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Fourier Analysis – Exercise sheet 3 – part I

As indicated in the lectures, we will need the notion of absolute continuity. *A function* $f : [a, b] \to \mathbb{C}$ *is called absolutely continuous if there exists* $g \in L^1([a, b])$ *such that* $f(t) = f(a) + \int_a^x g(s)\,ds$ *for all* $x \in [a, b]$. *It can be shown that this is equivalent to the property that*

$$\forall \epsilon > 0 \exists \delta > 0 \forall n \forall \text{disjoint subintervals } \{(a_i, b_i)\}_{i=1}^n : \quad \left(\sum_{k=1}^{n} |b_i - a_i| < \delta \implies \sum_{k=1}^{n} |f(b_i) - f(a_i)| < \epsilon \right)$$

(the latter is usually referred to as absolute continuity). Moreover, it then follows that f *is differentiable almost everywhere in* $[a, b]$ *and* $f' = g$.

Ex 3.0: *Show (using the above mentioned facts) that*

(a) *any absolutely continuous function is uniformly continuous;*

(b) *any Lipschitz-continuous function is absolutely continuous;*

(c) *any absolutely continuous function* f *is of bounded variation, i.e.*

$$\mathrm{Var}_{a,b}(f) := \sup \left\{ \sum_{i=1}^{N} |f(a_{i+1}) - f(a_i)| : N \in \mathbb{N}, a = a_1 < a_2 < .. < a_N = b \right\} < \infty$$

(d) *there exists a continuous function which is not absolutely continuous.*

Ex 3.1: *(Decay of Fourier coefficients) Show the following.*

(1) *If* $f \in C^1(\mathbb{T})$, *then* $\hat{f}(n) \in o(n^{-1})$ $(n \to \pm\infty)$ *and find a corresponding assertion for* $f \in C^k(\mathbb{T})$.

(2) *If* f *is absolutely continuous, then*

$$\hat{f}(n) = \frac{1}{in} \widehat{f'}(n), \quad n \in \mathbb{Z}.$$

(3) *If* $g \in L^1(\mathbb{T})$ *is such that* $\hat{g}(n) = -\hat{g}(-n) \geq 0$ *for all* $n \in \mathbb{N}_0$, *then* $(\frac{\hat{g}(n)}{n})_{n \in \mathbb{N}} \in \ell^1(\mathbb{N})$.
(Hint: Use (2) and the theorem on the pointwise convergence of the Fejér means $F_n * f$*)*

Ex 3.2: *(The space* $\mathrm{A}(\mathbb{T})$*) Let* $\mathrm{A}(\mathbb{T})$ *denote the space of functions* f *in* $L^1(\mathbb{T})$ *with absolutely summable Fourier coefficients, i.e.* $\hat{f} \in \ell^1(\mathbb{Z})$.

(a) *Show that* $\mathrm{A}(\mathbb{T}) \subset C(\mathbb{T})$ *and argue why the inclusion is strict.*[1]

(b) *Let* $\mathrm{A}(\mathbb{T})$ *be equipped with the norm* $\|f\|_{\mathrm{A}(\mathbb{T})} = \|\hat{f}\|_{\ell^1(\mathbb{Z})}$. *Argue why this is indeed well-defined and clarify on the relation to* $\|\cdot\|_{C(\mathbb{T})}$.

(c) *Recall that* $L^1(\mathbb{T})$ *is an algebra with the convolution* $*$ *and show that* $\mathrm{A}(\mathbb{T})$ *is an ideal of* $L^1(\mathbb{T})$ *with respect to* $*$.

(d) *Show that any absolutely continuous* f *with* $f' = g \in L^2(\mathbb{T})$ *lies in* $\mathrm{A}(\mathbb{T})$. *Also show the existence of an absolutely continuous* f *such that* $f' \notin L^2(\mathbb{T})$ *(Hint: Ex. 3.1).*

(e) *Is* $\mathrm{A}(\mathbb{T})$ *a homogeneous Banach space?*

Ex 3.3:

(1) *Let* $f \in L^1(\mathbb{T})$ *be defined by* $f(x) = x - \pi$. *Compute the Fourier series of* f *and discuss its convergence (pointwise, in* L^p, $C(\mathbb{T})$*).*

(2) *If* $f \in L^1(\mathbb{T})$ *is piecewise continuously differentiable, then the Fourier series of* f *converges to* f *pointwise* [2].

[1] as usual for L^1-functions (equivalence classes of functions equal λ-a.e.) we identify with the continuous representative if it exists.

[2] Here "piecewise continuously differentiable" means that except for finitely many points in $[0, 2\pi]$, f is differentiable with continuous derivative. At these finite points of exception the function the right and left limit of f and f' are assumed to exist (including the points 0 and 2π).

Ex 3.4: (revision from sheet 2)

(a) *Let $f, g \in L^1(\mathbb{T})$ and $h \in L^\infty(\mathbb{T})$. Show that $\int_{\mathbb{T}}(f * g)(s)h(s)\,ds = \int_{\mathbb{T}} f(s)(g * R(h))(s)\,ds$.* [3]
Note that this identity can be linked to the "dual operator (also called "conjugate operator") of

$$M_g : L^1(\mathbb{T}) \to L^1(\mathbb{T}), f \mapsto f * g.$$

Recall that the dual operator $T' : Y' \to X'$ of a bounded linear operator $T : X \to Y$ (X, Y Banach spaces) is defined through

$$\langle Tx, x'\rangle_{X,X'} = \langle x, T'x'\rangle_{X,X'} \qquad \forall x \in X, x' \in X'.$$

Hint: It suffices to consider simple functions h and moreover indicator functions on measurable subsets of $\mathbb{T}$.

(b) *Following the notation introduced in (a), determine the dual operator $(\mathcal{M}_g)'$ of $\mathcal{M}_g$.*

(c) *Using (a) prove that $C(\mathbb{T})$ is weak*-dense in L^∞*
(this exercise was already given in Ex. 2.3, but as there was a typo in (a) then, we consider it here again. However, there is not a big difference in solving it. [4]*.)*

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[3] There was a TYPO in the original formulation of the exercise in Ex. 2.3: There, on the right-hand-side h should have been replaced by $R(h) = h(-\cdot)$.

[4] you may look at Ex. 2.3 and follow the hint there