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Fourier Analysis – Exercise sheet 6
(to be discussed on June 25)

Exercises 6.1 and partially 6.2 deal with proofs of rather elementary facts that we have already introduced in the lectures. Although this reads a bit lengthy most of them are very short. The other exercises are about Fourier series and are supposed to round-off this subject. Exercise 6.4 and 6.5 combine several results we derived on the convergence of Fourier series and should hence summarize many important facts we discussed.

Ex 6.1: (**Basic properties of the Fourier transform**) Show that for any $f \in L^1(\mathbb{R})$ the Fourier transform $\mathcal{F}(f)$

$$(\mathcal{F}(f))(s) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-ist}\,dt$$

satisfies the properties listed below. Let us also use the following operators for fixed $\omega \in \mathbb{R} \setminus \{0\}$, $g \in L^1(\mathbb{R})$,

$$\tau_w f = f(\cdot + \omega), \qquad m_g f = fg, \qquad Rf = f(-\cdot), \qquad D_\omega f = f(w\cdot)$$

defined for $f \in L^p(\mathbb{R})$ for any $p \in [1, \infty]$.

(a) $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ is linear, bounded

(b) The range ran $\mathcal{F}$ of $\mathcal{F}$ lies in $C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) \colon \lim_{x\to\pm\infty} f(x) = 0\}$

(c) For $\mathrm{e}_{i\omega}(s) := \mathrm{e}^{i\omega s}$ we have

$$\mathcal{F} m_{\mathrm{e}_{-i\omega}} = \tau_\omega \mathcal{F} \qquad (\textit{Modulation})$$
$$\mathcal{F}\tau_\omega = m_{\mathrm{e}_{i\omega}} \mathcal{F} \qquad (\textit{Translation})$$
$$\mathcal{F}R = R\mathcal{F} \qquad (\textit{Reflection})$$
$$\mathcal{F}\,\bar{\cdot} = \overline{R\mathcal{F}} \qquad (\textit{Conjugation})$$
$$\mathcal{F}D_\omega = m_{1/|\omega|} D_{\frac{1}{\omega}} \mathcal{F} \qquad (\textit{Dilation})$$

considered on $L^1(\mathbb{R})$.

(d) $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$ (*Convolution Theorem*)

(e) for any $f \in C^k(\mathbb{R})$ such that $f^{(\ell)} \in L^1(\mathbb{R})$ for all $\ell = 0, .., k$,

$$\mathcal{F}(f^{(k)}) = m_{(i\mathbf{s})^k} \mathcal{F}(f),$$

where $i\mathbf{s}$ refers to the function $s \mapsto is$. Conversely, if $x \mapsto x^\ell f(x) \in L^1(\mathbb{R})$ for all $\ell = 0, .., k$,

$$[\mathcal{F}(f)]^{(k)} = \mathcal{F}(m_{(-i\mathbf{s})^k} f)$$

Ex 6.2: (**Schwartz functions and the Fourier transform**)

(1) Show that $x \mapsto \mathrm{e}^{-x^2}$ and any $C^\infty(\mathbb{R})$ function with compact support are in the Schwartz class.
(2) Show that convergence of a sequence in the Schwartz class $S(\mathbb{R})$ (as defined in the lectures, Def. 1.4)) implies that the sequence also converges in $L^p(\mathbb{R})$ for any $p \in [1, \infty]$.
(3) Show that the Fourier transform leaves $S(\mathbb{R})$ invariant, i.e. $\mathcal{F}f \in S(\mathbb{R})$ for all $f \in S(\mathbb{R})$.
(4) Show that with $f, g \in S(\mathbb{R})$ also fg and $f * g \in S(\mathbb{R})$. Conclude that $\mathcal{F}$ is an algebra homomorphism on $S(\mathbb{R})$ (with respect to the group action $*$).
(5) Show that $\mathcal{F}$ is continuous as mapping from $S(\mathbb{R})$ to $S(\mathbb{R})$ in the sense that

$$f_n \xrightarrow{S} f \quad \Longrightarrow \quad \mathcal{F}f_n \xrightarrow{S} \mathcal{F}f$$

(6) Adapt the definition of an approximate identity to functions in $L^1(\mathbb{R})$ and show that for any continuous function $g \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} g = 1$, the family $(\lambda g(\lambda \cdot))_{\lambda>0}$ defines such approximate identity. Use this together with a version of the results on approximate identities from the lectures to show that the C^∞ functions with compact support lie dense in $L^p(\mathbb{R})$ for any $p \in [1, \infty)$.

(7) *(extra) Show that the sequential convergence we introduced on $S(\mathbb{R})$ corresponds to a metric with respect to which $S(\mathbb{R})$ is complete.

Ex 6.3: *Hausdorff–Young inequality for Fourier series*

(a) Prove the following statement:

For $(p, q) \in [1, \infty]^2$ such that $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$, and $f \in L^p(\mathbb{T})$, it holds that $\hat{f} \in \ell^q(\mathbb{Z})$ with

$$\|\hat{f}\|_{\ell^q(\mathbb{Z})} \leq C\|f\|_{L^p(\mathbb{T})}$$

for some absolute constant $C > 0$.

(b) Analogously to (a) prove the following similar statement:

Let $(p, q) \in [1, \infty]^2$ such that $p \in [1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. For any $(a_n) \in \ell^p(\mathbb{Z})$ there exists $f \in L^q(\mathbb{T})$ such that $\hat{f}(n) = a_n$ for all $n \in \mathbb{Z}$ with

$$\|f\|_{L^q(\mathbb{T})} \leq C\|(a_n)_{n\in\mathbb{N}}\|_{\ell^p(\mathbb{Z})}$$

for some absolute constant $C > 0$.

Hint: Interpolation.

Ex. 6.4: Discuss the convergence of the Fourier series of the function 2π-periodic function defined by

$$f(t) = \pi - t \quad \forall t \in (0, 2\pi) \quad \text{and } f(0) = f(2\pi) = 0$$

in $L^1(\mathbb{T})$. More precisely, consider convergence in the following sense

(1) with respect to the norms $\|\cdot\|_{L^p}$ where $p \in [1, \infty]$

(2) absolutely (for all $t \in \mathbb{T}$), that is, in the norm $\|\cdot\|_{A(\mathbb{T})}$, see Ex. 3.2.

(3) pointwise for $t \in \mathbb{T}$ or for almost every $t \in \mathbb{T}$,

and derive the limit function if it exists.

You may also plot the first, say 5 to 10, partial sums of the Fourier series with matlab in order to get a feeling for the (uniform) convergence. *Hints: you may consider previous exercises on Fourier series, e.g. Ex. 5.1, Ex. 4.1, Ex. 3.2–3.3.*

Ex 6.5: Recapitulate what you can say about the convergence of Fourier series of the following (classes) of functions. Consider the same types of convergence as in Ex. 5.2.

(a) $f \in C(\mathbb{T})$

(b) $f : \mathbb{T} \to \mathbb{C}$ differentiable (and 2π-periodic)

(c) $f \in C^1(\mathbb{T})$