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Fourier Analysis – Exercise sheet 5 (incl. sketch of some solutions)
(to be discussed on June 11)

Ex 5.1: *(Pointwise convergence of Dirchlet means for differentiable functions)*
Let $f \in L^1(\mathbb{T})$ be differentiable at $t_0 \in \mathbb{T}$. Then the partial sums $D_n * f$ of the Fourier series of f converge to $f(t_0)$ at t_0.
Hint: Use Ex. 4.1.

Ex 5.2: *(The maximum principle for entire functions) — an alternative approach due to* Orr-Shalit[1]
(*This exercise may be well-known for those who are familiar with basic complex analysis*)
A function $f : \mathbb{C} \to \mathbb{C}$ is called *entire* if f is a complex power series with radius of convergence equal to ∞, i.e. there exists $(a_n)_{n\in\mathbb{N}}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \forall z \in \mathbb{C}.$$

(a) Show that for any entire f,

$$\max_{z\in\overline{\mathbb{D}}} |f(z)| = \max_{|z|=1} |f(z)|.$$

Hint: Follow the steps

(i) Reduce the claim to complex polynomials f of degree larger than 1.
Idea: For given $\epsilon > 0$ chose $p(z) = \sum_{n=0}^{N} a_n z^n$ with sufficiently large N such that $\|f - p\|_{\infty,\overline{\mathbb{D}}} < \epsilon$. Assuming that the claim holds for polynomials we derive

$$\max_{z\in\overline{\mathbb{D}}} |f(z)| \le \max_{z\in\overline{\mathbb{D}}} |f(z) - p(z)| + \max_{z\in\overline{\mathbb{D}}} |p(z)| \le \epsilon + \max_{|z|=1} |p(z)| \le 2\epsilon + \max_{|z|=1} |f(z)|.$$

(ii) Let $n > 1$ and $z \in \mathbb{D}$. Consider $U \in \mathbb{C}^{(n+1)\times(n+1)}$ defined by

$$U = \begin{pmatrix} z & 0 & \dots & 0 & \sqrt{1-|z|^2} \\ \sqrt{1-|z|^2} & 0 & \dots & 0 & -\bar{z} \\ 0 & & & & 0 \\ \vdots & & I_{n-1} & & \vdots \\ 0 & & & & 0 \end{pmatrix}$$

where I_{n-1} denotes the identity matrix of dimension $(n-1) \times (n-1)$.
Show that U is unitary, i.e. $U^*U = UU^* = I_{n+1}$ [2] and that for polynomials f with $deg(f) = n$,

$$f(z) = P_1 f(U) P_1^T$$

where $P_1 = (1, 0, \dots, 0) \in \mathbb{C}^{1\times(n+1)}$. Conclude that $|f(z)| \le \|f(U)\|_{2\to 2}$ where the operator norm is induced by the Euclidean norm.

(iii) Conclude the assertion by arguing why $\|f(U)\|_{2\to 2} \le \max_{|z|=1} |f(z)|$ (use that U is unitary and the spectral theorem from linear algebra).

(b) Show that in (a) the set $\mathbb{D}$ can be replaced by any bounded, open, connected set Ω in $\mathbb{C}$, i.e.

$$\max_{z\in\overline{\Omega}} |f(z)| = \max_{\partial\Omega} |f(z)|,$$

where $\partial\Omega$ denotes the boundary of the open set Ω.
Hint: Assume that there exists $z \in \Omega$ such that $|f(z)| \ge \max_{\tilde{z}\in\overline{\Omega}} |f(\tilde{z})|$.

[1] Orr Shalit *A sneaky proof of the maximum modulus principle*, The American Mathematical Monthly Vol. 120, No. 4, pp. 359-362, 2013; see also `https://arxiv.org/abs/1304.5839`.

[2] where $T^* = (\overline{t_{j,i}})_{i,j}$ denotes the hermitian transpose of the matrix $T = (t_{i,j})_{i,j}$.

Note that there is a gap here — it seems that this cannot be concluded from part (a) without any further effort. I will comment on this in the coming lectures/exercise classes. Assume we knew that part (a) implies the following stronger statement

If $z_0 \in \mathbb{D}$ is such that $|f(z_0)| = \max_{z\in\overline{\mathbb{D}}} |f(z)|$ then $|f(\cdot)|$ is constant on $\overline{\mathbb{D}}$,

then we could proceed as follows. Assume there exists z in Ω such that $|f(z)| \geq \max_{z\in\overline{\Omega}} |f(z)|$. By the above stronger statement, we conclude that $|f(\cdot)|$ has to be constant on this ball. This argument can be extended to arbitrary points in $\tilde{z} \in \Omega$. Since Ω is connected (which, as as open subspaces of $\mathbb{C}$, are *path-connected*) we find a (continuous) path γ from z to $\tilde{z}$ which is of finite length as Ω is bounded. For any point y on this path we can find a ball contained in Ω, with center y and radius ϵ, the latter being independent of y. By compactness of the path, finitely many balls will already cover the path and on the union of these finite sets, f is constant by the argument mentioned in the beginning. Thus, $|f(z) = |f(\tilde{z})|$ and hence $|f(\cdot)|$ is constant on Ω which shows the assertion. Alternatively, use the definition of connectedness directly: In particular this means that the only clopen (open and closed) sets in Ω are Ω and the empty set. Show that the set on which $|f(\cdot)|$ is constant is clopen (it's open by the first part of the proof and closed by continuity).

(c) (for people familiar with basic complex analysis) Show above statements for functions f that are analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ (or Ω and $\overline{\Omega}$ respectively). **Note that if f is analytic on $\mathbb{D}$ and continuous on the closure $\overline{\mathbb{D}}$, then the power series of f (centered at 0) need not converge on $\partial\mathbb{D}$ — but such examples are delicate to construct). However, any such f can be approximated by polynomials uniformly on $\overline{\mathbb{D}}$.** To see this, recall first that the power series (centered at 0) of f converges uniformly to f on any smaller disc $r\overline{\mathbb{D}}$, $r < 1$. Hence, for every $r < 1$, $f_r = f(r\cdot)$ has a uniformly converging power series on $\overline{\mathbb{D}}$. Let $r_n \nearrow 1$ as $n \to \infty$ and consider for each n the corresponding polynomial from the truncated power series p_n corresponding to f_{r_n} such that $\|p_n - f_{r_n}\|_{\infty,\overline{\mathbb{D}}} \leq 2^{-n}$. The statement that for given $\epsilon > 0$, we find $N \in \mathbb{N}$ such that $\|f - p_n\|_{\infty,\overline{\mathbb{D}}} < \epsilon$ for all $n \geq N$ now follows by triangle inequality and continuity of f.

Ex. 5.3: (Isoperimetric inequality in 2D) The goal of this exercise is to show the statement

For any closed, regular, nonself-intersecting, positively orientated C^1-curve Γ in $\mathbb{R}^2$ [3] of length L and with enclosing area A the inequality

$$4\pi A \leq L^2 \tag{$*$}$$

holds with equality if and only if the curve is a circle. Here, **regular** *means that $\gamma'(t) \neq 0$ for all $t \in \mathbb{T}$ for any C^1-parametrization $\gamma : \mathbb{T} \to \mathbb{R}^2$ of Γ.*

For that consider the following steps, where $\gamma : [0, 2\pi] \to \mathbb{R}^2$, with components γ_1 and γ_2, denotes a C^1-parametrization of Γ, see [1].

(a) Show that the area A enclosed by Γ equals

$$A = \frac{1}{2}\int_0^{2\pi} \gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s)\, ds.$$

Idea: By Green's theorem (applied in the last identity)

$$A = \int\int_A \frac{1}{2} + \frac{1}{2}\, dxdy = \int\int_A \frac{1}{2}\frac{\partial}{\partial x}x - \frac{1}{2}\frac{\partial}{\partial y}(-y)\, dx\, dy = \frac{1}{2}\int_\Gamma (-y)\, dx + x\, dy$$

Using the parametrization $(x, y) = (\gamma_1(t), \gamma_2(t))$, this leads to the assertion.

(b) *(Poincaré–Wirtinger inequality in 1D)*
Show that for $f \in C^1(\mathbb{T})$ (or more generally, for f being absolutely continuous with $f' \in L^2$)

[3] here we mean that there exists a continuously differentiable $\gamma : [0, 2\pi] \to \mathbb{R}^2$ such that γ is injective on $[0, 2\pi)$, $\gamma'(t) \neq 0$ for all $t \in \mathbb{T}$, $\gamma(0) = \gamma(2\pi)$ and $\Gamma = \gamma(\mathbb{T})$. The length (or perimeter) L of Γ can be expressed as $L = \int_0^{2\pi} \|\gamma'(t)\|_2\, ds$.

it holds that

$$\|f - \hat{f}(0)\|_{L^2(\mathbb{T})} \leq \|f'\|_{L^2(\mathbb{T})}.$$

Proof: Since f and f' are in L^2, we know that the respective Fourier series converge (in L^2) and by $\widehat{f'}(n) = -in\hat{f}(n)$ and Parseval's identity (twice) we obtain

$$\|f'\|_{L^2}^2 = 2\pi\|\widehat{f'}\|_{\ell^2(\mathbb{Z})}^2 \geq 2\pi \sum_{k\neq 0} |\hat{f}(k)|^2 = \|f - \hat{f}(0)\|_{L^2}^2.$$

(c) Show $(*)$ in the case that $\|\gamma'(t)\|_2 = (\gamma_1(t)^2 + \gamma_2(t)^2)^{\frac{1}{2}} = 1$ for all $t \in \mathbb{T}$.
(Hint: Consider $f(s) = \gamma_1(s) + i\gamma_2(s)$, show that $A = \frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} f'(s)\overline{f(s)}\,ds$ and note that $\int_{\mathbb{T}} f'(s)\,ds = 0$)

Proof: with the suggested choice of f, we have

$$\begin{aligned}\operatorname{Im}(f'(s)\overline{f(s)}) &= \frac{1}{2i}[(\gamma_1'(s) + i\gamma_2'(s))(\gamma_1(s) - i\gamma_2(s)) - (\gamma_1'(s) - i\gamma_2'(s))(\gamma_1(s) + i\gamma_2(s))] \\ &= \gamma_1(s)\gamma_2'(s) - \gamma_1'(s)\gamma_2(s).\end{aligned}$$

Hence, by (a) and (b)

$$\begin{aligned}A &= \frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} f'(s)\overline{f(s)}\,ds \\ &= \frac{1}{2}\operatorname{Im}\int_{\mathbb{T}} f'(s)\overline{(f(s) - \hat{f}(0))}\,ds \\ &\leq \frac{1}{2}\|f'\|_{L^2(\mathbb{T})}^2 = \pi\end{aligned}$$

where the last identity follows from the assumption that $\gamma_1(t)^2 + \gamma_2(t)^2 = 1$. Since $L = \int_0^{2\pi}\|\gamma'(t)\|_2(t)\,ds = 2\pi$, the assertion follows ($\pi = L^2/4\pi$).

(d) Show why the assumption in (c) on γ can always be made by reparametrizing.
(Hint: $\gamma \rightsquigarrow \gamma \circ h^{-1}$ where $h(t) = \frac{1}{L}\int_0^t \|\gamma'(s)\|_2\,ds$.)

Proof: Since the curve is regular, h is bijective (and C^1) on $[0,1]$. Using the chain rule on deduces that $(\gamma \circ h^{-1})'(t) = 1$ for all $t \in \mathbb{T}$. Thus $\gamma \circ h^{-1}$ is a suitable parametrization of Γ.

(e) Show the statement on the equality by investigating when equality holds in (b) and the inequalities in the proof of (c).
To have equality in (b) we have to have that $\hat{f}(m) = 0$ for all $|m| \geq 2$. Thus f has the form

$$f(t) = \hat{f}(-1)\mathrm{e}^{-it} + \hat{f}(0) + \hat{f}(1)\mathrm{e}^{it}.$$

By our assumption in (c), that $\|\gamma'(t)\| = 1$ — which is equivalent to $|f'(t)| = 1$ — for all $t \in \mathbb{T}$, we have that for all $t \in \mathbb{T}$

$$1 = |\hat{f}(-1)|^2 + |\hat{f}(1)|^2 - 2\Re\hat{f}(-1)\overline{\hat{f}(1)}\mathrm{e}^{2it},$$

and similarly by $\|f'\|_{L^2}^2 = \int_0^{2\pi}\|\gamma'(t)\|_2 dt = 2\pi$ and Parseval, $2\pi = 2\pi(|\hat{f}(-1)|^2 + |\hat{f}(1)|^2)$. This yields that

$$2\Re\hat{f}(-1)\overline{\hat{f}(1)}\mathrm{e}^{2it} = 0.$$

Finally using the choices $t = 0$ and $t = \pi/4$, we conclude that both the real part and the imaginary part of $\hat{f}(-1)\overline{\hat{f}(1)}$ equal zero. Thus either $\hat{f}(1)$ or $\hat{f}(-1)$ is zero, which, by the form of $f(t)$, implies that the curve Γ has to be a circle.

Ex. 5.4: (Young's inequality for convolutions) Prove Theorem 2.3 from the lecture for $\Omega = \mathbb{T}$.

Let $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ for $p, q, r \in [1, \infty]$ and $f \in L^p(\Omega)$, $g \in L^q(\Omega)$. Then $f * g \in L^r(\Omega)$ and

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Proof: Use Riesz–Thorin's theorem (and also Minkowski's inequality, Ex. 2.4).

Let us first discuss some special cases: If $q = 1$, then $p = r$ and the statement follows directly from Minkowski's inequality, Ex. 2.4. Obviously, not both p and q can equal to ∞ simultaneously, so that by symmetry we can always assume that $q < \infty$. Thus in the following let $q \in (1, \infty)$. For fixed $g \in L^q$ consider the operator T defined by $Tf = f * g$. By Minkowski's inequality,

$$T : L^1 \to L^q$$

is well-defined and bounded with $\|T\|_{L^1 \to L^q} \leq \|q\|_{L^q}$. Therefore, also the dual operator $T' : L^{q'} \to L^\infty$ is bounded by the same constant (where q' denotes the Hoelder conjugate, $\frac{1}{q} + \frac{1}{q'} = 1$, and where we used the identification of the dual L^p spaces). We claim that — with a similar argument than in Ex. 3.4, see below — the latter implies that

$$T : L^{q'} \to L^\infty$$

is well-defined the operator norm can again be estimated by $\|g\|_{L^q}$. Assume we have this, then, by Riesz–Thorin theorem,

$$T : L^p \to L^{\tilde{q}}$$

is also well-defined and bounded for all $(p, \tilde{q})$ such that

$$\frac{1}{p} = \frac{1-\theta}{1} + \frac{\theta}{q'}, \quad \frac{1}{\tilde{q}} = \frac{1-\theta}{q} + \frac{\theta}{\infty} = \frac{1-\theta}{q}$$

for some $\theta \in (0, 1)$. Let us express θ in terms of p and q. Since $\frac{1}{q'} = 1 - \frac{1}{q}$, we get by the first equation that $\theta = q - \frac{q}{p}$. Inserting this in the second equation yields

$$\frac{1}{\tilde{q}} = \frac{1 - (q - \frac{p}{q}}{q} = \frac{1}{q} - 1 + \frac{1}{p}$$

Since the right-hand side equals $\frac{1}{r}$ by assumption, we conclude that T is a bounded operator from L^p to L^r. Moreover, still by Riesz–Thorin theorem, we have that the following estimate for the operator norm (recall that $\theta = q - \frac{q}{p}$)

$$\|Tf\|_{L^{\tilde{q}}} = \|Tf\|_{L^r} \leq \|T\|_{L^1 \to L^q}^{1-\theta} \|T\|_{L^{q'} \to L^\infty}^{\theta} \|f\|_{L^q} \leq \|g\|_{L^q}^{1-(q-\frac{q}{p})} \|g\|_{L^q}^{q-\frac{q}{p}} \|f\|_{L^q} = \|g\|_{L^q} \|f\|_{L^p}.$$

Since $Tf = f * g$, this proves the assertion.

It remains to argue why T is bounded from $L^{q'}$ to L^∞ is bounded. For that, one shows analogously to Ex. 3.4 that for $f \in L^1, g \in L^q$ and $h \in L^{q'}$ that

$$\int_{\mathbb{T}} (f * g)(s) h(s)\, ds = \int_{\mathbb{T}} f(s)(g * R(h))\, ds, \tag{1}$$

(note that the difference to Ex. 3.4 only was that there $g, h \in L^1$.) where $R(h) = h(-\cdot)$ is the reflection of h. With this, we get from the definition of T' (and identifying duals of L^p-spaces by the usual isomorphism)

$$\int_{\mathbb{T}} (T'h)(s) f(s)\, ds = \int_{\mathbb{T}} h(s)(Tf)(s)\, ds \qquad \forall f \in L^1, h \in L^{q'}.$$

By (1), we conclude that $T'h = g * R(h)$. Since $g * Rh$. Since $T'h = TR(h)$ for $h \in L^1 \cap L^{q'}$, we conclude since R is an isometric isomorphism, i.e. $\|R \cdot\|_{L^{q'}} = \|\cdot\|_{L^{q'}}$ and R invertible, that $T = T'R^{-1}$ is bounded from $L^{q'}$ to L^∞ with norm

$$\|T\|_{L^{q'} \to L^\infty} = \|T'R^{-1}\|_{L^{q'} \to L^\infty} = \|T'\|_{L^{q'} \to L^\infty} = \|T\|_{L^1 \to L^q} \leq \|g\|_{L^q}.$$

The statement also holds for $\Omega = \mathbb{R}$ as the Riesz–Thorin theorem remains valid and Minkowski's inequality can be proved analogously in this case.