcal{Z}^j$).

Observe that by selecting appropriately the order in which the cop players play and her respective choice R of each round, the robber can force all positions $((Z^1,R^1),\ldots,(Z^i,R^i))$, where $(Z^1,\ldots,Z^i)$ can be any transversal from the families

$\mathcal{Z}^1,\ldots,\mathcal{Z}^i$ and satisfying $\bigcap_{j=1}^{i} Z^j \neq \emptyset$, if the cop players play according to σ. Since σ is a winning strategy for the cop players, we have that $\max\{|\bigcap_{j=1}^{i} Z^j_{t_j}| \mid t_j \in V(T^j)\} \leqslant k$ and the proof is complete by Theorem 6.7. ∎

7.2 *i* Cop Players vs an Invisible Robber

To describe the *'i-Cops and invisible Robber'* game, where the robber is invisible to the cop players, we will need to state it in a slightly alternative fashion. Positions, moves, rounds and plays are defined as in Section 7.1. The initial position of the game is always $((Z_0^1, R_0^1),\ldots,(Z_0^i, R_0^i)) = ((\emptyset, V(G)),\ldots,(\emptyset, V(G))$. As in the case of the visible robber, the l-th *round* of a play starts from a position $((Z_{l-1}^1, R_{l-1}^1),\ldots,(Z_{l-1}^i, R_{l-1}^i))$. Compared to steps (i)-(iv) from Section 7.1, the round plays as follows:

(a) same as (i)

(b) same as (ii)

(c) if $R_{l-1}^j \subseteq Z_l^j$, the *robber player* keeps the same $R_l^j = R_{l-1}^j$, otherwise the robber is automatically assigned with R_l^j being the set of all vertices connected to R_{l-1}^j with a path in the graph $G \setminus (Z_{l-1}^j \cap Z_l^j)$

(d) same as (iv).

The winning conditions of the game are exactly the same as the ones of Section 7.1.

If the cop players have a winning strategy, we say that 'i teams of cops *without vision* can search the graph with cooperation at most k'. If the cop players can, in addition, always win in such a way that $R_{l+1}^j \subseteq R_l^j$ for every $j = 1,\ldots,i$ and every round l, we say that 'i teams of cops *without vision* can *monotonely* search the graph with cooperation at most k'.

Pathwidth corresponds to the 1-latticewidth and for $i = 1$, the game becomes the classical 'Cops and Robber' game with one cop player and an invisible robber, that characterises pathwidth. Similarly, monotone winning strategies for the i cop players characterise i-latticewidth.

Theorem 7.2 *A graph G can be monotonely searched with cooperation at most k by i teams of cops* without vision *if and only if* $\mathrm{lw}_i(G) \leqslant k$.

Proof. The proof is a direct adaptation of the proof of Theorem 7.1. We still briefly sketch it for the sake of completeness. Let $\mathrm{lw}_i(G) \leqslant k$. By Theorem 6.15, there are path decompositions $(P^1, \mathcal{Z}^1), \ldots, (P^i, \mathcal{Z}^i)$ with

$$\max\{|\bigcap_{j=1}^{i} Z^j_{u_j}| \mid u_j \in V(P^j)\} \leqslant k.$$

For $j = 1, \ldots, i$, let $P^j = (u^j_1, \ldots, u^j_{n_j})$. If the j-th cop player plays successively $Z^j_{u^j_1}, \ldots, Z_{u^j_{n_j}}$ each time he is chosen by the robber, then the i cop players win monotonely.

Conversely, suppose that the cop players have a monotone winning strategy. Since the cop players are invisible to each other, the overall strategy of the cop players comprises individual strategies of each cop player. For $j = 1, \ldots, i$, this individual strategy of the j-th cop player can be viewed as a sequence $(Z^j_1, \ldots, Z^j_{n_j})$, which he will successively follow each time he is chosen to play again. Let R^j_m be the set assigned to the robber player in step (c), after the j-th cop player has chosen Z^j_m in step (b). By the definition of monotonicity, we have $R^j_m \subseteq R^j_{m-1}$. By (c), this implies that $Z^j_{m-1} \cap Z^j_m$ separates $R^j_m \cup Z^j_m$ from $V(G) \setminus R^j_m$.

By letting $P^j = (u^j_1, \ldots, u^j_{n_j})$ and $Z^j_{u^j_m} := Z^j_m$, we can then easily see that $(P, \mathcal{Z}^j = (Z^j_{u^j_m})_{u^j_m \in P^j}) \in \mathcal{P}^G$. Even though the robber can choose any order with which the cop players will play and force any position $((Z^1, R^1), \ldots, (Z^i, R^i))$ with $(Z^1, \ldots, Z^i)$ an arbitrary transversal of $(\mathcal{Z}^1, \ldots, \mathcal{Z}^i)$ satisfying $\bigcap_{j=1}^{i} Z^j \neq \emptyset$, the cop players still always win with cooperation at most k. By Theorem 6.15, the path decompositions $(P^1, \mathcal{Z}^1), \ldots, (P^i, \mathcal{Z}^i)$ show that $\mathrm{lw}_i(G) \leqslant k$.

∎

Talking nonsense is the sole privilege mankind possesses over the other organisms. It's by talking nonsense that one gets to the truth! I talk nonsense, therefore I'm human.

Fyodor Dostoyevsky

8 Concluding Remarks and Further Research

8.1 Comparing the Two Parts

Maybe surprisingly, the seemingly incomparable main notions of the two parts of this dissertation are not orthogonal to each other. Indeed, let us highlight even more the fact that the 'high-dimensional' character of medianwidth parameters makes them a structural tool probably more suitable for classifying dense graph classes.

We say that a graph class $\mathcal{G}$ has *bounded medianwidth* if there is an $i_0 \geqslant 1$ such that $\mathcal{G}$ has bounded i_0-medianwidth. Clearly, by Lemma 6.9, all bipartite graphs have 2-medianwidth at most 2, so the class of complete bipartite graphs has bounded medianwidth, but it is not even nowhere dense. It turns out that for a graph class $\mathcal{G}$, the property of bounded expansion is strictly stronger than the property of bounded medianwidth. This follows directly by Lemma 6.9 and the fact that all graphs in a bounded expansion class have bounded chromatic number.

The observation above actually allows us to prove something stronger. We say that a graph class $\mathcal{G}$ has *all-depth bounded medianwidth* if for every $i \geqslant 1$, the class

$\mathcal{G}\widetilde{\nabla} i$ has bounded medianwidth: there are functions $f, g : \mathcal{G} \to \mathbb{N}$ such that for every $i \geqslant 1$ and every $H \in \mathcal{G}\widetilde{\nabla} i$, it is $\mathrm{mw}_{g(i)}(H) \leqslant f(i)$.

Lemma 8.1 *Let $\mathcal{G}$ be a graph class of bounded expansion. Then $\mathcal{G}$ has all-depth bounded medianwidth .*

Proof. Since $\mathcal{G}$ has bounded expansion, for every $i \geqslant 1$ there is a $c(i)$ such that $\widetilde{\nabla}_i(\mathcal{G}) \leqslant c(i)$. This means that every graph $H \in \mathcal{G}\widetilde{\nabla} i$ has average degree at most $2c(i)$ and hence, $\chi(H) \leqslant \lceil 2c(i) \rceil$. Then, by Lemma 6.9 we have

$$\mathrm{mw}_{\lceil 2c(i) \rceil}(H) \leqslant \mathrm{mw}_{\chi(H)}(H) \leqslant \chi(H) \leqslant \lceil 2c(i) \rceil.$$

■

Consider again the class $\mathcal{C}$ of all bipartite graphs, which served as an example separating bounded medianwidth from bounded expansion. Now, already $\mathcal{C}\widetilde{\nabla} 1$ is the class of all graphs, which of course has unbounded medianwidth. To this end, we do not know if the property of all-depth bounded medianwidth is strictly weaker than the property of bounded expansion for a graph class. We believe this an interesting question in an attempt to further compare and understand the interplay between the two notions.

8.2 Neighbourghood Complexity and Nowhere Dense Graph Classes

One should note that in Theorem 4.7 the derived bound is *exponential* in the measure wcol_{2r}. Consequently, we cannot use neighbourhood complexity to characterise nowhere dense classes: recall that in these classes, the quantity wcol_r can only be bounded by $\mathcal{O}(|G|^{o(1)})$ which only results in superpolynomial bounds for ν_r.

This constitutes an unusual phenomenon in the following sense: so far, every known characterisation of bounded expansion translated to a direct characterisation of nowhere denseness, but this has not yet been the case for neighbourhood complexity. It would be remarkable if one could only characterise the property of bounded expansion through neighbourhood complexity and not that of nowhere denseness. So far, it is only known that ν_1 is bounded by $\mathcal{O}(|G|^{o(1)})$ in nowhere dense classes [41]. We pose as an interesting open question whether this holds true for ν_r for all r, or whether nowhere dense classes can indeed have a neighbourhood complexity that cannot be bounded by such a function.

There are numerous directions worth looking into that stem from the development of the theory of part two. We highlight some of the ones that we consider the most important.

8.3 Brambles

In a graph G, we say that two subsets of $V(G)$ *touch* if they have a vertex in common or there is an edge in G between them. A *bramble* $\mathcal{B}$ is a set of mutually touching connected vertex sets of G. A subset of $V(G)$ is said to *cover* $\mathcal{B}$ if it meets every element of $\mathcal{B}$. The least number of vertices that cover a bramble is the *order* of that bramble. We denote the set of all brambles of G with $\mathscr{B}^G$.

Brambles are canonical obstructions to small treewidth, as shown by the following Theorem of [87], sometimes also called the *treewidth duality Theorem*.

Theorem 8.2 (Seymour & Thomas) *Let $k \geqslant 0$ be an integer. A graph has treewidth at least k if and only if it contains a bramble of order strictly greater than k.*

Inspired by Theorem 6.7 and its proof, one might think that brambles with large minimum intersections of covers are the corresponding obstructions to i-medianwidth. Using Theorem 6.7, it is not difficult to prove that the quantity

$$\max_{\mathcal{B}^1,\ldots,\mathcal{B}^i \in \mathscr{B}^G} \min\{|\bigcap_{j=1}^{i} X^j| \mid X^j \text{ covers } \mathcal{B}^j\}$$

is a lower bound for $\mathrm{mw}_i(G)$.

However, it is unknown to us if $\mathrm{mw}_i(G)$ can be upper-bounded by such a quantity and thus, we do not know if this is the correct obstructing notion characterizing large i-medianwidth. We believe this is an important question towards a better comprehension of this theory.

8.4 Graph Relations and Orderings

As we have already stressed in this dissertation, tree decompositions were the most fundamental structural tool throughout the whole body of the Graph Minor Theory, but tree notions were also very relevant in terms of Sparsity as in [71]. Let us recall that (one of) the ultimate goals of Robertson and Seymour was to prove Theorem 2.4, which also involved as an initial step that graphs of bounded treewidth are well-quasi-ordered. One of the main reasons that tree decompositions are really appropriate as a structural tool towards studying graph minors is that treewidth itself is a minor-closed parameter.

Lovasz concludes his survey on Graph Minors [60] that the Excluded Minor Theorem can be interpreted as follows: graphs excluding a fixed minor are essentially 2-dimensional and vice-versa. He then asks 'whether there is a similar description of '3-dimensional' graphs and whether there is a general notion of 'minor' that would correspond to graphs whose structure we feel is 3-dimensional'.

Based on the above intuition, let us note that in the Excluded Graph Minor Theorem, tree decompositions are the '1-dimensional' decompositions that are used to decompose the graph into its '2-dimensional' parts. Since median decompositions can be seen as generalisations of tree decompositions of multidimensional character, we feel that the above questions as stated by Lovasz are remotivated in the following more specific way: are there 'high-dimensional' generalisations of the minor-relation that make the respective i-medianwidth parameters 'minor-closed'? The existence of such a notion, which unfortunately still eludes us, would spark a large variety of direct analogues from the multitude of questions stemming from Graph Minor Theory and Graph Sparsity, one of them being (well-quasi-)orderings of graphs, along with all the natural extensions of methods so successfully employed for treewidth and tree decompositions throughout the literature.

8.5 Towards the Chromatic Number

A median decomposition $(M, \mathcal{X})$ is called *weakly-Θ-smooth* if for every Θ-class F_{ab} of M, we have that both $Z_{ab} \setminus Z_{ba}$ and $Z_{ba} \setminus Z_{ab}$ are non-empty, and whenever $|Z_{ab}| \leqslant |Z_{ba}|$, there is an injective function $s_{ab} : Z_{ab} \setminus Z_{ba} \to Z_{ba} \setminus Z_{ab}$ such that:

- $X^{-1}(v) \cup X^{-1}(s_{ab}(v))$ is convex in M,
- for every $xy \in F_{ab}$ with $x \in U_{ab}$ and $y \in U_{ba}$,

$$v \in X_x \text{ if and only if } s_{ab}(v) \in X_y.$$

As is easily seen, tree decompositions are always weakly-Θ-smooth. Moreover, every Θ-smooth median decomposition can be seen to be weakly-Θ-smooth, by defining s_{ab} to send the single element of $Z_{ab} \setminus Z_{ba}$ to the single element of $Z_{ba} \setminus Z_{ab}$.

Consider the following variation of a third axiom in the definition of median decompositions:

(M3') $\mathcal{D}$ is weakly-Θ-smooth.

Let the *weakly-smooth-medianwidth* ws-mw(G) of G to be the minimum width over all median decompositions of G that additionally satisfy (M3'). A direct adaptation of the proof of Lemma 5.14 shows that it is still the case that ws-mw$(G) = \chi(G)$. Nevertheless, even though *weak* Θ*-smoothness* is indeed a weaker notion than Θ-smoothness, it does not seem to enhance substantially more our understanding of the chromatic number compared to Θ-smoothness.

In the end, the third axiom ensures the following: if you add edges to a graph to make every bag of a median decomposition of it a clique, the new graph will be *perfect*, one whose clique number and chromatic number coincide. We believe though, that if there is a substantially better notion than smoothness that captures this intuition, it will be a much less artificial one than weak Θ-smoothness.

8.6 Non-Monotonicity in the i-Cops and Robber Game

One of the most notable facts about the classical Cops and Robber game is that the cop player can search a graph with k cops if and only if he can search it with k cops *monotonely*. The equivalence in the strength of non-monotone and monotone strategies in the classical Cops and Robber game is obtained either from knowledge of the obstructing notion for the respective width parameter, such as brambles being the obstructions for small treewidth [87], or by arguments making use of the *submodularity* of an appropriate connectivity function (whose definition we omit), such as the size of the *border* θX, the set of vertices in a vertex set X adjacent to the complement of X (for example, see [10, 11]).

It is a fundamental question to see if non-monotone winning strategies for the cop players in the i-Cops and Robber game are stronger than monotone ones, unlike the case for $i = 1$. However, as already argued in Section 8.3, we have no access yet to obstructing notions of i-medianwith of i-latticewidth, whose presence might provide certificates for winning strategies for the robber as in the case of treewidth and pathwidth. To this end, we also don't know if the notion of submodularity can be properly adjusted to provide similar results for any $i > 1$ in the fashion it does for $i = 1$.

8.7 Algorithmic Considerations

Even though treewidth is known to have a wide variety of algorithmic applications using dynamic programming techniques, this can in general not be the case for i-medianwidth when $i \geqslant 2$: by Lemma 6.9, all bipartite graphs have 2-medianwidth at most 2 and most of the graph problems considered on graphs of bounded treewidth remain as hard in the bipartite case as in the general case.

However, it might still be meaningful to study Minimum Vertex Cover (or Maximum Independent Set) on graphs of bounded i-medianwidth, which are known to be efficiently solvable on bipartite graphs.

Lastly, by [13], deciding the treewidth of a graph (which is the 1-medianwidth) is fixed-parameter tractable, while by [32], deciding the clique number (which is the infinite version of i-medianwidth) is complete for the complexity class W[1]. It is unknown to us what the (parametrised) complexity of deciding the i-medianwidth of a graph is for any fixed $i \geqslant 2$.

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