ON HAMILTONIAN CYCLES IN HYPERGRAPHS WITH DENSE LINK GRAPHS

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Dedicated to Endre Szemerédi on the occasion of his 80^{th} birthday

ABSTRACT. We show that every k-uniform hypergraph on n vertices whose minimum $(k-2)$-degree is at least $(5/9+o(1))n^2/2$ contains a Hamiltonian cycle. A construction due to Han and Zhao shows that this minimum degree condition is optimal. The same result was proved independently by Lang and Sahueza-Matamala.

§1. INTRODUCTION

Hamiltonian cycles are a central theme in graph theory and extremal combinatorics. Dirac's classic result [5] states that every graph on $n \geqslant 3$ vertices whose minimum degree is at least $\frac{n}{2}$ contains a Hamiltonian cycle. The present work continues the investigation of hypergraph generalisations of Dirac's theorem – an area of research owing many deep insights to Endre Szemerédi.

1.1. **Hypergraphs and Hamiltonian cycles.** For $k \geqslant 2$ a *k-uniform hypergraph* is defined to be a pair $H = (V, E)$ consisting of a (finite) *set of vertices* V and a set

$$E \subseteq V^{(k)} = \{U \subseteq V : |U| = k\}$$

of edges. A k-uniform hypergraph $H = (V, E)$ with n vertices is said to contain a *Hamiltonian cycle* if its vertex set admits a cyclic enumeration $V = \{x_i : i \in \mathbb{Z}/n\mathbb{Z}\}$ such that $\{x_i, x_{i+1}, \dots, x_{i+k-1}\} \in E$ holds for all $i \in \mathbb{Z}/n\mathbb{Z}$. Observe that this naturally generalises the familiar notion of Hamiltonian cycles in graphs.

In contrast to the graph case, there are several interesting minimum degree notions for hypergraphs. For a k-uniform hypergraph $H = (V, E)$ and a set $S \subseteq V$ the *degree of S in H* is defined by

$$d_H(S) = |\{e \in E : S \subseteq e\}|.$$

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Moreover, for an integer i with $1 \leqslant i < k$ the number

$$\delta_i(H) = \min\{d_H(S)\colon S \in V^{(i)}\}$$

is called the *minimum i-degree of H*.

The research on minimum i-degree conditions guaranteeing the existence of Hamiltonian cycles in hypergraphs was initiated by Katona and Kierstead [12]. The main problem is to determine, for any two given integers $k \geqslant 2$ and $i \in [k-1]$, the optimal minimum i-degree condition which for k-uniform hypergraphs ensures the existence of a Hamiltonian cycle. Notice that Dirac's aforementioned theorem solves the case $(k, i) = (2, 1)$.

In general, if $i < j$, then a minimum j-degree condition seems to reveal more structural information about a hypergraph than a minimum i-degree condition. For this reason, it is reasonable to suspect that the difficulty of the problem we are interested in increases with $k - i$. The first case, $i = k - 1$, was solved more than a decade ago by Rödl, Ruciński, and Szemerédi [18].

Theorem 1.1. *For every integer $k \geqslant 2$ and every $\alpha > 0$ there exists an integer n_0 such that every k-uniform hypergraph H on $n \geqslant n_0$ vertices with $\delta_{k-1}(H) \geqslant \left(\frac{1}{2} + \alpha\right) n$ contains a Hamiltonian cycle.*

Similarly as for Dirac's theorem, slightly unbalanced bipartite hypergraphs show that this result is asymptotically best possible. Our main result addresses the next case, $i = k - 2$.

Theorem 1.2. *For every integer $k \geqslant 3$ and every $\alpha > 0$, there exists an integer n_0 such that every k-uniform hypergraph H on $n \geqslant n_0$ vertices with $\delta_{k-2}(H) \geqslant \left(\frac{5}{9} + \alpha\right) \frac{n^2}{2}$ contains a Hamiltonian cycle.*

In previous articles written in collaboration with Ruciński, Schacht, and Szemerédi [15,17] we solved the cases $k = 3$ and $k = 4$. The general case was also obtained by Lang and Sanhueza-Matamala [13] in independent research. A construction due to Han and Zhao [11] reproduced in the introduction of [15] shows that the number $\frac{5}{9}$ appearing in Theorem 1.2 is optimal.

We would like to conclude this subsection by pointing to some problems for future investigations. First and foremost, it remains an intriguing question whether for $k \geqslant 4$ the minimum $(k-3)$-degree condition $\delta_{k-3}(H) \geqslant \left(\frac{5}{8} + o(1)\right) \frac{n^3}{6}$ enforces the existence of a Hamiltonian cycle. Here the number $\frac{5}{8}$ would again match the construction of Han and Zhao [11].

Another possible area of research would be to extend the work of Pósa [16] and Chvátal [4], who in the graph case studied which conditions on the degree sequence (rather than just on the minimum degree) guarantee the existence of Hamiltonian cycles. Such degree sequence versions have recently been obtained for the Hajnal-Szemerédi theorem [10] by Treglown [24] and for Pósa's conjecture (see [8, Problem 9]) by Staden and Treglown [22]. It would be very interesting to find similar theorems for Hamiltonian cycles in hypergraphs. For first results in this direction we refer to [19].

1.2. **Organisation and Overview.** We use the *absorption method* developed by Rödl, Ruciński, and Szemerédi and surveyed by Szemerédi himself in [23]. Therefore, the proof decomposes in the usual way into a Connecting Lemma, an Absorbing Path Lemma, and a Covering Lemma.

Very roughly speaking, the Absorbing Path Lemma reduces the task of proving Theorem 1.2 to the much easier problem of finding 'almost spanning' cycles in k-uniform hypergraphs H satisfying $\delta_{k-2}(H) \geqslant \left(\frac{5}{9}+\alpha\right)\frac{|V(H)|^2}{2}$. Such an almost spanning cycle is build in two main steps: First, the covering lemma asserts that we can cover almost all vertices by means of long paths. Second, the Connecting Lemma allows us to connect these 'pieces' into one long cycle.

In our earlier articles we stored all information about H that became relevant in the course of the proof in various 'setups' and the complexity of these setups got somewhat out of control. To avoid this in the present work, we abandon the setups and replace them by the much more flexible notion of a *constellation* (see Definition 2.10 below).

Section 2 lays out a systematic treatment of these constellations and contains several auxiliary results that will assist us later. The subsequent Sections 3 – 6 deal with the main lemmata enumerated above: connecting, absorbing, and covering. Lastly, in Section 7 we derive Theorem 1.2 from these results.

§2. Preliminaries

2.1. **Graphs.** In our earlier articles [15, 17] dealing with the 3- and 4-uniform case of Theorem 1.2 we inductively reduced connectability properties of the hypergraphs under discussion to connectability properties of their 2-uniform link graphs. Here we pursue the same strategy and the present subsection contains the graph theoretic preliminaries that we require for this purpose. The central notion we work with in this context is taken from [17, Definition 2.2] and reappeared as [15, Definition 2.1].

Definition 2.1. *Given $\beta > 0$ and $\ell \in \mathbb{N}$ a graph R is said to be (β, ℓ)-robust if for any two distinct vertices x and y of R the number of x-y-paths of length ℓ is at least $\beta|V(R)|^{\ell-1}$.*

It turns out that every graph whose edge density is larger than 5/9 possesses a robust subgraph containing more than two thirds of its vertices that is quite disconnected from the rest of the graph. The following statement to this effect was proved in [15, Proposition 2.2] (marginally strengthening [17, Proposition 2.3]).

Proposition 2.2. *Given α, $\mu > 0$, there exist $\beta > 0$ and an odd integer $\ell \geqslant 3$ such that for sufficiently large n, every n-vertex graph $G = (V, E)$ with $|E| \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ contains a (β, ℓ)-robust induced subgraph $R \subseteq G$ satisfying*

(i) $|V(R)| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n$,

(ii) *and* $e_G\big(V(R), V \smallsetminus V(R)\big) \leqslant \mu n^2$.

Remark 2.3. We shall usually apply Proposition 2.2 with $\mu \leqslant \frac{\alpha}{4}$. In this case, clause *(ii)* yields

$$e(R) \geqslant \left(\frac{5}{9} + \frac{\alpha}{2}\right)\frac{n^2}{2} - \frac{(n - |V(R)|)^2}{2} \overset{(i)}{\geqslant} \left(\frac{4}{9} + \frac{2}{3}\alpha\right)\frac{n^2}{2}.$$

Originally, this estimate was included as a third clause into [15, Proposition 2.2], but it seems preferable to omit this part.

In Section 5 below we need to render our absorbers connectable. To this end we shall utilise a consequence of the following graph theoretic lemma.

Lemma 2.4. *Let $\alpha > 0$ and let G be a graph with n vertices and at least $\left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ edges. If*

$$A = \big\{x \in V(G) \colon d(x) < n/3\big\}$$

and

$$B = \big\{x \in V(G) \colon |N(x) \smallsetminus A| \leqslant \alpha n/3\big\},$$

then

$$e(A \cup B) \leqslant \frac{n^2}{18}.$$

Proof. In the special case that $|A| < (\frac{1}{3} - \frac{\alpha}{3})n$, every vertex $x \in B$ satisfies

$$d(x) \leqslant |N(x) \smallsetminus A| + |A| < \frac{n}{3},$$

which yields $B \subseteq A$ and the desired inequality

$$e(A \cup B) = e(A) \leqslant \frac{1}{2}|A|^2 \leqslant \frac{1}{18}n^2.$$

So henceforth we may suppose that

$$|A| \geqslant \left(\frac{1}{3} - \frac{\alpha}{3}\right)n. \tag{2.1}$$

Now the definition of A implies

$$\frac{5}{9}n^2 \leqslant 2e(G) = \sum_{x \in V(G)} d(x) \leqslant \frac{1}{3}|A|n + (n - |A|)n = n^2 - \frac{2}{3}|A|n,$$

i.e.,

$$|A| \leqslant \frac{2}{3}n\,, \tag{2.2}$$

and

$$e(G-A) \geqslant \Big(\frac{5}{9}+\alpha\Big)\frac{n^2}{2} - \frac{1}{3}|A|n\,. \tag{2.3}$$

Setting $X = V(G) \smallsetminus (A \cup B)$ we conclude from the definition of B that

$$2e(B \smallsetminus A) + e(B \smallsetminus A, X) = \sum_{x \in B \smallsetminus A} |N(x) \smallsetminus A| \leqslant |B \smallsetminus A| \cdot \frac{\alpha}{3}n \leqslant \frac{\alpha}{3}n^2\,, \tag{2.4}$$

which together with (2.3) yields

$$\begin{aligned} |X|^2 \geqslant 2e(X) &= 2e(G-A) - 2e(B \smallsetminus A) - 2e(B \smallsetminus A, X) \\ &\geqslant \Big(\frac{5}{9}+\alpha\Big)n^2 - \frac{2}{3}|A|n - \frac{2}{3}\alpha n^2 \geqslant \frac{5}{9}n^2 - \frac{2}{3}|A|n\,. \end{aligned}$$

In view of (2.2) this entails

$$|X|^2 \geqslant \frac{4}{9}n^2 - \frac{2}{3}|A|n + \frac{1}{4}|A|^2 = \Big(\frac{2}{3}n - \frac{1}{2}|A|\Big)^2\,,$$

wherefore

$$|X| \geqslant \frac{2}{3}n - \frac{1}{2}|A|\,. \tag{2.5}$$

Next, we claim that

$$\frac{1}{3}|A|n + |B \smallsetminus A||A| + \frac{1}{2}|X|^2 \leqslant \Big(\frac{1}{3}+\frac{\alpha}{6}\Big)n^2\,. \tag{2.6}$$

In view of $|A| + |B \smallsetminus A| + |X| = n$ the left side of this estimate rewrites as

$$\frac{1}{3}|A|n + (n - |A| - |X|)|A| + \frac{1}{2}|X|^2 = \frac{4}{3}|A|n - \frac{3}{2}|A|^2 + \frac{1}{2}(|A| - |X|)^2\,.$$

By (2.5) and $X \subseteq V(G) \smallsetminus A$ we have

$$\frac{2}{3}n - \frac{3}{2}|A| \leqslant |X| - |A| \leqslant n - 2|A|$$

and, hence,

$$(|A| - |X|)^2 \leqslant \max\Big\{(n-2|A|)^2, \Big(\frac{2}{3}n - \frac{3}{2}|A|\Big)^2\Big\}\,.$$

So to conclude the proof of (2.6) it suffices to observe that

$$\frac{4}{3}|A|n - \frac{3}{2}|A|^2 + \frac{1}{2}(n-2|A|)^2 = \frac{n^2}{3} + \frac{1}{6}(n-|A|)(n-3|A|) \overset{(2.1)}{\leqslant} \Big(\frac{1}{3}+\frac{\alpha}{6}\Big)n^2$$

and, similarly,

$$\frac{4}{3}|A|n - \frac{3}{2}|A|^2 + \frac{1}{2}\Big(\frac{2}{3}n - \frac{3}{2}|A|\Big)^2 = \frac{n^2}{3} - \Big(\frac{1}{3}n - \frac{1}{2}|A|\Big)^2 - \frac{1}{8}|A|^2 \leqslant \frac{n^2}{3}\,.$$

Having thus established (2.6) we appeal to the definition of A again and observe

$$e(A) + e(G) = \sum_{x \in A} d(x) + e(G-A) \leqslant \frac{1}{3}|A|n + e(G-A)\,.$$

Consequently,

$$e(A \cup B) + e(G) \leqslant \frac{1}{3}|A|n + e(B \smallsetminus A, A) + e(B \smallsetminus A) + e(G - A)$$

and (2.4) leads to

$$e(A \cup B) + e(G) \leqslant \frac{1}{3}|A|n + |B \smallsetminus A||A| + e(X) + \frac{\alpha}{3}n^2\,.$$

Owing to (2.6) we deduce

$$e(A \cup B) + e(G) \leqslant \Big(\frac{1}{3} + \frac{\alpha}{6}\Big)n^2 + \frac{\alpha}{3}n^2 = \Big(\frac{2}{3} + \alpha\Big)\frac{n^2}{2} \leqslant \frac{1}{18}n^2 + e(G)\,,$$

which implies the desired estimate $e(A \cup B) \leqslant \frac{1}{18}n^2$. □

Remark 2.5. The set A already had an appearance in [15] and Lemma 2.3 there is roughly equivalent to the weaker estimate $e(A) \leqslant \frac{n^2}{18}$. Concerning the set B one can prove $|B| \leqslant \frac{n}{3}$, but this fact is not going to be exploited in the sequel.

The following consequence of Lemma 2.4 will later be generalised to k-uniform hypergraphs (see Lemma 2.7) and constitutes the base case of an induction on k.

Corollary 2.6. *Let $\alpha > 0$, and let V be a set of n vertices. If G, G' are two graphs with $V(G)$, $V(G') \subseteq V$ and*

$$e(G),\, e(G') \geqslant \Big(\frac{5}{9} + \alpha\Big)\frac{n^2}{2}\,,$$

then there are at least $\frac{\alpha^2}{3}n^3$ triples $(x, y, z) \in V^3$ such that

- *xyz is a walk in G,*
- *$xy \in E(G')$,*
- *and $d_G(y)$, $d_G(z) \geqslant \frac{n}{3}$.*

Proof. By adding some isolated vertices to G and G' if necessary, we may assume that $V(G) = V(G') = V$. The sieve formula yields

$$|E(G) \cap E(G')| \geqslant 2\Big(\frac{5}{9} + \alpha\Big)\frac{n^2}{2} - \frac{n^2}{2} = \Big(\frac{1}{18} + \alpha\Big)n^2\,.$$

Define the sets A and B with respect to G as in Lemma 2.4. In view of that lemma itself, there are at least αn^2 unordered pairs $xy \in E(G) \cap E(G')$ for which $x, y \in A \cup B$ fails. Consequently, there are at least αn^2 ordered pairs $(x, y) \in V^2$ such that $xy \in E(G) \cap E(G')$ and $y \notin A \cup B$. For each of them there are, by the definition of B, at least $\frac{\alpha}{3}n$ vertices z with $yz \in E(G)$ and $z \notin A$. Altogether, this yields at least $\frac{\alpha^2}{3}n^3$ triples (x, y, z) with the desired properties. □

2.2. **Hypergraphs.** In this subsection we introduce our terminology and some preliminary results on hypergraphs. When there is no danger of confusion we shall omit several parentheses, braces, and commas. For instance, we write $x_1 \dots x_k$ for the edge $\{x_1, \dots, x_k\}$ of a k-uniform hypergraph.

Walks, paths, and cycles. A k-uniform *walk* W of length $\ell \geqslant 0$ is a hypergraph whose vertices can, possibly with repetitions, be enumerated as $(x_1, \dots, x_{\ell+k-1})$ in such a way that $e \in E(W)$ if and only if $e = x_i \dots x_{i+k-1}$ for some $i \in [\ell]$. The ordered $(k-1)$-tuples $(x_1, \dots, x_{k-1})$ and $(x_{\ell+1}, \dots, x_{\ell+k-1})$ are called the *end-tuples* of W and we say that W is a $(x_1 \dots x_{k-1})$-$(x_{\ell+1} \dots x_{\ell+k-1})$-walk. This notion of end-tuples is not symmetric and implicitly fixes a direction of W. Sometimes we refer to $(x_1, \dots, x_{k-1})$ and $(x_{\ell+1}, \dots, x_{\ell+k-1})$ as the *starting $(k-1)$-tuple* and *ending $(k-1)$-tuple* of W, respectively. We call $x_k, \dots, x_\ell$ the *inner vertices* of W. Counting them with their multiplicities we say for $\ell \geqslant k-1$ that a walk of length ℓ has $\ell - k + 1$ inner vertices. We often identify a walk with the sequence of its vertices $x_1 x_2 \dots x_{\ell+k-1}$. If the vertices $x_1, \dots, x_{\ell+k-1}$ are distinct we call the walk W a *path.* For $\ell > k$ a *cycle of length ℓ* is a hypergraph C whose vertices and edges can be represented as $V(C) = \{x_i \colon i \in \mathbb{Z}/\ell\mathbb{Z}\}$ and $E(C) = \{x_i \dots x_{i+k-1} \colon i \in \mathbb{Z}/\ell\mathbb{Z}\}$.

Link hypergraphs. Given a k-uniform hypergraph $H = (V, E)$ and a set $S \subseteq V$ with $|S| \leqslant k-2$ we define the $(k-|S|)$-uniform *link hypergraph* H_S by $V(H_S) = V(H)$ and

$$E(H_S) = \{e \smallsetminus S \colon S \subseteq e \in E\}\,.$$

Clearly the vertices in S are isolated in H_S and sometimes it is convenient to remove them. In such cases, we write $\overline{H}_S = H_S - S$. For instance, we have $H_\varnothing = \overline{H}_\varnothing = H$ for every hypergraph H. If $S = \{v\}$ consists of a single vertex, we abbreviate $H_{\{v\}}$ to H_v.

A lemma with two hypergraphs. Our next step is to generalise Corollary 2.6 to hypergraphs.

Lemma 2.7. *Suppose that $k \geqslant 2$, $\alpha > 0$, and that V is a set of n vertices. If H, H' are two k-uniform hypergraphs satisfying*

$$V(H), V(H') \subseteq V$$

and

$$\delta_{k-2}(H),\, \delta_{k-2}(H') \geqslant \Big(\frac{5}{9} + \alpha\Big)\frac{n^2}{2}\,,$$

then the number of $(2k-1)$-tuples $(x_1, \dots, x_{2k-1}) \in V^{2k-1}$ such that

- *$x_1 \dots x_{2k-1}$ is a walk in H,*
- *$\{x_1, \dots, x_k\} \in E(H')$,*
- *and $d_H(x_2, \dots, x_k)$, $d_H(x_{k+1}, \dots, x_{2k-1}) \geqslant \frac{n}{3}$*

is at least $\left(\frac{\alpha}{2}\right)^{2^{k-1}} n^{2k-1}$.

Proof. For $k = 2$ this follows from Corollary 2.6. Proceeding by induction on k, we assume $k \geqslant 3$ and that the assertion holds for $k-1$ in place of k. Construct an auxiliary bipartite graph Γ with vertex classes V and V^{2k-3} by drawing an edge between $x \in V$ and

$$(x_1, \dots, x_{k-2}, x_k, \dots, x_{2k-2}) \in V^{2k-3}$$

if and only if

(a) $x_1 \dots x_{k-2} x_k \dots x_{2k-2}$ is a walk in $\overline{H}_x$,
(b) $\{x_1, \dots, x_{k-2}, x_k\} \in E(\overline{H}'_x)$,
(c) $d_{\overline{H}_x}(x_2, \dots, x_{k-2}, x_k) \geqslant \frac{n}{3}$ and $d_{\overline{H}_x}(x_{k+1}, \dots, x_{2k-2}) \geqslant \frac{n}{3}$.

The induction hypothesis, applied to the hypergraphs $\overline{H}_x$ and $\overline{H}'_x$, reveals that every vertex $x \in V$ has at least degree $\left(\frac{\alpha}{2}\right)^{2^{k-2}} n^{2k-3}$ in Γ. Thus

$$e(\Gamma) \geqslant \left(\frac{\alpha}{2}\right)^{2^{k-2}} n^{2k-2}$$

and the Cauchy-Schwarz inequality implies

$$\sum_{\vec{x} \in V^{2k-3}} |N_\Gamma(\vec{x})|^2 \geqslant \frac{e(\Gamma)^2}{n^{2k-3}} \geqslant \left(\frac{\alpha}{2}\right)^{2^{k-1}} n^{2k-1} \,,$$

where $N_\Gamma(\vec{x})$ denotes the neighbourhood of the vertex $\vec{x}$ in Γ. Now if

$$\vec{x} = (x_1, \dots, x_{k-2}, x_k, \dots, x_{2k-2}) \in V^{2k-3} \quad \text{and} \quad x_{k-1}, x_{2k-1} \in N_\Gamma(\vec{x})$$

are arbitrary, then $(x_1, \dots, x_{2k-1})$ has the desired properties. □

Walks in dense hypergraphs. For later use we now quote a lower bound on the number of walks of given length in a given dense hypergraph, that is somewhat related to Sidorenko's conjecture [20, 21]. It is well known that this conjecture holds for paths in graphs, i.e., that for $d \in [0, 1]$ and $\ell \in \mathbb{N}$ every graph $G = (V, E)$ satisfying $|E| \geqslant d|V|^2/2$ contains at least $d^\ell |V|^{\ell+1}$ walks of length ℓ (see [3] for a proof based on linear algebra and [1, Lemma 3.8] for a different approach using vertex deletions and the tensor power trick). The latter argument generalises in a straightforward manner to partite hypergraphs (see Lemma 2.8 below). An alternative proof based on the entropy method was worked out by Fitch [9, Lemma 7] and by Lee [14, Theorems 2.6 and 2.7].

Lemma 2.8. *Suppose $k \geqslant 2$, $d \in [0, 1]$, and that H is a k-partite k-uniform hypergraph with vertex partition $\{V_i \colon i \in \mathbb{Z}/k\mathbb{Z}\}$. If H has $d \prod_{i \in \mathbb{Z}/k\mathbb{Z}} |V_i|$ edges, then for every $r \geqslant k$ there are at least*

$$d^{r-k+1} \prod_{i \in [r]} |V_i|$$

walks $(x_1, \dots, x_r)$ in H with $x_1 \in V_1, \dots, x_k \in V_k$. □

By identifying the vertex classes one obtains the following, more standard, non-partite version of this lemma.

Corollary 2.9. *For $k \geqslant 2$ and $d \in [0,1]$ let $H = (V, E)$ be a k-uniform hypergraph. If $|E| \geqslant d|V|^k/k!$, then for every integer $r \geqslant k$ there are at least $d^{r-k+1}|V|^r$ walks $(x_1, \dots, x_r)$ in H.* □

2.3. **Abstract connectability.** Our intended way of using Proposition 2.2 is that given a k-uniform hypergraph H satisfying $\delta_{k-2}(H) \geqslant \big(\frac{5}{9} + \alpha\big)|V(H)|^2/2$ we can choose robust subgraphs of all the $\binom{|V(H)|}{k-2}$ link graphs. It will be convenient to collect the data thus arising into a single structure.

Definition 2.10. *For $k \geqslant 2$ a k-uniform constellation is a pair*

$$\Psi = \big(H, \big\{R_x \colon x \in V(H)^{(k-2)}\big\}\big)$$

consisting of a k-uniform hypergraph H and a family of induced subgraphs $R_x \subseteq H_x$ of the 2-uniform link hypergraphs that can be formed in H. We write $H(\Psi) = H$ for the underlying hypergraph of a constellation Ψ and use the abbreviations $V(\Psi) = V(H)$, $E(\Psi) = E(H)$ for its vertex set and edge set, respectively. For a constellation Ψ and $x \in V(\Psi)^{(k-2)}$ we denote the subgraph associated with x by $R_x^\Psi = R_x$.

Example 2.11. A 2-uniform constellation is determined by its underlying graph H and a distinguished induced subgraph $R_\varnothing \subseteq H_\varnothing = H$.

Notice that so far the induced subgraphs $R_x \subseteq H_x$ are completely arbitrary and at this moment there are no restrictions on their orders, sizes, and connectivity properties. This allows us to study constellations "axiomatically", adding further useful conditions in each of the following subsections. The central connectability notions are definable without any such assumptions and they will be introduced in the present subsection (see Definition 2.14 below). Of course one cannot prove a meaningful Connecting Lemma at this level of generality, so our way of organising the material may appear somewhat peculiar on first sight. When establishing the covering lemma in Section 6 however, we need to analyse connectability in random subconstellations and for such situations the abstract approach developed here turns out to be advantageous. Subconstellations themselves are defined in the expected way.

Definition 2.12. *Let*

$$\Psi = \big(H, \big\{R_x \colon x \in V(H)^{(k-2)}\big\}\big)$$

be a k-uniform constellation, where $k \geqslant 2$. For $X \subseteq V(\Psi)$ we call

$$\Psi[X] = \big(H[X], \big\{R_x[X] \colon x \in X^{(k-2)}\big\}\big)$$

the subconstellation of Ψ *induced by* X*. Moreover,* $\Psi - X = \Psi[V(\Psi) \smallsetminus X]$ *denotes the constellation obtained from* Ψ *by removing* X*.*

We can also form link constellations in the obvious way.

Definition 2.13. *Let* $k \geqslant 2$ *and let*

$$\Psi = \big(H, \big\{R_x \colon x \in V(H)^{(k-2)}\big\}\big)$$

be a k*-uniform constellation. If* $S \subseteq V(\Psi)$ *and* $|S| \leqslant k-2$*, then the* $(k-|S|)$*-uniform link constellation* Ψ_S *is defined to be*

$$\Psi_S = \big(\overline{H}_S, \big\{R_{x \cup S} - S \colon x \in (V(H) \smallsetminus S)^{(k-2-|S|)}\big\}\big)\,.$$

Next we tell which $(k-1)$-tuples of vertices of a k-uniform constellation are regarded as being ζ-leftconnectable for a given real number $\zeta > 0$. The definition progresses by recursion on k.

Definition 2.14. *Let* $k \geqslant 2$, $\zeta > 0$, *let*

$$\Psi = \big(H, \big\{R_x \colon x \in V(H)^{(k-2)}\big\}\big)$$

be a k*-uniform constellation, and let* $\vec{x} = (x_1, \dots, x_{k-1}) \in V(\Psi)^{k-1}$ *be a* $(k-1)$*-tuple of distinct vertices.*

(a) *If* $k = 2$ *we say that* $\vec{x} = (x_1)$ *is* ζ*-leftconnectable in* Ψ *if* $x_1 \in V(R_\varnothing)$*.*

(b) *If* $k \geqslant 3$ *we say that* $\vec{x}$ *is* ζ*-leftconnectable in* Ψ *if*

$$|U^\Psi_{\vec{x}}| \geqslant \zeta |V(\Psi)|\,,$$

where

$$U^\Psi_{\vec{x}} = \big\{z \in V(\Psi) \colon x_1 \dots x_{k-1} z \in E(\Psi) \text{ and } (x_2, \dots, x_{k-1}) \text{ is } \zeta\text{-leftconnectable in } \Psi_z\big\}\,.$$

We remark that this is a "new" concept in the sense that in the earlier articles [15, 17] we managed to work with symmetric notions of connectability. For this reason, we need to be careful when quoting the Connecting Lemma from [17] later.

Example 2.15. Let (x_1, x_2) be a pair of distinct vertices from a 3-uniform constellation Ψ and let $\zeta > 0$. According to part (b) of Definition 2.14 the pair (x_1, x_2) is ζ-leftconnectable in Ψ if and only if $|U^\Psi_{(x_1,x_2)}| \geqslant \zeta |V(\Psi)|$. Due to part ($a$) the definition of this set unravels to

$$U^\Psi_{(x_1,x_2)} = \big\{z \in V(\Psi) \colon x_1 x_2 z \in E(\Psi) \text{ and } x_2 \in V(R^\Psi_z)\big\}\,.$$

There is a dual notion of rightconnectability obtained by reversing the ordering of the vertices.

Definition 2.16. *Let $k \geqslant 2$, $\zeta > 0$, Ψ, and $\vec{x} \in V(\Psi)^{k-1}$ be as in Definition 2.14.*

(a) If the reverse tuple $(x_{k-1}, \dots, x_1)$ is ζ-leftconnectable, then $\vec{x}$ itself is said to be ζ-rightconnectable.

(b) Further, $\vec{x}$ is called ζ-connectable if it is ζ-leftconnectable and ζ-rightconnectable.

Some readers may react negatively to our choice of the specifiers 'left' and 'right' in these notions, arguing that the definition of leftconnectability of $\vec{x}$ pivots on the right end-segment of $\vec{x}$. The reason for our terminological choice is that the Connecting Lemma (Proposition 3.1 below) will assert that under reasonable assumptions every leftconnectable tuple can be connected to every rightconnectable tuple in such a way that the leftconnectable tuple is 'on the left side' in the resulting path, while the rightconnectable tuple is 'on the right side'.

The following observation follows by a straightforward induction from Definition 2.14. In later sections we will often use it either tacitly or by referring to 'monotonicity'.

Fact 2.17. *For a k-uniform constellation Ψ and $\zeta > \zeta' > 0$ every ζ-leftconnectable $(k-1)$-tuple is also ζ'-leftconnectable. Similarly statements hold for rightconnectability and connectability.*

Proof. It suffices to display the argument for leftconnectability. We argue by induction on k. In the base case $k = 2$ the definition of ζ-leftconnectability does not depend on ζ and there is nothing to prove. Now let $k \geqslant 3$ and suppose that the assertion is true for $k-1$ playing the rôle of k.

Let $\zeta > \zeta' > 0$, let $\Psi = \big(H, \big\{R_x \colon x \in V(H)^{(k-2)}\big\}\big)$ be a k-uniform constellation, and let $\vec{x} = (x_1, \dots, x_{k-1}) \in V(\Psi)^{k-1}$ be a ζ-leftconnectable $(k-1)$-tuple. We are to prove that $\vec{x}$ is ζ'-leftconnectable as well. To this end we consider the sets

$$U = \{z \in V(\Psi) \colon x_1 \dots x_{k-1} z \in E(\Psi) \text{ and } (x_2, \dots, x_{k-1}) \text{ is } \zeta\text{-leftconnectable in } \Psi_z\}$$

and

$$W = \{z \in V(\Psi) \colon\ x_1 \dots x_{k-1} z \in E(\Psi) \text{ and } (x_2, \dots, x_{k-1}) \text{ is } \zeta'\text{-leftconnectable in } \Psi_z\}\,.$$

The induction hypothesis discloses $U \subseteq W$ and the assumption that $\vec{x}$ is ζ-leftconnectable means that $|U| \geqslant \zeta |V(\Psi)|$. So altogether we have

$$|W| \geqslant |U| \geqslant \zeta |V(\Psi)| \geqslant \zeta' |V(\Psi)|\,,$$

for which reason $\vec{x}$ is indeed ζ'-leftconnectable. $\square$

Next, we study connectability in subconstellations.

Fact 2.18. *Suppose that Ψ is a k-uniform constellation, that $\Psi' = \Psi[X]$ is a subconstellation induced by some $X \subseteq V(\Psi)$ with $|X| \geqslant \frac{1}{2}\big(|V(\Psi)| + k - 2\big)$. If $\vec{x} \in V(\Psi')^{k-1}$ is (2ζ)-leftconnectable in Ψ', then it is ζ-leftconnectable in Ψ as well. Similar statements hold for 'rightconnectability' and 'connectability'.*

Proof. Again we only display the argument for leftconnectability and proceed by induction on k. The base case $k = 2$ is trivial. For the induction step from $k-1$ to k we recall that the assumption entails $|U| \geqslant 2\zeta|V(\Psi')| \geqslant \zeta|V(\Psi)|$, where

$$U = \big\{z \in V(\Psi')\colon x_1 \dots x_{k-1}z \in E(\Psi') \text{ and } (x_2, \dots, x_{k-1}) \text{ is } (2\zeta)\text{-leftconnectable in } \Psi'_z\big\}\,.$$

Now consider an arbitrary vertex $z \in U$. Since

$$|V(\Psi'_z)| = |V(\Psi')| - 1 \geqslant \frac{1}{2}\big(|V(\Psi)| + k - 4\big) = \frac{1}{2}\big(|V(\Psi_z)| + k - 3\big)\,,$$

the induction hypothesis is applicable to the constellation Ψ_z, its subconstellation Ψ'_z, and to the (2ζ)-leftconnectable $(k-2)$-tuple $(x_2, \dots, x_{k-1})$. It follows that

$$U \subseteq \big\{z \in V(\Psi)\colon x_1 \dots x_{k-1}z \in E(\Psi) \text{ and } (x_2, \dots, x_{k-1}) \text{ is } \zeta\text{-leftconnectable in } \Psi_z\big\}$$

and together with $|U| \geqslant \zeta|V(\Psi)|$ this shows that $\vec{x}$ is indeed ζ-leftconnectable in Ψ. □

We shall frequently have the situation that for some edge $x_1 \dots x_k$ of a k-uniform constellation Ψ we know $x_k \in V(R^{\Psi}_{x_1 \dots x_{k-2}})$ and we would like to conclude from this state of affairs that $(x_2, \dots, x_k)$ is ζ-leftconnectable in Ψ. While such deductions are invalid in general, it turns out that for small values of ζ there are only few exceptions to this rule of inference. More precisely, we have the following result (cf. [17, Fact 4.1] and [15, Lemma 3.7] for similar statements).

Lemma 2.19. *Let $k \geqslant 2$ and $\zeta > 0$ be given. If Ψ is a k-uniform constellation, then there exist at most $(k-2)\zeta|V(\Psi)|^k$ k-tuples $(x_1, \dots, x_k) \in V(\Psi)^k$ such that*

- *(a) $\{x_1, \dots, x_k\} \in E(\Psi)$,*
- *(b) $x_k \in V(R^{\Psi}_{x_1 \dots x_{k-2}})$,*
- *(c) and $(x_2, \dots, x_k)$ fails to be ζ-leftconnectable in Ψ.*

Proof. We argue by induction on k. In the base case $k = 2$ the demands (b) and (c) contradict each other and, hence, there are indeed no such pairs. Now let $k \geqslant 3$ and suppose that the lemma is true for $k-1$ in place of k. Define $A \subseteq V(\Psi)^k$ to be the set of all k-tuples satisfying (a)–(c), set

$$A' = \big\{(x_1, \dots, x_k) \in A\colon x_1 \in U^{\Psi}_{(x_2, \dots, x_k)}\big\}$$

and define

$$A''_x = \big\{(x_2, \dots, x_k) \in V(\Psi)^{k-1} \colon (x, x_2, \dots, x_k) \in A \smallsetminus A'\big\}$$

for every $x \in V(\Psi)$. Since

$$|A| = |A'| + \sum_{x \in V(\Psi)} |A''_x|\,,$$

it suffices to show

(*1*) $|A'| \leqslant \zeta |V(\Psi)|^k$

(*2*) and $|A''_x| \leqslant (k-3)\zeta |V(\Psi_x)|^{k-1}$ for every $x \in V(\Psi)$.

Now (*1*) follows from the fact that for $(x_1, \dots, x_k) \in A' \subseteq A$ we have

$$\big|U^{\Psi}_{(x_2,\dots,x_k)}\big| < \zeta |V(\Psi)|$$

by (*c*) and the definition of ζ-leftconnectability. For the proof of (*2*) we apply the induction hypothesis to the link constellation Ψ_x. Notice that if $(x_2, \dots, x_k) \in A''_x$, then

- $\{x_2, \dots, x_k\} \in E(\Psi_x)$
- and $x_k \in V(R^{\Psi_x}_{x_2 \dots x_{k-2}})$

follow from (*a*), (*b*), and the definition of Ψ_x. Moreover $(x, x_2, \dots, x_k) \in A \smallsetminus A'$ yields $x \notin U^{\Psi}_{(x_2,\dots,x_k)}$, which together with $\{x, x_2, \dots, x_k\} \in E(\Psi)$ reveals that

$$(x_3, \dots, x_k) \text{ fails to be } \zeta\text{-leftconnectable in } \Psi_x\,.$$

So altogether the induction hypothesis leads to (*2*) and the induction step is complete. □

We proceed with a similar statement that will ultimately assist us in the construction of the absorbing path.

Lemma 2.20. *For $k \geqslant 2$, $\zeta > 0$, and a k-uniform constellation Ψ, there are at most $(k-2)\zeta |V(\Psi)|^{2k-3}$ walks $x_1 \dots x_{2k-3}$ in $H(\Psi)$ such that*

(*a*) $x_{k-1} \in V(R^{\Psi}_{x_k \dots x_{2k-3}})$

(*b*) *but $(x_1, \dots, x_{k-1})$ fails to be ζ-leftconnectable.*

Proof. Again we argue by induction on k. In the base case $k = 2$ condition (*a*) reads $x_1 \in V(R^{\Psi}_{\varnothing})$, whereas (*b*) demands that (x_1) fails to be ζ-leftconnectable in Ψ. As these requirements contradict each other, there are indeed no 1-vertex walks with the required properties.

Now let $k \geqslant 3$ and assume that the lemma is true for $k-1$ instead of k. Let $A \subseteq V(\Psi)^{2k-3}$ be the set of all walks $x_1 \dots x_{2k-3}$ satisfying (*a*) and (*b*), set

$$A' = \big\{(x_1, \dots, x_{2k-3}) \in A \colon x_k \in U^{\Psi}_{(x_1,\dots,x_{k-1})}\big\}$$

and put

$$A''_{x,y} = \big\{(x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_{2k-3}) \in V(\Psi)^{2k-5} \colon \\ (x, x_2, \dots, x_{k-1}, y, x_{k+1}, \dots, x_{2k-3}) \in A \smallsetminus A'\big\}$$

for all $x, y \in V(\Psi)$. In view of

$$|A| = |A'| + \sum_{(x,y)\in V(\Psi)^2} |A''_{x,y}|$$

it suffices to prove

(1) $|A'| \leqslant \zeta |V(\Psi)|^{2k-3}$

(2) and $|A''_{x,y}| \leqslant (k-3)\zeta|V(\Psi_y)|^{2k-5}$ for all $x, y \in V(\Psi)$.

The estimate *(1)* follows from the fact that due to *(b)* every $(x_1, \dots, x_{2k-3}) \in A' \subseteq A$ has the property $\big|U^{\Psi}_{(x_1,\dots,x_{k-1})}\big| < \zeta\big|V(\Psi)\big|$. For the proof of *(2)* we intend to apply the induction hypothesis to Ψ_y. Consider any $(2k-5)$-tuple

$$\vec{x} = (x_2, \dots, x_{k-1}, x_{k+1}, \dots, x_{2k-3}) \in A''_{x,y}\,.$$

Since $(x_2, \dots, x_{k-1}, y, x_{k+1}, \dots, x_{2k-3})$ is a walk in $H(\Psi)$, we know that $\vec{x}$ itself is a walk in $H(\Psi_y)$. Moreover, *(a)* rewrites as

$$x_{k-1} \in V(R^{\Psi_y}_{x_{k+1}\dots x_{2k-3}})\,.$$

Finally, $y \notin U^{\Psi}_{(x,x_2,\dots,x_{k-1})}$ and $\{x, x_2, \dots, x_{k-1}, y\} \in E(\Psi)$ imply that

$$(x_2, \dots, x_{k-1}) \text{ fails to be } \zeta\text{-leftconnectable in } \Psi_y\,.$$

Altogether, the $(2k-5)$-tuples in $A''_{x,y}$ have the required properties for applying the induction hypothesis to Ψ_y. This proves *(2)* and completes the induction step. □

We conclude this subsection by introducing one further notion.

Definition 2.21. *Given $k \geqslant 2$, $\zeta > 0$, and a k-uniform constellation*

$$\Psi = \big(H, \big\{R_x \colon x \in V(H)^{(k-2)}\big\}\big)\,,$$

a k-tuple $(x_1, \dots, x_k) \in V(\Psi)^k$ is said to be a ζ-bridge in Ψ if

(a) *$\{x_1, \dots, x_k\} \in E(\Psi)$,*

(b) *$(x_1, \dots, x_{k-1})$ is ζ-rightconnectable,*

(c) *and $(x_2, \dots, x_k)$ is ζ-leftconnectable.*

Such bridges will help us later to construct connecting paths between given $(k-1)$-tuples of vertices. The fundamental existence result for such bridges (see Corollary 2.28 below) asserts, roughly speaking, that under natural assumptions k-uniform constellations contain many ζ-bridges for sufficiently small values of ζ.

2.4. **On (α,μ)-constellations.** In this subsection we study some properties of constellations that can be deduced from the order and size restrictions *(i)* and *(ii)* in Proposition 2.2 alone without taking the (β,ℓ)-robustness into account. We are thus led to the following concept.

Definition 2.22. *Let $k \geqslant 2$ and $\alpha,\mu > 0$. A k-uniform constellation Ψ is said to be an (α,μ)-constellation if*

$$\delta_{k-2}\big(H(\Psi)\big) \geqslant \Big(\frac{5}{9}+\alpha\Big)\frac{|V(\Psi)|^2}{2}$$

and every $x \in V(\Psi)^{(k-2)}$ satisfies

(*a*) $|V(R_x^\Psi)| \geqslant \big(\frac{2}{3}+\frac{\alpha}{2}\big)|V(\Psi)|$
(*b*) *as well as* $e_{H(\Psi_x)}\big(V(R_x^\Psi), V(\Psi)\smallsetminus V(R_x^\Psi)\big) \leqslant \mu|V(\Psi)|^2$.

It turns out that the level of generality provided by this concept is fully appropriate for discussing the key parts of our absorbing mechanism and for constructing an important building block entering the proof of the Connecting Lemma. Before reaching those results we record a couple of easier observations.

Fact 2.23. *If Ψ denotes a k-uniform $(\alpha,\frac{\alpha}{9})$-constellation for some $\alpha > 0$, then*

$$e\big(H(\Psi_x)\big) - e(R_x^\Psi) \leqslant \frac{|V(\Psi)|^2}{18}$$

holds for every $x \in V(\Psi)^{(k-2)}$.

Proof. Using both parts of Definition 2.22 we obtain

$$\begin{aligned} e\big(H(\Psi_x)\big) - e(R_x^\Psi) &= e_{H(\Psi_x)}\big(V(\Psi)\smallsetminus V(R_x^\Psi)\big) + e_{H(\Psi_x)}\big(V(R_x^\Psi), V(\Psi)\smallsetminus V(R_x^\Psi)\big) \\ &\leqslant \Big(\frac{1}{3}-\frac{\alpha}{2}\Big)^2\frac{|V(\Psi)|^2}{2} + \frac{\alpha}{9}|V(\Psi)|^2 = \Big(\frac{1}{18}+\frac{\alpha^2}{8}-\frac{\alpha}{18}\Big)|V(\Psi)|^2 \end{aligned}$$

and it remains to observe that the minimum $(k-2)$-degree condition imposed on $H(\Psi)$ is only satisfiable for $\alpha \leqslant \frac{4}{9}$. □

Fact 2.24. *Suppose that Ψ is a k-uniform (α,μ)-constellation. If $x \in V(\Psi)^{(k-2)}$ is arbitrary, then there are at most $\frac{2\mu}{\alpha}|V(\Psi)|$ vertices $z \in V(\Psi)\smallsetminus V(R_x^\Psi)$ with $d_{H(\Psi_x)}(z) > \frac{1}{3}(|V(\Psi)|-2)$.*

Proof. Definition 2.22 (*a*) tells us that $|V(\Psi)\smallsetminus V(R_x^\Psi)| \leqslant (\frac{1}{3}-\frac{\alpha}{2})|V(\Psi)|$. Consequently, the number of edges that every vertex z from the set

$$Z = \big\{z \in V(\Psi)\smallsetminus V(R_x^\Psi)\colon d_{H(\Psi_x)}(z) > \tfrac{1}{3}(|V(\Psi)|-2)\big\}$$

sends to $V(R^{\Psi}_x)$ is at least

$$d_{H(\Psi_x)}(z) - \big|V(\Psi) \smallsetminus \big(V(R^{\Psi}_x) \cup \{z\}\big)\big| \geqslant \frac{1}{3}\big(|V(\Psi)| - 2\big) - \Big(\frac{1}{3} - \frac{\alpha}{2}\Big)|V(\Psi)| + 1$$
$$> \frac{\alpha}{2}|V(\Psi)|\,.$$

In combination with Definition 2.22 *(b)* this yields

$$\frac{\alpha}{2}|V(\Psi)||Z| \leqslant e_{H(\Psi_x)}\big(V(R^{\Psi}_x), V(\Psi) \smallsetminus V(R^{\Psi}_x)\big) \leqslant \mu|V(\Psi)|^2$$

and the upper bound $|Z| \leqslant \frac{2\mu}{\alpha}|V(\Psi)|$ we are aiming for follows. □

Next, there is an obvious monotonicity statement.

Fact 2.25. *For $k \geqslant 2$, $\alpha \geqslant \alpha' > 0$, and $\mu' \geqslant \mu > 0$, every k-uniform (α, μ)-constellation is an (α', μ')-constellation as well.* □

Link constellations 'almost' inherit being (α, μ)-constellations, but since we are slightly shrinking the vertex set we need to be careful with clause *(b)* of Definition 2.22.

Fact 2.26. *Given $k \geqslant 2$, $\alpha > 0$, and $\mu' > \mu > 0$ there exists a natural number n_0 with the following property. If Ψ denotes a k-uniform (α, μ)-constellation having at least n_0 vertices and $S \subseteq V(\Psi)$ with $|S| \leqslant k-2$ is arbitrary, then Ψ_S is a $(k - |S|)$-uniform (α, μ')-constellation.* □

Now we estimate the number of walks of any short length in Ψ, whose starting $(k-1)$-tuple is rightconnectable and whose ending $(k-1)$-tuple is leftconnectable. Later we will use these walks in the proof of the Connecting Lemma thus gaining control over the length of the connections modulo k.

Lemma 2.27. *For $k \geqslant 2$ and $\alpha > 0$ let Ψ be a k-uniform $(\alpha, \frac{\alpha}{9})$-constellation. Provided that $|V(\Psi)| \geqslant \frac{k^2}{\alpha}$, there are for every positive integer r at least $\frac{1}{3^{r+1}}|V(\Psi)|^{r+k-1}$ walks $x_1x_2\dots x_{r+k-1}$ of length r in $H(\Psi)$ starting with a $\frac{1}{k3^{r+1}}$-rightconnectable $(k-1)$-tuple $(x_1, \dots, x_{k-1})$ and ending with a $\frac{1}{k3^{r+1}}$-leftconnectable $(k-1)$-tuple $(x_{r+1}, \dots, x_{r+k-1})$.*

Proof. Consider the auxiliary k-partite k-uniform hypergraph $\mathscr{A}$ whose vertex classes $V_1, \dots, V_k$ are copies of $V(\Psi)$ and whose edges $\{x_1, \dots, x_k\} \in E(\mathscr{A})$ with

$$x_1 \in V_1, \dots, x_k \in V_k$$

signify that

(1) $\{x_1, \dots, x_k\} \in E(\Psi)$,
(2) $x_1x_2 \in E(R^{\Psi}_{x_3\dots x_k})$,
(3) and $x_{r+k-2}x_{r+k-1} \in E(R^{\Psi}_{x_r\dots x_{r+k-3}})$,

where the indices in *(3)* are to be read modulo k.

In view of $|V(\Psi)| \geqslant \frac{k^2}{\alpha}$ and $\delta_{k-2}(H(\Psi)) \geqslant \left(\frac{5}{9}+\alpha\right)\frac{|V(\Psi)|^2}{2}$ there are at least

$$(|V(\Psi)|-k)^{k-2}\cdot\left(\frac{5}{9}+\alpha\right)|V(\Psi)|^2 \geqslant \frac{5}{9}|V(\Psi)|^k$$

possibilities $(x_1,\dots,x_k) \in V_1\times\dots\times V_k$ satisfying *(1)*. Among them, there are by Fact 2.23 at most $\frac{1}{9}|V(\Psi)|^k$ violating *(2)* and at most the same number violating *(3)*. Consequently, $e(\mathscr{A}) \geqslant \frac{1}{3}|V(\Psi)|^k$ and Lemma 2.8 applied to $\mathscr{A}$ and $d=\frac{1}{3}$ shows that there are at least $\frac{1}{3^r}|V(\Psi)|^{r+k-1}$ walks

$$x_1x_2\dots x_{r+k-1}$$

of length r in $\mathscr{A}$ with $x_1\in V_1,\dots,x_k\in V_k$. Among them, there are by *(2)* and Lemma 2.19 applied to $\zeta=\frac{1}{k3^{r+1}}$ at most

$$\frac{k-2}{k3^{r+1}}|V(\Psi)|^{r+k-1} < \frac{1}{3^{r+1}}|V(\Psi)|^{r+k-1}$$

walks for which $(x_1,\dots,x_{k-1})$ fails to be $\frac{1}{k3^{r+1}}$-rightconnectable. Similarly *(3)* and Lemma 2.19 ensure that at most $\frac{1}{3^{r+1}}|V(\Psi)|^{r+k-1}$ of our walks have the defect that $(x_{r+1},\dots,x_{r+k-1})$ fails to be $\frac{1}{k3^{r+1}}$-leftconnectable. This leaves us with at least

$$\left(\frac{1}{3^r}-\frac{2}{3^{r+1}}\right)|V(\Psi)|^{r+k-1} = \frac{|V(\Psi)|^{r+k-1}}{3^{r+1}}$$

walks of the desired form. □

Corollary 2.28. *Given $k\geqslant 2$ and $\alpha>0$ let Ψ be a k-uniform $(\alpha,\frac{\alpha}{9})$-constellation. If Ψ has at least $\frac{k^2}{\alpha}$ vertices, then the number of its $\frac{1}{9k}$-bridges is at least $\frac{1}{9}|V(\Psi)|^k$.*

Proof. Plug $r=1$ into Lemma 2.27. □

The following lemma builds a device that will assist us in the inductive proof of the Connecting Lemma in the next section.

Lemma 2.29. *Given $k\geqslant 4$, $\alpha>0$, and $\zeta\in\left(0,\frac{1}{3^{k+2}}\right]$, there exists an integer n_0 such that the following holds for every k-uniform $(\alpha,\frac{\alpha}{10})$-constellation Ψ on $n\geqslant n_0$ vertices.*

If two subsets $U,W\subseteq V(\Psi)$ satisfy $|U|,|W|\geqslant\zeta n$, then there are at least ζ^3n^{2k-2} $(2k-2)$-tuples $(u,q_1,\dots,q_{2k-4},w)\in V(\Psi)^{2k-2}$ such that

(i) $u\in U$ *and* $w\in W$ *are distinct,*
(ii) $q_1\dots q_{2k-4}$ *is a walk in* $H(\Psi_{uw})$,
(iii) $(q_1,\dots,q_{k-2})$ *is* ζ^3*-rightconnectable in* Ψ_u,
(iv) and $(q_{k-1},\dots,q_{2k-4})$ *is* ζ^3*-leftconnectable in* Ψ_w.

Proof. Assuming that n_0 has been chosen sufficiently large for the subsequent arguments, we commence by considering the $(2k-2)$-tuples $(u, q_1, \dots, q_{2k-4}, w) \in V(\Psi)^{2k-2}$ satisfying *(i)*, *(ii)* as well as the conditions

(v) $(q_1, \dots, q_{k-3})$ is ζ^3-rightconnectable in Ψ_{uw},

(vi) $(q_k, \dots, q_{2k-4})$ is ζ^3-leftconnectable in Ψ_{uw}.

First of all, by $|U|, |W| \geqslant \zeta n$ and $n \geqslant n_0 \geqslant 2/\zeta$ there are at least $\frac{1}{2}\zeta^2 n^2$ pairs (u, w) in $U \times W$ with $u \neq w$. For each of these pairs Fact 2.26 tells us that Ψ_{uw} is a $(k-2)$-uniform $(\alpha, \frac{\alpha}{9})$-constellation. Applying the case $r = k-1$ of Lemma 2.27 to this constellation we learn that the number of $(2k-4)$-tuples $(q_1, \dots, q_{2k-4}) \in V(\Psi_{uw})^{2k-4}$ obeying *(ii)*, *(v)*, and *(vi)* is at least $\frac{1}{3^k}(n-2)^{2k-4} \geqslant \frac{6}{3^{k+2}} n^{2k-4} \geqslant 6\zeta n^{2k-4}$.

Summarising, the number of $(2k-2)$-tuples $(u, q_1, \dots, q_{2k-4}, w)$ satisfying *(i)*, *(ii)*, *(v)*, and *(vi)* is at least $\frac{1}{2}\zeta^2 n^2 \cdot 6\zeta n^{2k-4} = 3\zeta^3 n^{2k-2}$. So it suffices to prove that among all $(2k-2)$-tuples $(u, q_1, \dots, q_{2k-4}, w) \in V(\Psi)^{2k-2}$ there are

(1) at most $\zeta^3 n^{2k-2}$ with *(ii)*, *(v)*, $\neg$*(iii)*

(2) and at most $\zeta^3 n^{2k-2}$ with *(ii)*, *(vi)*, $\neg$*(iv)*.

For reasons of symmetry we only need to establish *(2)*. To this end it is enough to check that for fixed vertices $w, q_1, \dots, q_{2k-4} \in V(\Psi)$ the number of vertices u such that

- $\{u, q_{k-1}, \dots, q_{2k-4}\} \in E(\Psi_w)$,
- *(vi)*, but $\neg$*(iv)*.

is at most $\zeta^3 n$. Now by Definition 2.14, the first bullet, and *(vi)* these vertices satisfy $u \in U^{\Psi_w}_{(q_{k-1}, \dots, q_{2k-4})}$ and by $\neg$*(iv)* the latter set has size at most $\zeta^3 |V(\Psi_w)|$. □

The last lemma of this subsection will help us to exchange arbitrary vertices by 'absorbable' ones in Section 5. Roughly speaking it asserts that for $\mu \ll \alpha, k^{-1}$, with few exceptions, the links of two vertices in a k-uniform (α, μ)-constellation intersect in a substantial number of connectable $(k-1)$-tuples.

Lemma 2.30. *Given $k \geqslant 3$ and $\alpha > 0$ set $\mu = \frac{1}{10k}\left(\frac{\alpha}{2}\right)^{2^{k-3}+1}$. If Ψ denotes a k-uniform (α, μ)-constellation on n vertices and $\zeta > 0$ is arbitrary, then there is a set $X \subseteq V(\Psi)$ of size $|X| \leqslant \frac{\zeta}{\mu} n$ such that for every $a \in V(\Psi)$ and every $x \in V(\Psi) \smallsetminus (X \cup \{a\})$ the number of ζ-connectable $(k-1)$-tuples $(x_1, \dots, x_{k-1})$ with $\{x_1, \dots, x_{k-1}\} \in E(\Psi_a) \cap E(\Psi_x)$ is at least $\mu |V(\Psi)|^{k-1}$.*

Proof. Set

$$\eta = \frac{1}{10}\left(\frac{\alpha}{2}\right)^{2^{k-3}} \tag{2.7}$$

and $V = V(\Psi)$. Since $\mu = \frac{\alpha\eta}{2k}$, we have

$$\max\left\{\frac{2\mu}{\alpha},\, 2k\mu\right\} \leqslant \eta\,. \tag{2.8}$$

The choice of X. With every $x \in V$ we shall associate two exceptional sets, the idea being that on average these sets can be proved to be small. So there will only be few vertices for which one of the exceptional sets is very large and these 'unpleasant vertices' will form the set X. For every vertex not belonging to X, we will then be able to show that its link constellation intersect the link constellations of all other vertices in the desired way.

For an arbitrary $x \in V$ the first of the exceptional sets A_x consists of all $(k-1)$-tuples $(x_1, \dots, x_{k-1}) \in V^{k-1}$ satisfying

- $\{x_1, \dots, x_{k-1}, x\} \in E(\Psi)$
- and $x_1 \in V(R^{\Psi}_{x_3 \dots x_{k-1} x})$
- that fail to be ζ-rightconnectable in Ψ.

We would like to point out that the second bullet does not involve the vertex x_2. Moreover, in the special case $k = 3$ the condition just means that $x_1 \in V(R^{\Psi}_x)$.

The second exceptional set B_x comprises all $(2k-4)$-tuples $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2k-3})$ in V^{2k-4} such that

- $x_1 \dots x_{k-1} x x_{k+1} \dots x_{2k-3}$ is a walk in $H(\Psi)$
- and $x_{k-1} \in V(R^{\Psi}_{x x_{k+1} \dots x_{2k-3}})$,
- for which $(x_1, \dots, x_{k-1})$ fails to be ζ-leftconnectable in Ψ.

Now we define

$$X' = \left\{x \in V \colon |A_x| > 2k\mu|V|^{k-1}\right\},$$
$$X'' = \left\{x \in V \colon |B_x| > 2k\mu|V|^{2k-4}\right\},$$

and set $X = X' \cup X''$. By Lemma 2.19 and double counting we have

$$2k\mu|X'||V|^{k-1} \leqslant \sum_{x \in X'} |A_x| \leqslant (k-2)\zeta|V|^k\,,$$

whence $|X'| \leqslant \frac{\zeta}{2\mu}|V|$. Similarly, Lemma 2.20 yields

$$2k\mu|X''||V|^{2k-4} \leqslant \sum_{x \in X''} |B_x| \leqslant (k-2)\zeta|V|^{2k-3}\,,$$

which shows that $|X''| \leqslant \frac{\zeta}{2\mu}|V|$ holds as well. Altogether we arrive at the desired estimate

$$|X| \leqslant |X'| + |X''| \leqslant \frac{\zeta}{\mu}|V|\,.$$

For the rest of the proof we fix two distinct vertices $a, x \in V$ with $x \notin X$. We are to show that the number of ζ-connectable $(k-1)$-tuples $(x_1, \dots, x_{k-1})$ such that

$$\{x_1, \dots, x_{k-1}\} \in E(\Psi_a) \cap E(\Psi_x)$$

is at least $\mu|V|^{k-1}$. The smallest case $k = 3$ receives a separate treatment.

The special case $k = 3$. We know that both of the graphs $H(\Psi_a)$ and $H(\Psi_x)$ have at least $\left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ edges and thus they have at least $\left(\frac{1}{9} + 2\alpha\right)\frac{n^2}{2}$ edges in common. Owing to Fact 2.23 this shows that $H(\Psi_a)$ and R_x^Ψ have at least αn^2 common edges or, in other words, that there are at least $2\alpha n^2$ ordered pairs (x_1, x_2) such that $x_1x_2 \in E(\Psi_a) \cap E(R_x^\Psi)$. Due to $|A_x| \leqslant 6\mu n^2$ at most $6\mu n^2$ of these pairs fail to be ζ-rightconnectable. By symmetry, at most the same number of pairs under consideration fails to be ζ-leftconnectable. Altogether, this demonstrates that among the ordered pairs (x_1, x_2) with $x_1x_2 \in E(\Psi_a) \cap E(\Psi_x)$ there are at least $(2\alpha - 12\mu)n^2$ which are ζ-connectable. Because of $\mu = \frac{\alpha^2}{120} < \frac{\alpha}{7}$ this is more than what we need.

The general case $k \geqslant 4$. Our first goal is to count ζ-leftconnectable $(k-1)$-tuples in the intersection of $H(\Psi_a)$ and $H(\Psi_x)$ that satisfy a certain minimum degree condition.

Claim 2.31. *The number of ζ-leftconnectable $(k-1)$-tuples $(x_1, \dots, x_{k-1})$ such that*

(1) $\{x_1, \dots, x_{k-1}\} \in E(\Psi_a) \cap E(\Psi_x)$
(2) and $d(x_2, \dots, x_{k-1}, x) \geqslant \frac{n-2}{3}$

is at least $3\eta n^{k-1}$.

Proof. For every vertex $x_{k-1} \in V \smallsetminus \{a, x\}$ we apply Lemma 2.7 to the $(k-2)$-uniform hypergraphs $H(\Psi_{xx_{k-1}})$ and $H(\Psi_{ax_{k-1}})$. This yields a lower bound on the number of $(2k-4)$-tuples

$$(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2k-3}) \in V^{2k-4}$$

such that

(a) $x_1 \dots x_{k-1}xx_{k+1} \dots x_{2k-3}$ is a walk in $H(\Psi)$
(b) $\{x_1, \dots, x_{k-1}\} \in E(\Psi_a)$
(c) $d(x_2, \dots, x_{k-1}, x) \geqslant \frac{n-2}{3}$
(d) and $d(x_{k-1}, x, x_{k+1}, \dots, x_{2k-3}) \geqslant \frac{n-2}{3}$.

Notably, there are $n-2$ possibilities for x_{k-1} and for each of them Lemma 2.7 yields

$$\left(\frac{\alpha}{2}\right)^{2^{(k-2)-1}} n^{2(k-2)-1} \overset{(2.7)}{=} 10\eta n^{2k-5}$$

possibilities for remaining $2k-5$ vertices. Therefore the number of $(2k-4)$-tuples

$$(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{2k-3}) \in V^{2k-4}$$

satisfying (a)–(d) is at least $10\eta(n-2)n^{2k-5}$.

Because of the minimum $(k-2)$-degree condition Ψ needs to have at least one edge, whence $n \geqslant k \geqslant 4$. As this implies $n-2 \geqslant \frac{1}{2}n$, the total number of $(2k-4)$-tuples satisfying *(a)*–*(d)* is at least $5\eta n^{2k-4}$.

In view of *(d)* and Fact 2.24 applied to $\{x, x_{k+1}, \dots, x_{2k-3}\}$ here in place of x there we know that all but at most $\frac{2\mu}{\alpha} n^{2k-4}$ of these $(2k-4)$-tuples satisfy

(e) $x_{k-1} \in V(R^{\Psi}_{x x_{k+1} \dots x_{2k-3}})$.

Now $x \notin X''$ yields $|B_x| \leqslant 2k\mu n^{2k-4}$. So at most $2k\mu n^{2k-4}$ of the $(2k-4)$-tuples satisfying *(a)* and *(e)* violate

(f) $(x_1, \dots, x_{k-1})$ is ζ-leftconnecctable.

Summarising, the number of $(2k-4)$-tuples satisfying *(a)*–*(f)* is at least

$$(5\eta - \tfrac{2\mu}{\alpha} - 2k\mu) n^{2k-4} \overset{(2.8)}{\geqslant} 3\eta n^{2k-4} .$$

Ignoring the vertices $x_{k+1}, \dots, x_{2k-3}$ as well as the conditions *(d)*, *(e)* we arrive at the desired conclusion. □

Now we keep working with the ζ-leftconnectable $(k-1)$-tuples satisfying *(1)* and *(2)* obtained in Claim 2.31. According to *(2)* and Fact 2.24 applied $\{x_3, \dots, x_{k-1}, x\}$ here in place of x there all but at most $\frac{2\mu}{\alpha} n^{k-1}$ of them have the property

(3) $x_2 \in V(R^{\Psi}_{x_3 \dots x_{k-1} x})$.

Moreover, by Definition 2.22 *(b)* applied to the $(k-2)$-set $\{x_3, \dots, x_{k-1}, x\}$ at most μn^{k-1} tuples of length $k-1$ satisfy *(1)* and *(3)* but not

(4) $x_1 \in V(R^{\Psi}_{x_3 \dots x_{k-1} x})$.

Finally, $x \notin X'$ implies $|A_x| \leqslant 2k\mu n^{k-1}$, so among the ζ-leftconnectable $(k-1)$-tuples satisfying *(1)*–*(4)* there are at most $2k\mu n^{k-1}$ for which

(5) $(x_1, \dots, x_{k-1})$ is ζ-rightconnectable

fails. In particular, the number of ζ-leftconnectable $(k-1)$-tuples $(x_1, \dots, x_{k-1})$ with *(1)* and *(5)* is at least

$$\left(3\eta - \frac{2\mu}{\alpha} - \mu - 2k\mu\right) n^{k-1} \overset{(2.8)}{\geqslant} \mu n^{k-1} .$$

Altogether this shows that the number of $(k-1)$-tuples $(x_1, \dots, x_{k-1})$ that are ζ-leftconnectable, ζ-rightconnectable, and satisfy $\{x_1, \dots, x_{k-1}\} \in E(\Psi_a) \cap E(\Psi_x)$ is at least μn^{k-1}. In view of Definition 2.16*(b)* this concludes the proof of Lemma 2.30. □

The 'connectable' edges in $E(\Psi_a) \cap E(\Psi_x)$ considered in the previous lemma can be used to build paths.

Corollary 2.32. *For given $k \geqslant 3$ and $\alpha > 0$ there exists a natural number n_0 such that if $\mu = \frac{1}{10k}\left(\frac{\alpha}{2}\right)^{2^{k-3}+1}$, Ψ is a k-uniform (α, μ)-constellation on $n \geqslant n_0$ vertices, and $\zeta > 0$ then there exists a set $X \subseteq V(\Psi)$ with $|X| \leqslant \frac{\zeta}{\mu} n$ such that the following holds. For every $a \in V(\Psi)$ and every $x \in V(\Psi) \smallsetminus (X \cup \{a\})$ the number of $(k-1)$-uniform paths $b_1 b_2 \dots b_{2k-2}$ in $H(\Psi_a) \cap H(\Psi_x)$ such that $(b_1, \dots, b_{k-1})$ and $(b_k, \dots, b_{2k-2})$ are ζ-connectable in Ψ is at least $\frac{1}{2}\mu^k n^{2k-2}$.*

Proof. Let X be the set produced by Lemma 2.30. Consider two distinct vertices $a, x \in V(\Psi)$ with $x \notin X$. Form an auxiliary $(k-1)$-partite $(k-1)$-uniform hypergraph

$$\mathscr{B} = (V_1 \cupdot \dots \cupdot V_{k-1}, E_{\mathscr{B}})$$

whose vertex classes are $k-1$ disjoint copies of $V(\Psi)$ and whose edges $\{v_1, \dots, v_{k-1}\} \in E_{\mathscr{B}}$ with $v_i \in V_i$ for $i \in [k-1]$ correspond to ζ-connectable $(k-1)$-tuples $(v_1, \dots, v_{k-1})$ such that $\{v_1, \dots, v_{k-1}\} \in E(\Psi_a) \cap E(\Psi_x)$.

Lemma 2.30 tells us that

$$|E_{\mathscr{B}}| \geqslant \mu n^{k-1}\,.$$

Thus Lemma 2.8 applied to $\mathscr{B}$ with $(k-1, \mu, 2k-2)$ here in place of (k, d, r) there yields at least $\mu^k n^{2k-2}$ walks $(b_1, \dots, b_{2k-2})$ in $\mathscr{B}$ with $b_1 \in V_1, \dots, b_{k-1} \in V_{k-1}$. By the definition of $\mathscr{B}$ each of these walks corresponds to a walk in $H(\Psi_a) \cap H(\Psi_x)$ whose first and last $k-1$ vertices form a ζ-connectable $(k-1)$-tuple in Ψ. At most $O(n^{2k-3})$ of these walks can have repeated vertices and, hence, there are at least

$$\mu^k n^{2k-2} - O(n^{2k-3}) \geqslant \frac{\mu^k}{2} n^{2k-2}$$

paths of the desired from. $\square$

2.5. **On $(\alpha, \beta, \ell, \mu)$-constellations.** This subsection is dedicated to (α, μ)-constellations Ψ whose distinguished graphs R_x^Ψ have the robustness property delivered by Proposition 2.2.

Definition 2.33. *Let $k \geqslant 2$, $\alpha, \beta, \mu > 0$ and let $\ell \geqslant 3$ be odd. A k-uniform constellation Ψ is said to be an $(\alpha, \beta, \ell, \mu)$-constellation if*

(a) it is an (α, μ)-constellation,
(b) and for all $x \in V(\Psi)^{(k-2)}$ and all distinct $y, z \in V(R_x^\Psi)$ the number of y-z-paths in R_x^Ψ of length ℓ is at least $\beta |V(\Psi)|^{\ell-1}$.

The main result of this subsection shows how to expand sufficiently large k-uniform hypergraphs whose minimum $(k-2)$-degree is at least $\left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ for appropriate choices of the parameters to such $(\alpha, \beta, \ell, \mu)$-constellations. Essentially, the proof of this observation proceeds by applying Proposition 2.2 to all link graphs.

Fact 2.34. *For all $k \geqslant 2$ and $\alpha, \mu > 0$ there exist $\beta = \beta(\alpha, \mu) > 0$ and an odd integer $\ell = \ell(\alpha, \mu) \geqslant 3$ such that for sufficiently large n every k-uniform n-vertex hypergraph H with $\delta_{k-2}(H) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ expands to an $(\alpha, \beta, \ell, \mu)$-constellation.*

Notice that this result is the reason why the study of $(\alpha, \beta, \ell, \mu)$-constellations conducted in the subsequent sections sheds light on Theorem 1.2.

Proof of Fact 2.34. For α and μ Proposition 2.2 delivers some constant $\beta' > 0$ and an odd integer $\ell \geqslant 3$. We contend that $\beta = (2/3)^{\ell-1}\beta'$ and ℓ have the desired property.

To see this, we consider a sufficiently large k-uniform hypergraph H on n vertices satisfying $\delta_{k-2}(H) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$. For every $x \in V(H)^{(k-2)}$ Proposition 2.2 applies to the link graph H_x and yields a (β', ℓ)-robust induced subgraph $R_x \subseteq H_x$ satisfying

- (i) $|V(R_x)| \geqslant \left(\frac{2}{3} + \frac{\alpha}{2}\right)n$
- (ii) and $e_{H_x}(V(R_x), V(H) \smallsetminus V(R_x)) \leqslant \mu n^2$.

We shall show that

$$\Psi = \left(H, \left\{R_x \colon x \in V(H)^{(k-2)}\right\}\right)$$

is the desired $(\alpha, \beta, \ell, \mu)$-constellation. By Definition 2.22 and (i), (ii) above, Ψ is an (α, μ)-constellation, meaning that part (a) of Definition 2.33 holds.

Moving on to the second part we fix an arbitrary $(k-2)$-set $x \subseteq V(H)$ as well as two distinct vertices y, z of R_x. Since R_x is (β', ℓ)-robust, the number of y-z-paths in R_x of length ℓ is indeed at least

$$\beta' |V(R_x)|^{\ell-1} \overset{(i)}{\geqslant} \left(\frac{3}{2}\right)^{\ell-1} \beta \cdot \left(\frac{2}{3} + \frac{\alpha}{2}\right)^{\ell-1} n^{\ell-1} \geqslant \beta n^{\ell-1}\,. \qquad \square$$

The remainder of this subsection deals with the question to what extent being an $(\alpha, \beta, \ell, \mu)$-constellation is preserved under taking link constellations and removing a small proportion of the vertices. Let us observe that if Ψ denotes a k-uniform $(\alpha, \beta, \ell, \mu)$-constellation, then for each $x \in V(\Psi)^{(k-2)}$ the vertices in x are isolated in H_x, which by Definition 2.33 (b) implies that they cannot be vertices of R_x^{Ψ}. Thus we have $V(R_x^{\Psi}) \cap x = \varnothing$ for each $x \in V(\Psi)^{(k-2)}$.

Let us now consider for some $S \subseteq V(\Psi)$ of size $|S| \leqslant k-2$ the $(k-|S|)$-uniform link constellation Ψ_S. For every $x \in V(\Psi_S)^{(k-2-|S|)}$ we have $R_x^{\Psi_S} = R_{S \cup x}^{\Psi} \smallsetminus S = R_{S \cup x}^{\Psi}$. Therefore, Ψ_S inherits the property in Definition 2.33 (b) from Ψ and together with Fact 2.26 this leads to the following conclusion.

Fact 2.35. *Given $k \geqslant 2$, $\alpha, \beta > 0$, $\mu' > \mu > 0$ and an odd integer $\ell \geqslant 3$, there exists a natural number n_0 such that the following holds.*

If Ψ is a k-uniform $(\alpha, \beta, \ell, \mu)$-constellation with at least n_0 vertices and $S \subseteq V(\Psi)$ consists of at most $k-2$ vertices, then the $(k-|S|)$-uniform link constellation Ψ_S is an $(\alpha, \beta, \ell, \mu')$-constellation. $\square$

Next we deal with a similar result allowing vertex deletions as well.

Lemma 2.36. *Given $k \geqslant 2$, $\alpha, \beta, \mu > 0$ and an odd integer $\ell \geqslant 3$ set*

$$\vartheta = \min\left\{\frac{\alpha}{4}, \frac{\beta}{2\ell}\right\},$$

and let Ψ be a k-uniform $(\alpha, \beta, \ell, \mu)$-constellation on $n \geqslant 6k$ vertices. If $S, X \subseteq V(\Psi)$ are disjoint, $|S| \leqslant k-2$, and $|X| \leqslant \vartheta n$, then $\Psi_S - X$ is an $\big(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, 2\mu\big)$-constellation.

Proof. Let $\Psi = \big(H, \{R_x \colon x \in V(H)^{(k-2)}\}\big)$ be a k-uniform $(\alpha, \beta, \ell, \mu)$-constellation on $n \geqslant 6k$ vertices. Recall that this means

$$\delta_{k-2}(H) \geqslant \Big(\frac{5}{9} + \alpha\Big)\frac{n^2}{2}, \tag{2.9}$$

and that for every $x \in V(\Psi)^{(k-2)}$ the graph $R_x \subseteq H_x$ has the following properties:

- *(i)* $|V(R_x)| \geqslant \big(\frac{2}{3} + \frac{\alpha}{2}\big)n$,
- *(ii)* $e_{H_x}\big(V(R_x), V(\Psi) \smallsetminus V(R_x)\big) \leqslant \mu n^2$,
- *(iii)* and for all distinct $y, z \in V(R_x)$ the number of y-z-paths in R_x of length ℓ is at least $\beta n^{\ell-1}$.

Further, let $S, X \subseteq V(\Psi)$ be any disjoint sets such that $|S| \leqslant k-2$ and $|X| \leqslant \vartheta n$. We are to prove that

$$\Psi_\star = \Psi_S - X = \Big(\overline{H}_S - X, \big\{R_{x \cup S} - X \colon x \in \big(V(H) \smallsetminus (S \cup X)\big)^{(k-2-|S|)}\big\}\Big)$$

is a $(k-|S|)$-uniform $\big(\frac{\alpha}{2}, \frac{\beta}{2}, \ell 2\mu\big)$-constellation, i.e., that its underlying hypergraph satisfies an appropriate minimum degree conditions and that the distinguished subgraphs of its link graphs have properties analogous to *(i)*–*(iii)*.

Because of

$$\begin{aligned}\delta_{k-|S|-2}(\overline{H}_S - X) \geqslant \delta_{k-2}(H - X) &\geqslant \Big(\frac{5}{9} + \alpha\Big)\frac{n^2}{2} - |X|n \\ &\geqslant \Big(\frac{5}{9} + \alpha\Big)\frac{n^2}{2} - \vartheta n^2 \geqslant \Big(\frac{5}{9} + \frac{\alpha}{2}\Big)\frac{n^2}{2} \geqslant \Big(\frac{5}{9} + \frac{\alpha}{2}\Big)\frac{|V(\Psi_\star)|^2}{2},\end{aligned}$$

where we utilised $\vartheta \leqslant \frac{\alpha}{4}$ in the penultimate step, the minimum degree of the hypergraph $H(\Psi_\star) = \overline{H}_S - X$ is indeed as large as we need it to be.

Now let $x \in \big(V(\Psi_\star)\big)^{(k-2-|S|)}$ be arbitrary. Since $x \cup S \in \big(V(\Psi) \smallsetminus X\big)^{(k-2)}$, the above statement *(i)* entails

$$|V(R_x^{\Psi_\star})| = |V(R_{x\cup S} - X)| \geqslant \Big(\frac{2}{3} + \frac{\alpha}{2}\Big)n - |X| \\ \geqslant \Big(\frac{2}{3} + \frac{\alpha}{2}\Big)n - \vartheta n \geqslant \Big(\frac{2}{3} + \frac{\alpha}{4}\Big)n \geqslant \Big(\frac{2}{3} + \frac{\alpha}{4}\Big)|V(\Psi_\star)|\,,$$

which shows that the required variant of *(i)* holds for $\Psi_\star$.

Next, the graph $H(\Psi_\star)_x = (\overline{H}_S - X)_x$ is a subgraph of $H_{x\cup S}$, so *(ii)* tells us that

$$e_{H(\Psi_\star)_x}\big(V(R_x^{\Psi_\star}), V(\Psi_\star) \smallsetminus V(R_x^{\Psi_\star})\big) \leqslant e_{H_{x\cup S}}\big(V(R_{x\cup S}), V(\Psi) \smallsetminus V(R_{x\cup S})\big) \leqslant \mu n^2\,.$$

From $\vartheta \leqslant \frac{\alpha}{4} \leqslant \frac{1}{9}$ and $n \geqslant 6k$ we conclude

$$|V(\Psi_\star)| = n - |X| - |S| \geqslant \Big(1 - \frac{1}{9} - \frac{1}{6}\Big)n = \frac{13}{18}n > \frac{n}{\sqrt{2}}$$

and thus we arrive indeed at

$$e_{H(\Psi_\star)_x}\big(V(R_x^{\Psi_\star}), V(\Psi_\star) \smallsetminus V(R_x^{\Psi_\star})\big) \leqslant 2\mu|V(\Psi_\star)|^2\,,$$

which concludes the proof that the appropriate modification of *(ii)* holds for $\Psi_\star$. Altogether, we have thereby shown that $\Psi_\star$ is an $\big(\frac{\alpha}{2}, 2\mu\big)$-constellation.

Finally we consider distinct vertices $y, z \in V(R_{x\cup S} - X)$ and recall that by *(iii)* above the number of y-z-paths in $R_{x\cup S}$ is at least $\beta n^{\ell-1}$. At most $(\ell-1)|X|n^{\ell-2} \leqslant \frac{\beta}{2}n^{\ell-1}$ of these paths can have an inner vertex in X and, consequently, $R_{x\cup S} - X$ contains at least $\frac{\beta}{2}n^{\ell-1}$ such paths. Therefore $\Psi_\star$ is indeed an $\big(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, 2\mu\big)$-constellation. □

§3. The Connecting Lemma

In this section we establish the Connecting Lemma (Proposition 3.3). Given an $(\alpha, \beta, \ell, \mu)$-constellation with appropriate parameters this result allows us to connect every leftconnectable $(k-1)$-tuple to every rightconnectable $(k-1)$-tuple by means of a short path. In the course of proving Theorem 1.2 the Connecting Lemma gets used $\Omega(n)$ times and, essentially, it allows us to convert an almost spanning path cover into an almost spanning cycle. For some reasons related to our way of employing the absorption method, it will turn out to be enormously helpful later if we can guarantee that the number of left-over vertices outside this almost spanning cycle is a multiple of k. There are several possibilities how one might try to accomplish this and our approach is to prove a version of the Connecting Lemma with absolute control over the length of the connecting path modulo k. When closing the almost spanning cycle by means of a final application of the Connecting Lemma, we will then be able to prescribe in which residue class modulo k the number of left-over vertices is going to be. (For a different way to handle such a situation we refer to recent work of Schacht and his students [2]).

The following result is implicit in [17, Proposition 2.6] and after stating it we shall briefly explain how it can be derived from the argument presented there.

Proposition 3.1. *Depending on $\alpha, \beta, \zeta_\star > 0$ and an odd integer $\ell \geqslant 3$ there exist a constant $\vartheta_\star = \vartheta_\star(\alpha, \beta, \ell, \zeta_\star) > 0$ and a natural number n_0 with the following property.*

If Ψ is a 3-uniform $(\alpha, \beta, \ell, \frac{\alpha}{4})$-constellation on $n \geqslant n_0$ vertices, $\vec{a}, \vec{b} \in V(\Psi)^2$ are two disjoint pairs of vertices such that $\vec{a}$ is $\zeta_\star$-leftconnectable and $\vec{b}$ is $\zeta_\star$-rightconnectable, then the number of $\vec{a}$-$\vec{b}$-paths in $H(\Psi)$ with $3\ell + 1$ inner vertices is at least $\vartheta_\star n^{3\ell+1}$. □

Observe that the Setup 2.4 we are assuming in [17, Proposition 2.6] is tantamount to an $(\alpha, \beta, \ell, \frac{\alpha}{4})$-constellation. The connectabilty assumptions in [17] are slightly different. Writing $\vec{a} = (x, y)$ we were using in the proof of [17, Proposition 2.6] that a set called U_{xy} there, and defined to consist of all vertices u with $xy \in E(R_u^\Psi)$, has at least the size $\zeta|V(\Psi)|$. When working with vertices $u \in U_{xy}$, however, we were only using $y \in V(R_u^\Psi)$ and $xyu \in E(\Psi)$. For this reason, the entire proof can also be carried out with the set called $U_{(x,y)}^\Psi$ here, or in other words it suffices to suppose that $\vec{a}$ is ζ-leftconnectable. Similarly, we may assume that $\vec{b}$ is ζ-rightconnectable rather than being ζ-connectable in the sense of [17]. Now we introduce the function giving the number of inner vertices in our connections.

Definition 3.2. *Given integers $k \geqslant 3$, $0 \leqslant i < k$, and $\ell \geqslant 3$ we set*

$$f(k, i, \ell) = [4^{k-3}(2\ell + 4) - 2]k + i\,.$$

We are now ready to state the k-uniform Connecting Lemma.

Proposition 3.3 (Connecting Lemma)**.** *For all $k \geqslant 3$, $\alpha, \beta, \zeta > 0$, and odd integers $\ell \geqslant 3$ there exist $\vartheta > 0$ and $n_0 \in \mathbb{N}$ with the following property.*

If Ψ is a k-uniform $(\alpha, \beta, \ell, \frac{\alpha}{k+6})$-constellation on $n \geqslant n_0$ vertices, $\vec{a}, \vec{b} \in V(\Psi)^{k-1}$ are two disjoint $(k-1)$-tuples such that $\vec{a}$ is ζ-leftconnectable and $\vec{b}$ is ζ-rightconnectable, and $0 \leqslant i < k$, then the number of $\vec{a}$-$\vec{b}$-paths in $H(\Psi)$ with $f = f(k, i, \ell)$ inner vertices is at least ϑn^f.

The proof of this result occupies the remainder of this section and before we begin we provide a short overview over the main ideas. The plan is to proceed by induction on k. When we reach a certain value of k, most of the work is devoted to showing the weaker assertion (Φ_k) that there exists at least one number $f_\star = f_\star(k, \ell)$ such that the statement of the Connecting Lemma holds for connections with $f_\star$ inner vertices. Once we know (Φ_k) the induction can be completed by putting short 'connectable' walks as obtained by Lemma 2.27 in the middle and connecting them with two applications of (Φ_k) to $\vec{a}$ and $\vec{b}$.

The proof of (Φ_k) itself is more complicated and starts by applying Lemma 2.29 to $U^{\Psi}_{\vec{a}}$ and $U^{\Psi}_{\vec{b}}$ here in place of U and W there. This yields many $(2k-2)$-tuples $(u, q_1, \dots, q_{2k-4}, w)$ in $V(\Psi)^{2k-2}$ which, after some reordering, have good chances to end up being middle segments of the desired connections. Applying the induction hypothesis to Ψ_u and Ψ_w we can connect $\vec{a}$ and $\vec{b}$ by many $(k-1)$-uniform paths to these middle segments and it remains to 'augment' these connections to k-uniform paths, which can be done by averaging over many possibilities for u and w, respectively (see Figure 3.1).

Proof of Proposition 3.3. We proceed by induction on k, keeping α, β, and ℓ fixed.

Choice of constants. Due to monotonicity (see Fact 2.17) we may suppose that $\zeta \leqslant \frac{1}{k3^{2k}}$. By recursion on $k \geqslant 3$ we define for every $\zeta \in \big(0, \frac{1}{k3^{2k}}\big]$ a positive real number $\vartheta(k,\zeta)$. Starting with $k = 3$ we set

$$\vartheta(3,\zeta) = \zeta\big(\vartheta_\star(\alpha,\beta,\ell,\zeta)\big)^2 \quad \text{for} \quad \zeta \in \big(0, 3^{-7}\big],$$

where $\vartheta_\star(\alpha,\beta,\ell,\zeta)$ is given by Proposition 3.1. For $k \geqslant 4$ and $\zeta \in \big(0, \frac{1}{k3^{2k}}\big]$ we stipulate

$$\vartheta(k,\zeta) = \zeta^{6s+1}\big(\vartheta(k-1,\zeta^3)\big)^{4s}, \quad \text{where } s = 4^{k-4}(2\ell+4)\,. \tag{3.1}$$

Our goal is to prove the Connecting Lemma with $2\vartheta(k,\zeta)$ playing the rôle of ϑ.

The base case $k = 3$. Suppose that Ψ is a sufficiently large 3-uniform $(\alpha,\beta,\ell,\frac{\alpha}{9})$-constellation, $i \in \{0,1,2\}$, the pair $\vec{a} = (a_1,a_2) \in V(\Psi)^2$ is ζ-leftconnectable, $\vec{b} = (b_1,b_2)$ is ζ-rightconnectable, the four vertices a_1, a_2, b_1, and b_2 are distinct, and $\zeta \leqslant \frac{1}{3^7}$. Lemma 2.27 applied to $(3, i+2)$ here in place of (k,r) there tells us that there are at least $\frac{1}{3^{i+3}}n^{i+4}$ walks $x_1 \dots x_{i+4}$ of length $i+2$ in $H(\Psi)$ whose starting pair (x_1,x_2) is ζ-rightconnectable and whose ending pair (x_{i+3},x_{i+4}) is ζ-leftconnectable. Among these walks at least

$$\Big(\frac{1}{3^{i+3}} - \frac{4(i+4)}{n}\Big)n^{i+4} > \frac{n^{i+4}}{3^{i+4}} \geqslant \frac{n^{i+4}}{3^6} \geqslant 3\zeta n^{i+4}$$

avoid $\{a_1,a_2,b_1,b_2\}$.

Now for each of them two applications of Proposition 3.1 to the $(\alpha,\beta,\ell,\frac{\alpha}{9})$-constellation Ψ enable us to find in $H(\Psi)$ at least $\vartheta_\star n^{3\ell+1}$ paths $a_1a_2p_1\dots p_{3\ell+1}x_1x_2$ and at least $\vartheta_\star n^{3\ell+1}$ paths $x_{i+3}x_{i+4}r_1\dots r_{3\ell+1}b_1b_2$ where $\vartheta_\star = \vartheta_\star(\alpha,\beta,\ell,\zeta)$. Altogether, this reasoning leads to at least $3\zeta\vartheta_\star^2 n^f$ walks

$$a_1a_2p_1\dots p_{3\ell+1}x_1x_2\dots x_{i+3}x_{i+4}r_1\dots r_{3\ell+1}b_1b_2$$

with f inner vertices, where

$$f = 2(3\ell + 1) + (i + 4) = 6\ell + 6 + i = f(3, i, \ell)\,.$$

At most $f^2 n^{f-1} = o(n^f)$ of these walks fail to be paths and thus the assertion follows.

Induction Step. Suppose $k \geqslant 4$ and that the Connecting Lemma is already known for $k-1$ instead of k. Set

$$t = 2k(s-1) + 2 \quad \text{and} \quad \eta = \zeta^{3s}\big(\vartheta(k-1, \zeta^3)\big)^{2s}\,, \tag{3.2}$$

where, let us recall, $s = 4^{k-4}(2\ell + 4)$ was introduced in (3.1) while we chose our constants. Following the plan outlined above, our first step is to prove a Connecting Lemma for connections with t inner vertices.

Claim 3.4. *For any two disjoint $(k-1)$-tuples $\vec{a} = (a_1, \dots, a_{k-1})$ and $\vec{b} = (b_1, \dots, b_{k-1})$ such that $\vec{a}$ is ζ-leftconnectable and $\vec{b}$ is ζ-rightconnectable, the number of $\vec{a}$-$\vec{b}$-walks with t inner vertices in $H(\Psi)$ is at least $2\eta n^t$.*

Proof. The connectability assumptions mean that the sets

$$U = \{u \in V(\Psi)\colon a_1 \dots a_{k-1} u \in E(\Psi) \text{ and } (a_2, \dots, a_{k-1}) \text{ is } \zeta\text{-leftconnectable in } \Psi_u\}$$

and

$$W = \{w \in V(\Psi)\colon w b_1 \dots b_{k-1} \in E(\Psi) \text{ and } (b_1, \dots, b_{k-2}) \text{ is } \zeta\text{-rightconnectable in } \Psi_w\}$$

satisfy $|U|, |W| \geqslant \zeta n$. Now by $\frac{\alpha}{k+6} \leqslant \frac{\alpha}{10}$ and Fact 2.25 Ψ is an $(\alpha, \frac{\alpha}{10})$-constellation. Combined with $\zeta \leqslant \frac{1}{3^{k+2}}$ and Lemma 2.29 this shows that the number of $(2k-2)$-tuples

$$(u, \vec{q}, w) = (u, q_1, \dots, q_{2k-4}, w) \in U \times V(\Psi)^{2k-4} \times W$$

such that

(*a*) $u \neq w$,
(*b*) $q_1 \dots q_{2k-4}$ is a walk in $H(\Psi_{uw})$,
(*c*) $(q_1, \dots, q_{k-2})$ is ζ^3-rightconnectable in Ψ_u,
(*d*) and $(q_{k-1}, \dots, q_{2k-4})$ is ζ^3-leftconnectable in Ψ_w.

is at least $\zeta^3 n^{2k-2}$. For later reference we recall that $u \in U$ and $w \in W$ mean

(*e*) $(a_2, \dots, a_{k-1})$ is ζ-leftconnectable in Ψ_u,
(*f*) $\{a_1, \dots, a_{k-1}, u\} \in E(\Psi)$,
(*g*) $(b_1, \dots, b_{k-2})$ is ζ-rightconnectable in Ψ_w,
(*h*) and $\{w, b_1, \dots, b_{k-1}\} \in E(\Psi)$.

Notice that by Fact 2.35 the link constellation of every vertex is a $(k-1)$-uniform $(\alpha, \beta, \ell, \frac{\alpha}{k+5})$-constellation and that $f(k-1, 1, \ell) = (k-1)(s-2)+1$. Now for every $(2k-2)$-tuple $(u, \vec{q}, w)$ satisfying (a)–(h) we apply the induction hypothesis twice with $(\zeta^3, 1)$ here in place of (ζ, i) there. First, by (c) and (e) we can connect $(a_2, \dots, a_{k-1})$ to $(q_1, \dots, q_{k-2})$ in Ψ_u, thus getting at least $2\vartheta(k-1, \zeta^3)(n-1)^{(k-1)(s-2)+1}$

(i) walks $a_2 \dots a_{k-1} p_1 \dots p_{(k-1)(s-2)+1} q_1 \dots q_{k-2}$ in Ψ_u

with $f(k-1, 1, \ell)$ inner vertices. Second, (d) and (g) allow us to connect $(q_{k-1}, \dots, q_{2k-4})$ to $(b_1, \dots, b_{k-2})$ in Ψ_w by at least $2\vartheta(k-1, \zeta^3)(n-1)^{(k-1)(s-2)+1}$

(j) walks $q_{k-1} \dots q_{2k-4} r_{(k-1)(s-2)+1} \dots r_1 b_1 \dots b_{k-2}$ in Ψ_w.

Altogether, the number of $\big((k-1)(2s-2)+2\big)$-tuples

$$(u, \vec{p}, \vec{q}, \vec{r}, w) \in U \times V(\Psi)^{(k-1)(s-2)+1} \times V(\Psi)^{2k-4} \times V(\Psi)^{(k-1)(s-2)+1} \times W$$

with (a)–(j), where

$$\vec{p} = (p_1, \dots, p_{(k-1)(s-2)+1}) \quad \text{and} \quad \vec{r} = (r_{(k-1)(s-2)+1}, \dots, r_1)\,,$$

is at least $4\zeta^3\big(\vartheta(k-1,\zeta^3)\big)^2(n-1)^{(k-1)(2s-4)+2}n^{2k-2} \geqslant 2\zeta^3\big(\vartheta(k-1,\zeta^3)\big)^2 n^{(k-1)(2s-2)+2}$.

Roughly speaking, we plan to derive the $\vec{a}$-$\vec{b}$-paths we are supposed to construct from these $\big((k-1)(2s-2)+2\big)$-tuples by taking many copies of u and w and inserting them in appropriate positions into $(\vec{a}, \vec{p}, \vec{q}, \vec{r}, \vec{b})$. To analyse the number of ways of doing this, we consider the auxiliary 3-partite 3-uniform hypergraph $\mathscr{A}$ with vertex classes $U^\star$, M, and $W^\star$, where $U^\star$ and $W^\star$ are two disjoint copies of $V(\Psi)$, while $M = V(\Psi)^{(k-1)(2s-2)}$.

We represent the vertices in M as sequences

$$\vec{m} = (\vec{p}, \vec{q}, \vec{r}) = (p_1, \dots, p_{(k-1)(s-2)+1}, q_1, \dots, q_{2k-4}, r_{(k-1)(s-2)+1}, \dots, r_1)\,.$$

The edges of $\mathscr{A}$ are defined to be the triples $\{u, \vec{m}, w\}$ with $u \in U \subseteq U^\star$, $\vec{m} \in M$, and $w \in W \subseteq W^\star$, for which the $\big((k-1)(2s-2)+2\big)$-tuple $(u, \vec{m}, w)$ satisfies (a)–(j). We have just proved that

$$e(\mathscr{A}) \geqslant 2\zeta^3\big(\vartheta(k-1,\zeta^3)\big)^2 n^{(k-1)(2s-2)+2} = 2\zeta^3\big(\vartheta(k-1,\zeta^3)\big)^2 |U^\star||M||W^\star|\,. \tag{3.3}$$

By the (ordered) bipartite link graph of a vertex $\vec{m} \in M$ we mean the set of pairs

$$\mathscr{A}_{\vec{m}} = \big\{(u,w) \in U \times W \colon u\vec{m}w \in E(\mathscr{A})\big\}\,.$$

The convexity of the function $x \longmapsto x^s$ on $\mathbb{R}_{\geqslant 0}$ yields

$$\begin{aligned}\sum_{\vec{m} \in M} |\mathscr{A}_{\vec{m}}|^s &\geqslant |M|\left(\frac{e(\mathscr{A})}{|M|}\right)^s \overset{(3.3)}{\geqslant} n^{(k-1)(2s-2)}\big(2\zeta^3\big(\vartheta(k-1,\zeta^3)\big)^2 n^2\big)^s \\ &\geqslant 2\zeta^{3s}\big(\vartheta(k-1,\zeta^3)\big)^{2s} n^{k(2s-2)+2} \overset{(3.2)}{=} 2\eta n^t\,. \end{aligned} \tag{3.4}$$

In other words, the number of t-tuples

$$(\vec{u}, \vec{p}, \vec{q}, \vec{r}, \vec{w}) \in U^s \times V(\Psi)^{(k-1)(s-2)+1} \times V(\Psi)^{2k-4} \times V(\Psi)^{(k-1)(s-2)+1} \times W^s$$

with

$$(u_1, w_1), \ldots, (u_s, w_s) \in \mathscr{A}_{\vec{m}}\,,$$

where

$$\vec{u} = (u_1, \ldots, u_s)\,, \quad \vec{w} = (w_1, \ldots, w_s)\,, \quad \text{and} \quad \vec{m} = (\vec{p}, \vec{q}, \vec{r}) \in M\,,$$

is at least $2\eta n^t$. So to conclude the proof of Claim 3.4 it suffices to show that for every such t-tuple the sequence

$$a_1 \ldots a_{k-1} u_1 p_1 \ldots p_{k-1} u_2 p_k \ldots p_{2k-2} u_3 \ldots u_{s-2} p_{(k-1)(s-3)+1} \ldots p_{(k-1)(s-2)} u_{s-1} p_{(k-1)(s-2)+1}$$
$$q_1 \ldots q_{k-2} u_s w_s q_{k-1} \ldots q_{2k-4}$$
$$r_{(k-1)(s-2)+1} w_{s-1} r_{(k-1)(s-2)} \ldots r_{(k-1)(s-3)+1} w_{s-2} \ldots w_3 r_{2k-2} \ldots r_k w_2 r_{k-1} \ldots r_1 w_1 b_1 \ldots b_{k-1}$$

indicated in Figure 3.1 is an $\vec{a}$-$\vec{b}$-walk in $H(\Psi)$.

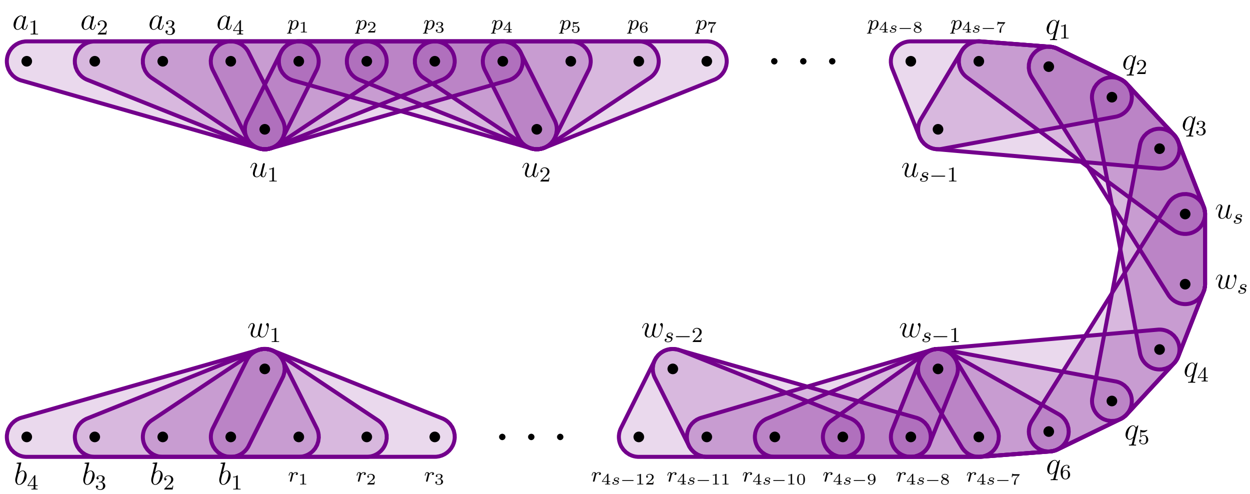


FIGURE 3.1. Connecting (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) in a 5-uniform constellation.

We shall now argue that this follows from the fact that for each $j \in [s]$ the conditions *(a)*–*(j)* hold for u_j and w_j here in place of u and w there.

The first of the required edges is provided by the case $u = u_1$ of *(f)*. Together with *(i)* this shows that the initial segment

$$a_1 a_2 \ldots a_{k-1} u_1 p_1 \ldots p_{k-1} u_2 p_k \ldots p_{2k-2} u_3 \ldots u_{s-2} p_{(k-1)(s-3)+1} \ldots p_{(k-1)(s-2)}$$
$$u_{s-1} p_{(k-1)(s-2)+1} q_1 \ldots q_{k-2} u_s$$

is a walk in $H(\Psi)$. Similarly, by (h) and (j) the terminal segment

$$w_s q_{k-1} \dots q_{2k-4} r_{(k-1)(s-2)+1} w_{s-1} r_{(k-1)(s-2)} \dots r_{(k-1)(s-3)+1} w_{s-2} \dots w_3$$
$$r_{2k-2} \dots r_k w_2 r_{k-1} \dots r_1 w_1 b_1 \dots b_{k-2} b_{k-1}$$

is a walk in $H(\Psi)$. Finally, the middle part

$$q_1 \dots q_{k-2} u_s w_s q_{k-1} \dots q_{2k-4}$$

is a walk in $H(\Psi)$, because by (b) we know that $q_1 \dots q_{2k-4}$ is a walk in $H(\Psi_{u_s w_s})$. □

Returning to the induction step, we consider $i \in \{0, 1, \dots, k-1\}$, a ζ-leftconnectable $(k-1)$-tuple $\vec{a} \in V(\Psi)^{k-1}$, and a ζ-rightconnectable $(k-1)$-tuple $\vec{b}$ such that $\vec{a}$ and $\vec{b}$ have no vertices in common. Plugging $r = i + k - 3$ into Lemma 2.27 we obtain at least $\frac{1}{3^{i+k-2}} n^{i+2k-4}$ walks $x_1 \dots x_{i+2k-4}$ of length $i + k - 3$ in $H(\Psi)$ that start with a ζ-rightconnectable $(k-1)$-tuple and end with a ζ-leftconnectable $(k-1)$-tuple. Of these walks, at least

$$\Big(\frac{1}{3^{i+k-2}} - \frac{2(k-1)(i+2k-4)}{n}\Big) n^{i+2k-4} > \frac{n^{i+2k-4}}{3^{i+k-1}} > \frac{n^{i+2k-4}}{3^{2k}} > \zeta n^{i+2k-4}$$

have no common vertices with $\vec{a}$ and $\vec{b}$. For each such walk, Claim 3.4 tells us that we can connect $\vec{a}$ to $(x_1, \dots, x_{k-1})$ in at least $2\eta n^t$ ways by a walk with t inner vertices, and the same applies to connections from $(x_{i+k-2}, \dots, x_{i+2k-4})$ to $\vec{b}$.

Altogether this reasoning leads to $4\zeta\eta^2 n^f = 4\vartheta(k,\zeta) n^f$ walks in $H(\Psi)$ from $\vec{a}$ to $\vec{b}$ with f inner vertices, where

$$\begin{aligned} f &= 2t + (i + 2k - 4) = 2(2ks - 2k + 2) + (i + 2k - 4) \\ &= (4s - 2)k + i = [4^{k-3}(2\ell + 4) - 2]k + i = f(k, i, \ell)\,. \end{aligned}$$

As usual, at most $O(n^{f-1})$ of these walks can fail to be paths. So, in particular, there exist at least $2\vartheta(k,\zeta) n^f$ paths from $\vec{a}$ to $\vec{b}$ possessing f inner vertices. This completes the induction step and, hence, the proof of the Connecting Lemma. □

§4. Reservoir Lemma

In this section we discuss a standard device occurring in many applications of the absorption method: the reservoir. The problem addressed by the Reservoir Lemma is that while the Connecting Lemma delivers many connections for any two disjoint connectable $(k-1)$-tuples, it gives us no control where the inner vertices of these connections are. Thus it might happen that each of these connections has an inner vertex which is 'unavailable' to us, because we already assigned a different rôle to it in the Hamiltonian cycle we are about to construct. To avoid this problem, one fixes a small random subset of the vertex set, called the reservoir, and decides that the vertices in the reservoir will only be used for the purpose of connecting pairs of $(k-1)$-tuples by means of short paths.

Proposition 4.1 (Reservoir Lemma)**.** *Suppose that $k \geqslant 3$, $\alpha, \beta, \xi, \zeta_{\star\star} > 0$, and that $\ell \geqslant 3$ is an odd integer. If $\vartheta_{\star\star} = \vartheta(k, \alpha, \beta, \ell, \zeta_{\star\star})$ is provided by Proposition 3.3, then there exists some $n_0 \in \mathbb{N}$ such that for every k-uniform $(\alpha, \beta, \ell, \frac{\alpha}{k+6})$-constellation Ψ on $n \geqslant n_0$ vertices there exists a subset $\mathcal{R} \subseteq V(\Psi)$ with the following properties.*

(i) We have $\frac{1}{2}\xi n \leqslant |\mathcal{R}| \leqslant \xi n$.
(ii) For all pairs of disjoint $(k-1)$-tuples $\vec{a}, \vec{b} \in V(\Psi)^{k-1}$ such that $\vec{a}$ is $\zeta_{\star\star}$-leftconnectable and $\vec{b}$ is $\zeta_{\star\star}$-rightconnectable in Ψ, and for every $i \in [0, k)$, the number of $\vec{a}$-$\vec{b}$-paths in $H(\Psi)$ with $f = f(k, i, \ell)$ inner vertices all of which belong to $\mathcal{R}$ is at least $\frac{1}{2}\vartheta_{\star\star}|\mathcal{R}|^f$.

Since the proof of this result is quite standard, we will only provide a brief sketch here. It suffices to prove that the binomial random subset $\mathcal{R} \subseteq V(\Psi)$ including every vertex independently with probability $\frac{3}{4}\xi$ a.a.s. has the properties *(i)* and *(ii)*. Now *(i)* is a straightforward consequence of Chernoff's inequality. As there are only polynomially many possibilities for $(\vec{a}, \vec{b}, i)$ in *(ii)*, it suffices to show that for each of them the probability that there are fewer than $\frac{1}{2}\vartheta_{\star\star}|\mathcal{R}|^f$ paths of the desired form is at most $\exp\big(-\Omega(n)\big)$. This can in turn be established by applying the Azuma-Hoeffding inequality to the at least $\vartheta_{\star\star}n^f$ such paths in $V(\Psi)^f$ delivered by Proposition 3.3. For further details we refer to [15, Proposition 4.1], where we gave a full account of the argument for $k = 4$.

Let us emphasise again that the set $\mathcal{R}$ provided by Proposition 4.1 is called the *reservoir*. The connections in *(ii)* whose inner vertices belong to $\mathcal{R}$ are called paths *through* $\mathcal{R}$.

In the proof of Theorem 1.2 we shall repeatedly connect suitable tuples through the reservoir. Whenever such a connection is made, some of the vertices of the reservoir are used and the part of $\mathcal{R}$ still available for further connections shrinks. Although the reservoir gets used $\Omega(|V(\Psi)|)$ times, we shall be able to keep an appropriate version of property *(ii)* of the reservoir intact throughout this process.

Corollary 4.2. *Let a sufficiently large k-uniform $(\alpha, \beta, \ell, \frac{\alpha}{k+6})$-constellation Ψ as well as a reservoir $\mathcal{R} \subseteq V(\Psi)$ as in Proposition 4.1 be given. Moreover, let $\mathcal{R}' \subseteq \mathcal{R}$ be an arbitrary set with $|\mathcal{R}'| \leqslant \frac{\xi\vartheta_{\star\star}}{4^k k\ell}n$. If $\vec{a}, \vec{b} \in V(\Psi)^{k-1}$ are two disjoint $(k-1)$-tuples such that $\vec{a}$ is $\zeta_{\star\star}$-leftconnectable and $\vec{b}$ is $\zeta_{\star\star}$-rightconnectable, then for every $i \in [0, k)$ there is an $\vec{a}$-$\vec{b}$-path through $\mathcal{R} \smallsetminus \mathcal{R}'$ with $f(k, i, \ell)$ inner vertices.*

Proof. Set $f = f(k,i,\ell)$ and recall that $f(k,i,\ell) = (4^{k-3}(2\ell+4)-2)k + i < 4^{k-2}k\ell$. So the lower bound in Proposition 4.1 (i) together with the bound on $|\mathcal{R}'|$ yields

$$|\mathcal{R}'| \leqslant \frac{\vartheta_{\star\star}|\mathcal{R}|}{4^{k-1}k\ell} \leqslant \frac{\vartheta_{\star\star}|\mathcal{R}|}{4f}\,.$$

Consider all $\vec{a}$-$\vec{b}$-paths through $\mathcal{R}$ with f inner vertices. On the one hand, there are at least $\frac{\vartheta_{\star\star}}{2}|\mathcal{R}|^f$ of them due to Proposition 4.1 (ii). On the other hand, there are at most

$$f|\mathcal{R}'||\mathcal{R}|^{f-1} \leqslant \frac{\vartheta_{\star\star}}{4}|\mathcal{R}|^f$$

such paths having an inner vertex in $\mathcal{R}'$. Consequently, at least $\frac{\vartheta_{\star\star}}{2}|\mathcal{R}|^f - \frac{\vartheta_{\star\star}}{4}|\mathcal{R}|^f > 0$ of our paths have all their inner vertices in $\mathcal{R} \smallsetminus \mathcal{R}'$. □

§5. The absorbing path

5.1. **Overview.** In this section we establish that for $\mu \ll \alpha$ every sufficiently large (α,β,ℓ,μ)-constellation contains an *absorbing path* P_A, whose main property is that it can 'absorb' an arbitrary but not too large set of vertices whose cardinality is a multiple of k. Thus the problem of proving Theorem 1.2 gets reduced to the simpler task of finding an almost spanning cycle containing the absorbing path and missing a number of vertices that is divisible by k. In order to have a realistic chance to include the absorbing path into such a cycle we make sure that its first and last $(k-1)$-tuple is connectable. Moreover, we will need to be able to work outside a forbidden 'reservoir set' that later will have been selected in advance.

Proposition 5.1 (Absorbing Path Lemma). *Given $k \geqslant 3$, $\alpha > 0$, $\beta > 0$, and an odd integer $\ell \geqslant 3$ there exist constants $\zeta = \zeta(\alpha,k)$, $\vartheta_\star = \vartheta_\star(k,\alpha,\beta,\ell,\zeta) > 0$ and an integer n_0 with the following property.*

Suppose that Ψ is a k-uniform (α,β,ℓ,μ)-constellation with $\mu = \frac{1}{10k}\left(\frac{\alpha}{2}\right)^{2^{k-3}+1}$ on $n \geqslant n_0$ vertices. If $\mathcal{R} \subseteq V(\Psi)$ with $|\mathcal{R}| \leqslant \vartheta_\star^2 n$ is arbitrary, then there exists a path $P_A \subseteq H(\Psi) - \mathcal{R}$ such that

- (i) *$|V(P_A)| \leqslant \vartheta_\star n$,*
- (ii) *the starting and ending $(k-1)$-tuple of P_A are ζ-connectable,*
- (iii) *and for every subset $Z \subseteq V(\Psi) \smallsetminus V(P_A)$ with $|Z| \leqslant 2\vartheta_\star^2 n$ and $|Z| \equiv 0 \pmod{k}$, there exists a path $Q \subseteq H(\Psi)$ with $V(Q) = V(P_A) \cup Z$ having the same end-$(k-1)$-tuples as P_A.*

Our absorbers will be analogous to those in [15] and we refer to [15, Section 5.1] for further motivation. Here we will only recall that the absorbers have two kinds of main components reflecting the following observations.

- A complete k-partite subhypergraph S of $H(\Psi)$ whose vertex classes $\{x_i, x_{i+k}, x_{i+2k}\}$ are of size 3 (where $i \in [k]$) contains a spanning path $P = x_1 \dots x_{3k}$. Moreover, S also contains the path $P' = x_1 \dots x_k x_{2k+1} \dots x_{3k}$, which has the same first and last $(k-1)$-tuple as P. Thus if the absorbing path contains P' as a subpath but avoids the vertices $x_{k+1}, \dots, x_{2k}$, then it can absorb these vertices simultaneously (see Figure 5.1a). However, not every k-element subset of $V(\Psi)$ is *absorbable* in this manner.
- If the links of two vertices a and x intersect in a $(k-1)$-uniform path $b_1 \dots b_{2k-2}$, then we can form two k-uniform paths in $H(\Psi)$, namely $P_a = b_1 \dots b_{k-1} a b_k \dots b_{2k-1}$ and $P_x = b_1 \dots b_{k-1} x b_k \dots b_{2k-1}$ (see Figure 5.1b). Now if the absorbing path contains P_x, then we can remove x and insert a instead. We call such a structure an (a, x)-*exchanger*.

Now the plan for absorbing an arbitrary set $\{a_1, \dots, a_k\}$ of k vertices is that we will find an 'absorbable' set $\{x_1, \dots, x_k\}$ such that for every $i \in [k]$ there is an (a_i, x_i)-exchanger. The main difficulty in executing this strategy is that we need to pay a lot of attention to connectability issues, because ultimately we need to connect all parts of the absorbers we are about to construct to the rest of the Hamiltonian cycle we intend to exhibit. For this reason, the definition of absorbers reads as follows.

Definition 5.2. *Suppose that $k \geqslant 3$, $\alpha, \mu, \zeta > 0$, that Ψ is a k-uniform (α, μ)-constellation, and that $\vec{a} = (a_1, \dots, a_k) \in V(\Psi)^k$ is a k-tuple consisting of distinct vertices. We say that*

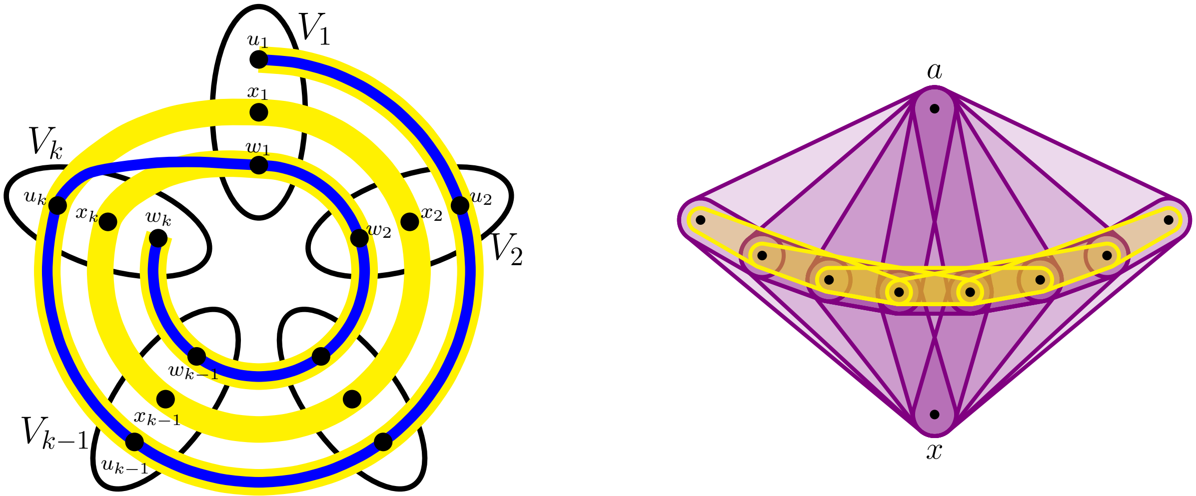
$$\vec{A} = (\vec{u}, \vec{x}, \vec{w}, \vec{b}_1, \dots, \vec{b}_k) \in V(\Psi)^{2k^2+k}$$

with $\vec{u} = (u_1, \dots, u_k)$, $\vec{x} = (x_1, \dots, x_k)$, $\vec{w} = (w_1, \dots, w_k)$, and $\vec{b}_i = (b_{i1}, \dots, b_{i(2k-2)})$ for $i \in [k]$ is an $(\vec{a}, \zeta)$-absorber in Ψ, if

- *(a) all $2k^2 + k$ vertices of $\vec{A}$ are distinct and different from those in $\vec{a}$,*
- *(b) $\vec{u}\vec{x}\vec{w}$ and $\vec{u}\vec{w}$ are paths in $H(\Psi)$,*
- *(c) $(u_1, \dots, u_{k-1})$ is ζ-rightconnectable and $(w_2, \dots, w_k)$ is ζ-leftconnectable in Ψ,*
- *(d) and for every $i \in [k]$ the $(2k-2)$-tuple $\vec{b}_i$ is a path in $H(\Psi_{a_i}) \cap H(\Psi_{x_i})$ whose first and last $(k-1)$-tuple is ζ-connectable in Ψ.*

We conclude this subsection with an explicit description how these absorbers are going to be utilised (see Figure 5.2). Suppose to this end that for some k-tuple $\vec{a} = (a_1, \dots, a_k)$ consisting of k distinct vertices and some $(\vec{a}, \zeta)$-absorber $(\vec{u}, \vec{x}, \vec{w}, \vec{b}_1, \dots, \vec{b}_k)$ it turns out that the paths

$$\vec{u}\vec{w} \quad \text{and} \quad b_{i1} \dots b_{i(k-1)} x_i b_{ik} \dots b_{i(2k-2)} \quad \text{for } i \in [k] \tag{5.1}$$



(a) The $K_k^{(k)}(3)$ with two paths

(b) A 5-uniform (a, x)-exchanger

FIGURE 5.1. The building blocks of an absorber.

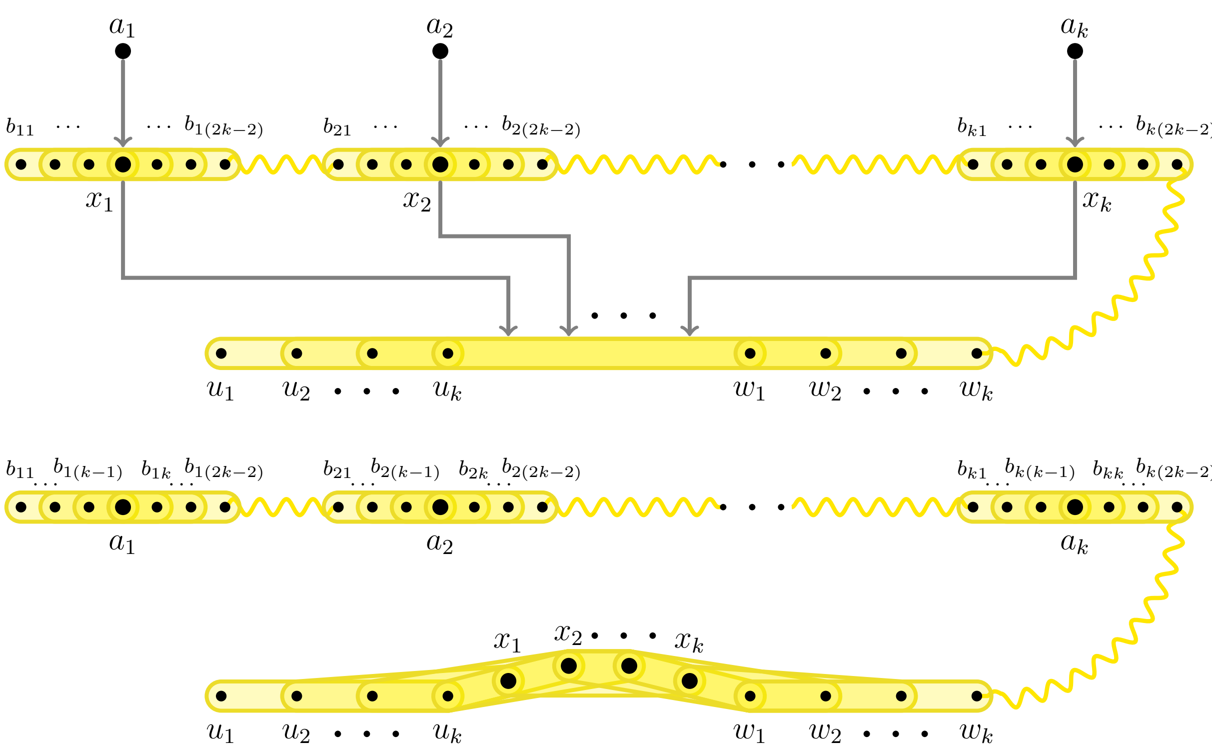


FIGURE 5.2. Absorber for $(a_1, \ldots, a_k)$ before and after absorption.

end up being subpaths of the absorbing path P_A we are about to construct, while $a_1, \ldots, a_k$ are not in $V(P_A)$. We may then replace for each $i \in [k]$ the path

$$b_{i1} \ldots b_{i(k-1)} x_i b_{ik} \ldots b_{i(2k-2)} \quad \text{by the path} \quad b_{i1} \ldots b_{i(k-1)} a_i b_{ik} \ldots b_{i(2k-2)} \,,$$

and then

$$\vec{u}\vec{w} \quad \text{by} \quad \vec{u}\vec{x}\vec{w}\,.$$

In this manner we transform P_A into a new path Q with $V(Q) = V(P_A) \cup \{a_1, \ldots, a_k\}$ having the same first and last $(k-1)$-tuple as P_A. We say in this situation that Q arises from P_A by *absorbing* $\{a_1, \ldots, a_k\}$. The $k+1$ paths enumerated in (5.1) are called the *pre-absorption paths* of the absorber $(\vec{u}, \vec{x}, \vec{w}, \vec{b}_1, \ldots, \vec{b}_k)$. So there is one pre-absorption path with $2k$ vertices, namely $\vec{u}\vec{w}$, and there are k pre-absorption paths with $2k-1$ vertices having a vertex x_i in the middle.

5.2. **Construction of the building blocks.** We commence with the first part $(\vec{u}, \vec{x}, \vec{w})$ of our absorbers consisting of $3k$ vertices. As we have already indicated, we shall find $(3k)$-tuples satisfying clause *(b)* of Definition 5.2 by looking for complete k-partite subhypergraphs of $H(\Psi)$ whose vertex classes are of size three.

Let us recall for this purpose that by a classic result of Erdős [6] the Turán density of every k-partite k-uniform hypergraph vanishes. This means that, given a k-partite k-uniform hypergraph F and a constant $\varepsilon > 0$, every sufficiently large k-uniform hypergraph H satisfying $|E(H)| \geqslant \varepsilon |V(H)|^k$ contains a copy of F. Due to the so-called 'supersaturation' phenomenon later discovered by Erdős and Simonovits [7], the same assumption actually implies that H contains $\Omega\big(|V(H)|^{|V(F)|}\big)$ copies of F. For later reference, we record this fact as follows.

Lemma 5.3. *Given a k-partite k-uniform hypergraph F and $\varepsilon > 0$, there are a constant $\xi > 0$ and a natural number n_0 such that every k-uniform hypergraph H on $n \geqslant n_0$ vertices with at least εn^k edges contains at least $\xi n^{|V(F)|}$ copies of F.* □

We shall now apply this result to $F = K_k^{(k)}(3)$, the complete k-partite hypergraph with vertex classes of size 3, and to an auxiliary hypergraph whose edges are derived from bridges. This will establish the following statement, whose conditions *(i)* and *(ii)* coincide with *(b)* and *(c)* from Definition 5.2.

Lemma 5.4. *For every $k \geqslant 2$ there exists $\xi = \xi(k) > 0$ such that for every $\alpha > 0$ there is an integer n_0 with the following property.*

For every k-uniform $(\alpha, \frac{\alpha}{9})$-constellation Ψ on $n \geqslant n_0$ vertices the number of $(3k)$-tuples $(\vec{u}, \vec{x}, \vec{w}) \in V(\Psi)^k \times V(\Psi)^k \times V(\Psi)^k$ such that writing $\vec{u} = (u_1, \ldots, u_k)$, $\vec{x} = (x_1, \ldots, x_k)$, and $\vec{w} = (w_1, \ldots, w_k)$

(i) both $\vec{u}\vec{x}\vec{w}$ and $\vec{u}\vec{w}$ are k-uniform paths in Ψ,

(ii) $(u_1, \ldots, u_{k-1})$ is $\frac{1}{9k}$-rightconnectable and $(w_2, \ldots, w_k)$ is $\frac{1}{9k}$-leftconnectable in Ψ

is at least ξn^{3k}.

Proof. Throughout the argument we assume that $\xi \ll k^{-1}$ is sufficiently small and that $n_0 \gg \alpha^{-1}, \xi^{-1}$ is sufficiently large. Let Ψ be a k-uniform $(\alpha, \frac{\alpha}{9})$-constellation on $n \geqslant n_0$ vertices. Construct an auxiliary k-partite k-uniform hypergraph $\mathscr{B} = (V_1 \cup \ldots \cup V_k, E_{\mathscr{B}})$ whose vertex classes are k disjoint copies of $V(\Psi)$ and whose edges $\{v_1, \ldots, v_k\} \in E_{\mathscr{B}}$ with $v_i \in V_i$ for $i \in [k]$ correspond to the $\frac{1}{9k}$-bridges $(v_1, \ldots, v_k)$ of Ψ. Corollary 2.28 tells us that

$$|E_{\mathscr{B}}| \geqslant \frac{1}{9}n^k = \frac{1}{9k^k}|V(\mathscr{B})|^k \,.$$

So Lemma 5.3 applied to $\mathscr{B}$ and $F = K_k^{(k)}(3)$ leads to $\Omega(n^{3k})$ copies of $K_k^{(k)}(3)$ in $\mathscr{B}$, where the implied constant only depends on k. In other words, for some constant $\xi = \xi(k)$ depending only on k there are at least ξn^{3k} tuples $(\vec{u}, \vec{x}, \vec{w}) \in V(\Psi)^k \times V(\Psi)^k \times V(\Psi)^k$ such that, writing $\vec{u} = (u_1, \ldots, u_k)$, $\vec{x} = (x_1, \ldots, x_k)$, and $\vec{w} = (w_1, \ldots, w_k)$, we have a copy of $K_k^{(k)}(3)$ in $\mathscr{B}$ with $u_i, x_i, w_i \in V_i$ for all $i \in [k]$. Clearly, these $(3k)$-tuples satisfy the demand *(i)* of the lemma and, since $\vec{u}$ and $\vec{w}$ are $\frac{1}{9k}$-bridges, they have property *(ii)* as well (cf. Definition 2.21). □

Armed with this result and with Corollary 2.32 we can now prove that if $\zeta, \mu \ll \alpha, k^{-1}$, then for every k-tuple $\vec{a}$ of distinct vertices from a sufficiently large (α, μ)-constellation the number of $(\vec{a}, \zeta)$-absorbers is at least $\Omega(n^{2k^2+k})$.

Lemma 5.5. *For every $k \geqslant 3$ and $\alpha > 0$ there exist constants $\zeta = \zeta(\alpha, k)$ and $\xi = \xi(\alpha, k)$ as well as an integer n_0 with the following property.*

If Ψ is a k-uniform (α, μ)-constellation on $n \geqslant n_0$ vertices, where $\mu = \frac{1}{10k}\big(\frac{\alpha}{2}\big)^{2^{k-3}+1}$, and $\vec{a} \in V(\Psi)^k$ is an arbitrary k-tuple of distinct vertices, then the number of $(\vec{a}, \zeta)$-absorbers in Ψ is at least ξn^{2k^2+k}.

Proof. Starting with the constant $\xi'' = \xi''(k) > 0$ provided by Lemma 5.4 we set

$$\xi' = \frac{\mu^k}{2}, \quad \zeta = \frac{\xi''\mu}{7k}, \quad \text{and} \quad \xi = \frac{1}{4}(\xi')^k\xi'' \tag{5.2}$$

and we suppose that n_0 is sufficiently large.

In order to show that ζ and ξ have the desired property, we consider a k-uniform (α, μ)-constellation Ψ on $n \geqslant n_0$ vertices as well as a k-tuple $\vec{a} = (a_1, \ldots, a_k) \in V(\Psi)^k$ consisting of distinct vertices. The set $X \subseteq V(\Psi)$ delivered by Corollary 2.32 (with the same meaning of Ψ, α, μ, and ζ as here) satisfies

$$|X| \leqslant \frac{\zeta}{\mu}n \overset{(5.2)}{=} \frac{\xi''}{7k}n \,. \tag{5.3}$$

By $\mu \leqslant \frac{\alpha}{9}$, $\zeta \leqslant \frac{1}{9k}$, and monotonicity, Lemma 5.4 yields at least $\xi'' n^{3k}$ paths $(\vec{u}, \vec{x}, \vec{w})$ in $V(\Psi)^{3k}$ with the properties *(i)* and *(ii)* of that lemma. Since the number of these paths sharing a vertex with $X \cup \{a_1, \dots, a_k\}$ can be bounded from above by

$$3k(|X| + k)n^{3k-1} \overset{(5.3)}{\leqslant} 3k\frac{\xi''}{7k}n^{3k} + 3k^2n^{3k-1} < \frac{\xi''}{2}n^{3k}\,,$$

there are at least $\frac{\xi''}{2}n^{3k}$ such paths avoiding both X and $\vec{a}$. Now it suffices to establish that each of them participates in at least $\frac{1}{2}(\xi')^k n^{2k^2-2k}$ absorbers.

For the rest of the proof we fix some such path $(\vec{u}, \vec{x}, \vec{w}) \in V(\Psi)^{3k}$ and, as usual, we write $\vec{x} = (x_1, \dots, x_k)$. Now we apply Corollary 2.32 for every $i \in [k]$ to the vertices a_i and x_i, thus obtaining $\xi' n^{2k-2}$ paths $\vec{b}_i = (b_{i1}, \dots, b_{i(2k-2)}) \in V(\Psi)^{2k-2}$ in $H(\Psi_{a_i}) \cap H(\Psi_{x_i})$ whose first and last $(k-1)$-tuples are ζ-connectable in Ψ. Altogether, this yields $(\xi')^k n^{2k^2-2k}$ possibilities for $(\vec{b}_1, \dots, \vec{b}_k)$ and for most of them $(\vec{u}, \vec{x}, \vec{w}, \vec{b}_1, \dots, \vec{b}_k)$ is an $(\vec{a}, \zeta)$-absorber. The only exceptions occur when some of these $2k^2 + k$ vertices coincide, but this can happen in at most $(2k^2 + k)(2k^2 - 2k)n^{(2k-2)k-1} < \frac{1}{2}(\xi')^k n^{2k^2-2k}$ ways. Thus $(\vec{u}, \vec{x}, \vec{w})$ is indeed extendable in at least $\frac{1}{2}(\xi')^k n^{2k^2-2k}$ distinct ways to an $(\vec{a}, \zeta)$-absorber. □

5.3. **Construction of the absorbing path.** After these preparations the Absorbing Path Lemma can be shown in a rather standard fashion. The argument starts by observing that a random selection of $(2k^2 + k)$-tuples contains, with high probability, for every k-tuple $\vec{a}$ a positive proportion of $(\vec{a}, \zeta)$-absorbers. Moreover, if we generate $\Theta(n)$ such random tuples with a small implied constant, then most of them will be disjoint to all others and it remains to connect the paths they consist of by means of the Connecting Lemma.

Proof of Proposition 5.1. Given to us are $k \geqslant 3$, $\alpha, \beta > 0$, an odd integer $\ell \geqslant 3$, and $\mu = \frac{1}{10k}\big(\frac{\alpha}{2}\big)^{2^{k-3}+1}$. Let $\zeta = \zeta(\alpha, k) > 0$ and $\xi = \xi(\alpha, k) > 0$ be the constants supplied by Lemma 5.5, let $\vartheta = \vartheta(k, \alpha, \beta, \ell, \zeta)$ be provided by Proposition 3.3, define an auxiliary constant by

$$\gamma = \min\Big\{\frac{\xi}{48k^2M^2}, \frac{\vartheta}{8kM^2}\Big\}, \quad \text{where} \quad M = 4^{k-2}k\ell \geqslant 12k\,, \tag{5.4}$$

and finally set

$$\vartheta_\star = 4kM\gamma\,.$$

We contend that ζ and $\vartheta_\star$ have the desired properties.

To verify this we consider a k-uniform $(\alpha, \beta, \ell, \mu)$-constellation Ψ on n vertices, where n is sufficiently large, as well as an arbitrary subset $\mathcal{R} \subseteq V(\Psi)$ whose size is at most $\vartheta_\star^2 n$. Let

$$t = 2k^2 + k < 3k^2$$

be the length of our absorbers. Since the desired absorbing path needs to be disjoint to $\mathcal{R}$, only the absorbers avoiding $\mathcal{R}$ are relevant in the sequel. For every k-tuple $\vec{a} \in V(\Psi)^k$

consisting of distinct vertices we denote the collection of appropriate absorbers by

$$\mathscr{A}(\vec{a}) = \big\{\vec{A} \in (V(\Psi) \smallsetminus \mathcal{R})^t : \vec{A} \text{ is an } (\vec{a},\zeta)\text{-absorber}\big\}\,.$$

Lemma 5.5 tells us that the total number of $(\vec{a},\zeta)$-absorbers is at least ξn^t and by subtracting those which meet $\mathcal{R}$ we obtain

$$|\mathscr{A}(\vec{a})| \geqslant \xi n^t - t|\mathcal{R}|n^{t-1} \geqslant (\xi - t\vartheta_\star^2)n^t \geqslant \frac{\xi}{2}n^t\,. \tag{5.5}$$

Let

$$\mathscr{A} = \bigcup\big\{\mathscr{A}(\vec{a}) : \vec{a} \in V(\Psi)^k \text{ consists of } k \text{ distinct vertices}\big\} \subseteq \big(V(\Psi) \smallsetminus \mathcal{R}\big)^t$$

be the set of all relevant absorbers. The probabilistic argument we have been alluding to earlier leads to the following result.

Claim 5.6. *There is a set $\mathscr{B} \subseteq \mathscr{A}$ of mutually disjoint absorbers of size $|\mathscr{B}| \leqslant 2\gamma n$ satisfying $|\mathscr{A}(\vec{a}) \cap \mathscr{B}| \geqslant \vartheta_\star^2 n$ for every k-tuple $\vec{a} \in V(\Psi)^k$ consisting of distinct vertices.*

Proof. Let $\mathscr{A}_p \subseteq \mathscr{A}$ be a random subset including every absorber in $\mathscr{A}$ independently with probability $p = \gamma n^{1-t}$. As $|\mathscr{A}_p|$ is binomially distributed with expectation $p|\mathscr{A}| \leqslant pn^t = \gamma n$, Markov's inequality yields

$$\mathbb{P}\big(|\mathscr{A}_p| \geqslant 2\gamma n\big) \leqslant \mathbb{P}\big(|\mathscr{A}_p| \geqslant 2p|\mathscr{A}|\big) \leqslant \frac{1}{2}\,. \tag{5.6}$$

Next we observe that the set

$$\big\{\{\vec{A}, \vec{A}'\} \in \mathscr{A}^{(2)} : \vec{A} \text{ and } \vec{A}' \text{ share a vertex}\big\}$$

of *overlapping pairs of absorbers* has at most the cardinality t^2n^{2t-1}. So the expected size of its intersection with $\mathscr{A}_p^{(2)}$ is at most $p^2t^2n^{2t-1} = \gamma^2t^2n$. Since

$$\gamma t \leqslant 3k^2\gamma \leqslant \tfrac{1}{4}\vartheta_\star\,,$$

a further application of Markov's inequality reveals

$$\mathbb{P}\Big(\big|\big\{\{\vec{A}, \vec{A}'\} \in \mathscr{A}_p^{(2)} : \ \vec{A} \text{ and } \vec{A}' \text{ share a vertex}\big\}\big| \geqslant \tfrac{1}{4}\vartheta_\star^2 n\Big) \leqslant \frac{1}{4}\,. \tag{5.7}$$

Finally, for every k-tuple $\vec{a} \in V(\Psi)^k$ of distinct vertices the random variable $|\mathscr{A}_p \cap \mathscr{A}(\vec{a})|$ is binomially distributed with expectation $p|\mathscr{A}(\vec{a})|$. By (5.5) we know that

$$p|\mathscr{A}(\vec{a})| \geqslant \frac{1}{2}\gamma\xi n \geqslant 24k^2M^2\gamma^2 n = \tfrac{3}{2}\vartheta_\star^2 n$$

and, therefore, Chernoff's inequality yields

$$\mathbb{P}\big(|\mathscr{A}_p \cap \mathscr{A}(\vec{a})| \leqslant \tfrac{5}{4}\vartheta_\star^2 n\big) \leqslant e^{-\Omega(n)} < \frac{1}{4n^k}\,.$$

As there are at most n^k possibilities for $\vec{a}$, the union bound leads to

$$\mathbb{P}\big(|\mathscr{A}_p \cap \mathscr{A}(\vec{a})| \leqslant \tfrac{5}{4}\vartheta_\star^2 n \text{ holds for some } \vec{a}\big) < \frac{1}{4}\,. \tag{5.8}$$

Taken together, the probabilities estimated in (5.6)–(5.8) amount to less than 1. Thus there exists a deterministic set $\mathscr{B}_\star \subseteq \mathscr{A}$ of size $|\mathscr{B}_\star| \leqslant 2\gamma n$ containing at most $\frac{1}{4}\vartheta_\star^2 n$ pairs of overlapping absorbers and satisfying $|\mathscr{B}_\star \cap \mathscr{A}(\vec{a})| \geqslant \frac{5}{4}\vartheta_\star^2 n$ for all k-tuples $\vec{a} \in V(\Psi)^k$ of distinct vertices.

Now it suffices to check that a maximal subcollection $\mathscr{B} \subseteq \mathscr{B}_\star$ of mutually disjoint absorbers has the desired properties. The upper bound $|\mathscr{B}| \leqslant |\mathscr{B}_\star| \leqslant 2\gamma n$ is clear and due to $|\mathscr{B}_\star \smallsetminus \mathscr{B}| \leqslant \frac{1}{4}\vartheta_\star^2 n$ we have

$$|\mathscr{B} \cap \mathscr{A}(\vec{a})| \geqslant \tfrac{5}{4}\vartheta_\star^2 n - \tfrac{1}{4}\vartheta_\star^2 n = \vartheta_\star^2 n$$

for every $\vec{a}$. $\square$

It remains to connect the absorbers we have just selected into a path. Recall that every member of $\mathscr{B}$ possesses $k+1$ pre-absorptions paths introduced in the last paragraph of Subsection 5.1. Each of these paths has at most $2k$ vertices, starts with a ζ-rightconnectable $(k-1)$-tuple, and ends with a ζ-leftconnectable $(k-1)$-tuple. In fact, most of the pre-absorptions paths even have ζ-connectable end-tuples (see Definition 5.2 *(d)*).

Setting $r = (k+1)|\mathscr{B}| \leqslant 4k\gamma n$, let $P_1, \dots, P_r$ be the pre-absorption paths of the absorbers in $\mathscr{B}$ enumerated in such a way that the end-tuples of P_1 and P_r are ζ-connectable. We shall construct our absorbing path P_A to be of the form

$$P_A = P_1 C_1 P_2 C_2 \dots P_{r-1} C_{r-1} P_r\,,$$

where $C_1, \dots, C_{r-1}$ are connections that will be provided by Proposition 3.3. Since we intend to use the Connecting Lemma with $i = 0$, each of these connections is going to have

$$f = f(k, 0, \ell) = [4^{k-3}(2\ell + 4) - 2]k \leqslant M - 2k$$

vertices, which will yield

$$|V(P_A)| \leqslant r\big(2k + (M - 2k)\big) = rM \leqslant 4kM\gamma n\,. \tag{5.9}$$

We will determine the connections $C_1, \dots, C_{r-1}$ one by one. When choosing C_j for some $j \in [r-1]$, the Connecting Lemma (Proposition 3.3) offers us at least ϑn^f possible ways to connect P_j with P_{j+1} by means of a path with f inner vertices. As we need to avoid both the already constructed parts of P_A and the set $\mathcal{R}$, there are at most

$$f(|\mathcal{R}| + 4kM\gamma n)n^{f-1} < (M\vartheta_\star^2 + 4kM^2\gamma)n^f \overset{(5.4)}{<} 8kM^2\gamma n^f \overset{(5.4)}{\leqslant} \vartheta n^f$$

potential connections we cannot use, and thus the choice of C_j is indeed possible. This concludes the description of the construction of P_A and it remains to check that the path we just defined has all required properties.

Condition *(i)* follows from (5.9) and *(ii)* is guaranteed by our choice of the enumeration $P_1, \dots, P_r$. For the proof of *(iii)* we consider any set $Z \subseteq V(\Psi) \smallsetminus V(P_A)$ satisfying $|Z| \leqslant 2\vartheta_\star^2 n$ and $|Z| \equiv 0 \pmod{k}$. Let $\vec{a}_1, \dots, \vec{a}_z \in V(\Psi)^k$ with $z = \frac{|Z|}{k} \leqslant \vartheta_\star^2 n$ be disjoint k-tuples with the property that every vertex from Z occurs in exactly one of them. By Claim 5.6 we can find distinct absorbers $\vec{A}_1, \dots, \vec{A}_z \in \mathscr{B}$ such that $\vec{A}_j$ is a $(\vec{a}_j, \zeta)$-absorber for every $j \in [z]$. It remains to utilise these absorbers one by one. □

§6. Covering

The aim of this section is to prove that under natural assumptions on the parameters almost all vertices of every large k-uniform $(\alpha, \beta, \ell, \mu)$-constellation can be covered by long paths whose first and last $(k-1)$-tuples are connectable. Before formulating the precise statements let us give an overview of the argument, which will proceed by induction on k.

In the induction step from $k-1$ to k we study a largest possible collection $\mathscr{C}$ of mutually vertex-disjoint M-vertex paths with connectable end-tuples and we denote the set of currently uncovered vertices by U. If U is not small enough already, i.e., if $|U| = \Omega(|V(\Psi)|)$, then we partition $V(\Psi)$ into sets of size M, the so-called blocks, such that the vertex set of each path in $\mathscr{C}$ is one such block. Next, we show by probabilistic arguments that there is a special selection of M blocks, called a useful society below, such that their union S has the property that for 'many' vertices $u \in U$ the induction hypothesis applies to $\Psi_u[S]$. For such vertices u we can then find $M+1$ (actually even more) long disjoint $(k-1)$-uniform paths in $\Psi_u[S]$ starting and ending with connectable $(k-2)$-tuples.

In fact, for some still not too small set $U'' \subseteq U'$ these paths will coincide for all $u \in U''$, meaning that inserting vertices from U'' at every k^{th} position will yield $M+1$ paths in Ψ with connectable end-tuples (see Figure 6.2). This allows us to take the original paths contained in S out of $\mathscr{C}$ and to add the newly constructed paths instead, thus increasing the size of $\mathscr{C}$. The following covering principle lies at the heart of this inductive argument.

Definition 6.1. *For $k \geqslant 3$ the statement $\heartsuit_k$ asserts that given $\alpha, \beta, \vartheta_\star > 0$ and an odd integer $\ell \geqslant 3$ there exists a constant $\zeta_{\star\star} > 0$ such that for every $M_0 \in \mathbb{N}$ there exist a natural number $M \geqslant M_0$ with $M \equiv -1 \pmod{k}$ and the following property:*

For every sufficiently large k-uniform $(\alpha, \beta, \ell, \frac{4\alpha}{17^k})$-constellation Ψ we can cover all but at most $\vartheta_\star^2 |V(\Psi)|$ vertices by mutually vertex-disjoint M-vertex paths whose first and last $(k-1)$-tuples are $\zeta_{\star\star}$-connectable.

For the base case $k = 3$ we quote [15, Lemma 2.14]. One needs to be a little bit careful here, because [15] uses a slightly different notion of $\zeta_{\star\star}$-connectable pairs in 3-uniform hypergraphs. However, every pair that is $\zeta_{\star\star}$-connectable in the sense of [15] is $\zeta_{\star\star}$-connectable in the sense of Definition 2.16 as well and, therefore, [15, Lemma 2.14] is strictly stronger than $\heartsuit_3$.

Fact 6.2. *The assertion $\heartsuit_3$ holds.* $\square$

There is one issue with the inductive proof of $\heartsuit_k$ sketched above: when applying the induction hypothesis to a $(k-1)$-uniform constellation of the form $\Psi_u[S]$, where S is the vertex set of a useful society, we would prefer to get a covering of almost all vertices in S by paths of length $\Omega(\sqrt{|S|})$ rather than $\Omega(1)$, but prima facie $\heartsuit_{k-1}$ does not seem to deliver this. For this reason we also have to deal with the following statement capable of providing coverings by very long paths.

Definition 6.3. *For $k \geqslant 3$ the covering principle $\spadesuit_k$ asserts that given $\alpha, \beta, \xi > 0$ and an odd integer $\ell \geqslant 3$, there exists an infinite arithmetic progression $P \subseteq k\mathbb{N}$ with the following property.*

If Ψ is a k-uniform $(\alpha, \beta, \ell, \frac{\alpha}{17^k})$-constellation, $M \in P$, and $\mathfrak{B} \subseteq V(\Psi)^k$ is a collection of ξ-bridges in Ψ with $|\mathfrak{B}| \geqslant \xi |V(\Psi)|^k$, then all but at most $\lfloor \xi |V(\Psi)| \rfloor + M$ vertices of Ψ can be covered with mutually disjoint M-vertex paths starting and ending with bridges from $\mathfrak{B}$.

Observe that for a fixed k-uniform $(\alpha, \beta, \ell, \frac{\alpha}{17^k})$-constellation Ψ we can apply $\spadesuit_k$ with every $M \in P$. For a larger value of M we have to cover fewer vertices, but, on the other hand, we need to cover them with longer paths. Thus there is no obvious monotonicity in M.

Now we plan to establish the implication $\heartsuit_{k-1} \Rightarrow \spadesuit_{k-1} \Rightarrow \heartsuit_k$, thus decomposing the induction step of the proof of $\heartsuit_k$ into two simpler tasks. They will be treated in Lemma 6.4 and Lemma 6.9, respectively.

Lemma 6.4. *If $k \geqslant 3$ and $\heartsuit_k$ holds, then so does $\spadesuit_k$.*

The idea behind the proof of this implication is the following (see Figure 6.1). Given an appropriate constellation Ψ, our first step is to take out a reservoir set $\mathcal{R}$. Next we decide which bridges from $\mathfrak{B}$ are going to appear at the ends of the paths we are supposed to construct. After these choices are made, we apply $\heartsuit_k$ to the constellation obtained from Ψ by removing $\mathcal{R}$ and the vertices reserved for the bridges, thus getting a covering of almost all remaining vertices with 'short' paths. Now we partition the set of these paths into groups of size p, where p denotes an arbitrary natural number. For each group we

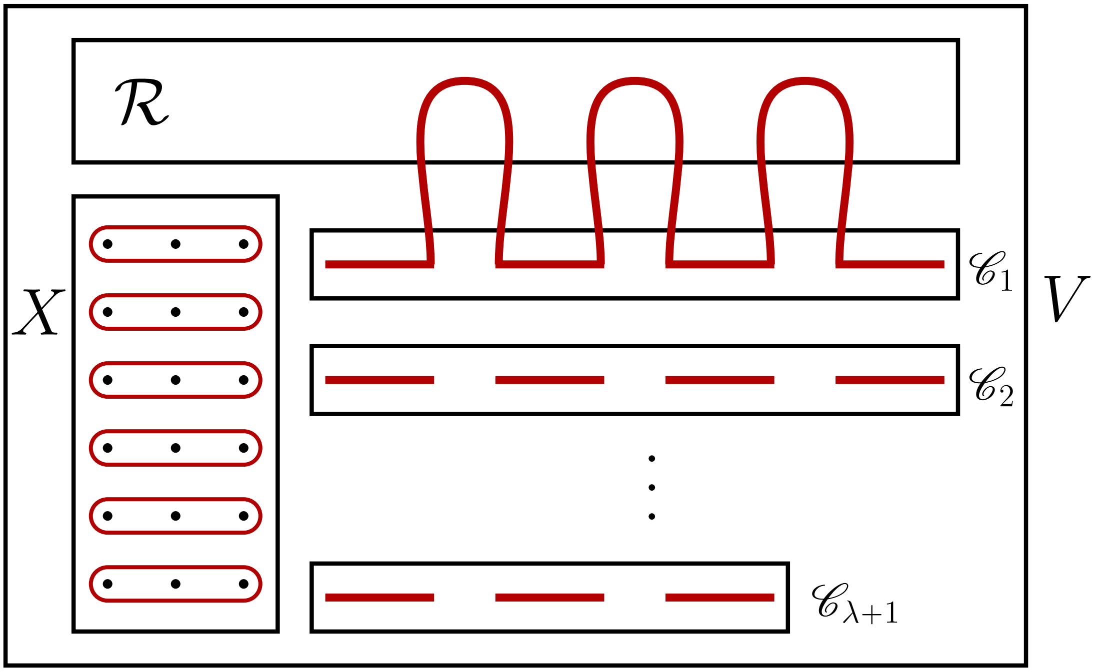


FIGURE 6.1. The case $k = 3$ of Lemma 6.4. The set X of vertices is reserved for bridges.

connect all its paths through the reservoir. Moreover, we connect the ends of the resulting paths to some of the bridges that have been put aside. In this manner we obtain a covering of almost all vertices of Ψ with longer paths, whose precise length depends linearly on p. Thus by varying p we can reach an arithmetic progression of possible lengths for the paths in the new covering.

Proof of Lemma 6.4. Let α, β, $\xi > 0$ and an odd integer $\ell \geqslant 3$ be given. Choose some auxiliary constants obeying the hierarchy

$$\alpha, \beta, \xi, k^{-1}, \ell^{-1} \gg \vartheta_\star \gg \zeta_{\star\star} \gg \vartheta_{\star\star} \gg M^{-1} \gg n_0^{-1},$$

where M is an integer with $M \equiv -1 \pmod{k}$.

We contend that

$$P = \{M' \in k\mathbb{N} \colon M' > n_0 \text{ and } M' \equiv f(k, 0, \ell) + 2k \pmod{M + f(k, 0, \ell)}\}$$

has the property demanded by $\spadesuit_k$.

By Definition 3.2 the number $f(k, 0, \ell)$ is divisible by k and, consequently, P is indeed an infinite arithmetic progression. Now let Ψ be a k-uniform $(\alpha, \beta, \ell, \frac{\alpha}{17^k})$-constellation with n vertices, let $M' \in P$ be arbitrary, and let $\mathfrak{B} \subseteq V(\Psi)^k$ be a set of ξ-bridges in Ψ with $|\mathfrak{B}| \geqslant \xi|V(\Psi)|^k$. We are to cover all but at most $\xi|V(\Psi)| + M'$ vertices of Ψ by mutually disjoint M'-vertex paths starting and ending with bridges from $\mathfrak{B}$. If $|V(\Psi)| \leqslant M'$, then the empty set is such a collection of paths. Thus, we may assume that $|V(\Psi)| > M' > n_0$.

Let $\mathcal{R} \subseteq V(\Psi)$ with $|\mathcal{R}| \leqslant \vartheta_\star n$ be a the reservoir set provided by Proposition 4.1 with $\vartheta_\star$, $\frac{\zeta_{\star\star}}{2}$ here in place of ξ, $\zeta_{\star\star}$ there. For later use we record that due to $\vartheta_{\star\star} \ll \vartheta_\star, k^{-1}, \ell^{-1}$ the case $i = 0$ of Corollary 4.2 yields:

($\star$) If $\mathcal{R}' \subseteq \mathcal{R}$ is an arbitrary set with $|\mathcal{R}'| \leqslant \vartheta_{\star\star}^2 |V(\Psi)|$, the $(k-1)$-tuple $\vec{a} \in V(\Psi)^{k-1}$ is $\frac{\zeta_{\star\star}}{2}$-leftconnectable, and $\vec{b} \in V(\Psi)^{k-1}$ is $\frac{\zeta_{\star\star}}{2}$-rightconnectable and disjoint to $\vec{a}$, then there is an $\vec{a}$-$\vec{b}$-path through $\mathcal{R} \smallsetminus \mathcal{R}'$ with $f(k, 0, \ell)$ inner vertices.

Let $b_1, \dots, b_r$ be a maximal sequence of bridges from $\mathfrak{B}$ that are mutually disjoint and disjoint to $\mathcal{R}$. Since the selected bridges and $\mathcal{R}$ together involve $kr + |\mathcal{R}|$ vertices, the maximality implies

$$k(kr + |\mathcal{R}|)|V(\Psi)|^{k-1} \geqslant |\mathfrak{B}| \geqslant \xi |V(\Psi)|^k\,,$$

whence

$$r \geqslant \frac{(\xi - k\vartheta_\star)|V(\Psi)|}{k^2} \geqslant \vartheta_\star |V(\Psi)|\,. \tag{6.1}$$

Set $x = \lfloor \vartheta_\star |V(\Psi)| \rfloor$ and let X be the set of vertices constituting $b_1, \dots, b_x$. Lemma 2.36 reveals that $\Psi' = \Psi - (X \cup \mathcal{R})$ is an $\big(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, \frac{2\alpha}{17^k}\big)$-constellation. Therefore, the principle $\heartsuit_k$ yields a family $\mathscr{C}$ of disjoint M-vertex paths in Ψ' which together cover all but at most $\vartheta_\star^2 |V(\Psi')|$ vertices of Ψ' and whose end-tuples are $\zeta_{\star\star}$-connectable in Ψ'. For later use we remark that owing to Fact 2.18 the end-tuples of the paths in $\mathscr{C}$ are $\frac{\zeta_{\star\star}}{2}$-connectable in Ψ.

By the definition of P there is a natural number p such that

$$M' = \big(M + f(k, 0, \ell)\big)p + f(k, 0, \ell) + 2k\,.$$

Fix an arbitrary partition $\mathscr{C} = \mathscr{C}_1 \cup \ldots \cup \mathscr{C}_{\lambda+1}$ with $|\mathscr{C}_1| = \dots = |\mathscr{C}_\lambda| = p > |\mathscr{C}_{\lambda+1}|$.

Now we declare our strategy for constructing vertex-disjoint paths $P_1, \dots, P_\lambda \subseteq H(\Psi)$ witnessing the conclusion of $\spadesuit_k$. For every $j \in [\lambda]$ we first intend to form a path P'_j by connecting the p paths in $\mathscr{C}_j$ through the reservoir $\mathcal{R}$. Subsequently, we plan to derive P_j from P'_j by connecting its ends with two bridges from the list $b_1, \dots, b_x$, say with b_{2j-1} and b_{2j}. For all $p+1$ connections required for this construction of P_j, we want to appeal to ($\star$). Clearly, if the paths $P_1, \dots, P_\lambda$ can be constructed, then each of them will consist of M' vertices.

Altogether, we are aiming for $(p+1)\lambda$ connections that require a total number of

$$(p + 1)f(k, 0, \ell)\lambda$$

vertices from the reservoir. If this number is less than $\vartheta_{\star\star}^2 n$, then repeated applications of $(\star)$ allow us to choose our connections disjointly. Since $M \gg \vartheta_{\star\star}^{-1} \gg k, \ell$, we have indeed

$$(p+1)f(k,0,\ell)\lambda \leqslant 2p \cdot 4^k k\ell \cdot \frac{|V(\Psi)|}{Mp} = \frac{2 \cdot 4^k k\ell |V(\Psi)|}{M} < \vartheta_{\star\star}^2 |V(\Psi)|\,.$$

Similarly,

$$2\lambda \leqslant \frac{2|V(\Psi)|}{Mp} \leqslant \frac{2|V(\Psi)|}{M} \leqslant \vartheta_\star |V(\Psi)|$$

proves that we have sufficiently many bridges at our disposal.

Altogether, the vertex-disjoint paths $P_1, \dots, P_\lambda \subseteq H(\Psi)$ can indeed be constructed. The number of vertices of Ψ they fail to cover can be bounded from above by

$$\begin{aligned}
|X| + |\mathcal{R}| + \Big|V(\Psi') \smallsetminus \bigcup_{P \in \mathscr{C}} V(P)\Big| + \Big|\bigcup_{P \in \mathscr{C}_{\lambda+1}} V(P)\Big| &\leqslant kx + \vartheta_\star |V(\Psi)| + \vartheta_\star^2 |V(\Psi)| + Mp \\
&\leqslant Mp + \big((k+1)\vartheta_\star + \vartheta_\star^2\big)|V(\Psi)| \\
&\leqslant M' + \xi |V(\Psi)|\,,
\end{aligned}$$

which concludes the proof of $\spadesuit_k$. $\square$

The proof of our next result involves some probabilistic arguments based on the following consequence of Janson's inequality (see [15, Corollary A.3]).

Lemma 6.5. *Let $m \geqslant k$ and M be positive integers, and let $\eta \in (0, \frac{1}{2k})$. Suppose that V is a finite set and that*

$$V = B_1 \cupdot \dots \cupdot B_\nu \cupdot Z$$

is a partition with $|B_1| = \dots = |B_\nu| = M < \eta|V|$, $|Z| < \eta|V|$, and $\nu \geqslant m$. Let $\mathscr{S} \subseteq \{B_1, \dots, B_\nu\}$ be an m-element subset chosen uniformly at random and set $S = \bigcup \mathscr{S}$. Further, let ξ be a real number with $\max(8k^2\eta, 16k^2/m) < \xi < 1$.

(*a*) *If $Q \subseteq V^k$ has size $|Q| = d|V|^k$, then*

$$\mathbb{P}\Big(\big||Q \cap S^k| - d(Mm)^k\big| \geqslant \xi (Mm)^k\Big) \leqslant 12\sqrt{m}\exp\left(-\frac{\xi^2 m}{48k^{2k+2}}\right)\,.$$

(*b*) *Similarly, if G denotes a k-uniform hypergraph with vertex set V and $d|V|^k/k!$ edges, then*

$$\mathbb{P}\Big(\big|e_G(S) - d(Mm)^k/k!\big| \geqslant \xi (Mm)^k/k!\Big) \leqslant 12\sqrt{m}\exp\left(-\frac{\xi^2 m}{48k^{2k+2}}\right)\,. \qquad \square$$

This has the following consequence on random subconstellations.

Lemma 6.6. *Given $k \geqslant 2$, $\alpha, \beta, \mu, \xi > 0$, and an odd integer $\ell \geqslant 3$ there exists a natural number M_0 such that the following holds for every $M \geqslant M_0$. If Ψ is a sufficiently large k-uniform $(\alpha, \beta, \ell, \mu)$-constellation,*

$$V(\Psi) = B_1 \charfusion[\mathbin]{\cup}{\cdot} \ldots \charfusion[\mathbin]{\cup}{\cdot} B_\nu \charfusion[\mathbin]{\cup}{\cdot} B'$$

is a partition with $|B_1| = \ldots = |B_\nu| = M$ and $|B'| < 2M$, and $\mathfrak{B} \subseteq V(\Psi)^k$ is a set of ξ-bridges in Ψ of size $|\mathfrak{B}| \geqslant \xi |V(\Psi)|^k$, then there are at least $\frac{3}{4}\binom{\nu}{M}$ sets $\mathscr{S} \subseteq \{B_1, \ldots, B_\nu\}$ of size M such that their union $S = \bigcup \mathscr{S}$ has the properties that $\Psi[S]$ is a $(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, 2\mu)$-constellation and

$$\mathfrak{B}_\star = \left\{\vec{x} \in \mathfrak{B} \cap S^k \colon \vec{x} \text{ is a } \tfrac{\xi}{2}\text{-bridge in } \Psi[S]\right\}$$

has at least the size $|\mathfrak{B}_\star| \geqslant \frac{\xi}{2}|S|^k$.

Proof. Let $M_0 \gg \alpha^{-1}, \beta^{-1}, \mu^{-1}, \xi^{-1}, k, \ell$ be sufficiently large. We call the sets $B_1, \ldots, B_\nu$ *blocks*. Choose a set $\mathscr{S} \subseteq \{B_1, \ldots, B_\nu\}$ of M blocks uniformly at random among all $\binom{\nu}{M}$ possibilities. We shall prove that the probability that $S = \bigcup \mathscr{S}$ fails to have the desired properties is at most $\exp\bigl(-\Omega(M)\bigr)$, where the implied constant only depends on α, β, μ, ξ, k, and ℓ. Hence, by choosing M_0 sufficiently large, this probability can be pushed below $\frac{1}{4}$, as desired. It will be convenient to set $V' = V \smallsetminus B'$. For $y \in V'$ we denote the unique block containing y by B_y.

Claim 6.7. *The event that $\Psi[S]$ fails to be a $(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, 2\mu)$-constellation has at most the probability $\exp\bigl(-\Omega(M)\bigr)$.*

Proof. We begin by estimating the probability of the unfortunate event $\mathfrak{U}$ that $\Psi[S]$ fails to be a $(\frac{\alpha}{2}, 2\mu)$-constellation. For an arbitrary set $x \in (V')^{(k-2)}$ we define

$$\mathscr{Z}_x = \{B_y \colon y \in x\}\,, \quad t_x = |\mathscr{Z}_x| \in [k-2]\,, \text{ and } \quad Z_x = \bigcup \mathscr{Z}_x\,.$$

Further, we consider the conditional probabilities

$$P_1(x) = \mathbb{P}\left(e_{\Psi_x}(S \smallsetminus Z_x) < \left(\frac{5}{9} + \frac{2\alpha}{3}\right)\frac{(M-t_x)^2M^2}{2} \;\middle|\; x \in S^{(k-2)}\right),$$

$$P_2(x) = \mathbb{P}\left(\bigl|V(R^\Psi_x[S])\bigr| < \left(\frac{2}{3} + \frac{\alpha}{3}\right)(M-t_x)M \;\middle|\; x \in S^{(k-2)}\right),$$

and

$$P_3(x) = \mathbb{P}\left(e_{\Psi_x[S]}\left(V(R^\Psi_x[S]), S \smallsetminus V(R^\Psi_x[S])\right) > 2\mu(M-t_x)^2M^2 \;\middle|\; x \in S^{(k-2)}\right)$$

and observe that

$$\mathbb{P}(\mathfrak{U}) \leqslant \sum_{x \in (V')^{(k-2)}} \mathbb{P}(x \in S^{(k-2)})\bigl(P_1(x) + P_2(x) + P_3(x)\bigr)\,. \tag{6.2}$$

So if we manage to prove

$$P_1(x), P_2(x), P_3(x) \leqslant \exp\bigl(-\Omega(M)\bigr)\,, \tag{6.3}$$

then

$$\mathbb{P}(\mathfrak{U}) \leqslant (M^2)^{k-2} \exp\bigl(-\Omega(M)\bigr) \leqslant \exp\bigl(-\Omega(M)\bigr) \tag{6.4}$$

will follow. Thus our next goal is to establish (6.3).

To this end, we will repeatedly apply Lemma 6.5 with

$$M - t_x\,, \frac{kM}{n}\,, B' \cup Z_x\,, \nu - t_x\,, \text{ and } \min\{\tfrac{\alpha}{6}, \mu\}$$

here in place of

$$m\,, \eta\,, Z\,, \nu\,, \text{ and } \xi$$

there and relocating the elements of $\mathscr{Z}_x$ to the exceptional set of the partition.

First, the minimum degree condition imposed on $H(\Psi)$ implies that the graph $H(\Psi_x)$ has at least $\bigl(\frac{5}{9} + \alpha\bigr) \frac{|V(\Psi)|^2}{2}$ edges. So Lemma 6.5 *(b)* applied with 2 and $H(\Psi_x)$ here in place of k and G there yields $P_1(x) \leqslant \exp\bigl(-\Omega(M)\bigr)$.

Second, we know that $|V(R_x^\Psi)| \geqslant \bigl(\frac{2}{3} + \frac{\alpha}{2}\bigr)\,|V(\Psi)|$, since Ψ is an (α,μ)-constellation. Hence, applying Lemma 6.5 *(a)* with 1 and $V(R_x^\Psi)$ here instead of k and Q there entails $P_2(x) \leqslant \exp\bigl(-\Omega(M)\bigr)$.

Lastly, from Ψ being a (α,μ)-constellation it also follows that

$$e_{\Psi_x}\bigl(V(R_x^\Psi), V \smallsetminus V(R_x^\Psi)\bigr) \leqslant \mu|V(\Psi)|^2\,.$$

Hence, Lemma 6.5 *(b)* applied to the bipartite subgraph of $H(\Psi_x)$ between $V(R_x^\Psi)$ and its complement tells us that $P_3(x) \leqslant \exp\bigl(-\Omega(M)\bigr)$. This concludes the proof of (6.3) and, hence, of (6.4). An analogous proof allows us to transfer part *(b)* of Definition 2.33 from Ψ to $\Psi[S]$ and we omit the details. □

It remains to prove that the event $|\mathfrak{B}_\star| \geqslant \frac{\xi}{2}|S|^k$ has high probability as well. Here we start with the estimate

$$\mathbb{P}\bigl(|\mathfrak{B}_\star| \leqslant \tfrac{\xi}{2}|S|^k\bigr) \leqslant \mathbb{P}\bigl(|\mathfrak{B} \cap S^k| \leqslant \tfrac{\xi}{2}|S|^k\bigr) + \mathbb{P}(\neg\mathfrak{E})\,,$$

where $\mathfrak{E}$ denotes the event that every ξ-bridge $\vec{x} \in \mathfrak{B} \cap S^k$ is a $\frac{\xi}{2}$-bridge in $\Psi[S]$. Another application of Lemma 6.5 *(a)* tells us that the first summand is at most $\exp\bigl(-\Omega(M)\bigr)$ and thus it remains to prove that

$$\mathbb{P}(\neg\mathfrak{E}) \leqslant \exp\bigl(-\Omega(M)\bigr)\,. \tag{6.5}$$

Towards this goal we analyse how connectability transfers to $\Psi[S]$.

Claim 6.8. *If $k' \in [k-1]$, $z, z' \in (V')^{(k-1-k')}$, and $\vec{x} \in \big(V' \smallsetminus (z \cup z')\big)^{k'}$ is a ξ-leftconnectable tuple in Ψ_z, then*

$$\mathbb{P}\big(\vec{x} \textit{ fails to be } \tfrac{\xi}{2}\textit{-leftconnectable in } \Psi_z[S] \mid \vec{x} \in S^{k'} \textit{ and } z' \subseteq S\big) \leqslant \exp\big(-\Omega(M)\big)\,.$$

Proof. We argue by induction on k'. In the base case $k' = 1$ the probability under consideration vanishes. This is because a 1-tuple $\vec{x} = (x)$ is ξ-leftconnectable in Ψ_z if and only if $x \in V(R_z^\Psi)$. Moreover, if $x \in S \smallsetminus z$, then (x) is $\frac{\xi}{2}$-leftconnectable in $\Psi_z[S]$ if and only if $x \in R_z^{\Psi[S]}$. Due to $R_z^{\Psi[S]} = R_z^\Psi[S]$ these two statements are equivalent to each other.

For the induction step from $k'-1$ to k' we write $\vec{x} = (x_1, \dots, x_{k'})$ and recall that the ξ-leftconnectability of $\vec{x}$ in Ψ_z means that $|U| \geqslant \xi |V(\Psi_z)|$, where

$$U = \big\{u \in V(\Psi_z) \colon x_1 \dots x_{k'} u \in E(\Psi_z) \text{ and } (x_2, \dots, x_{k'}) \text{ is } \xi\text{-leftconnectable in } \Psi_{zu}\big\}\,.$$

Assuming $\vec{x} \in S^{k'}$ the analogous set whose size decides whether $\vec{x}$ is $\frac{\xi}{2}$-leftconnectable in $\Psi_z[S]$ either contains $U \cap S$ as a subset, or it does not. Accordingly, if $\vec{x}$ fails to be $\frac{\xi}{2}$-leftconnectable in $\Psi_z[S]$, then either $|U \cap S| \leqslant \frac{\xi}{2}|V(\Psi_z[S])|$ or the event $\mathfrak{A}$ that for some $u \in S \cap U$ the $(k'-1)$-tuple $(x_2, \dots, x_{k'})$ fails to be $\frac{\xi}{2}$-leftconnectable in $\Psi_{zu}[S]$ occurs. For this reason, it suffices to prove

$$\mathbb{P}\big(|U \cap S| \leqslant \tfrac{\xi}{2}|S| \mid \vec{x} \in S^{k'} \text{ and } z' \subseteq S\big) \leqslant \exp\big(-\Omega(M)\big) \tag{6.6}$$

$$\text{and} \quad \mathbb{P}\big(\mathfrak{A} \mid \vec{x} \in S^{k'} \text{ and } z' \subseteq S\big) \leqslant \exp\big(-\Omega(M)\big)\,. \tag{6.7}$$

Now (6.6) follows in the usual way from Lemma 6.5 *(a)*. To prove (6.7) we observe that the induction hypothesis yields

$$\begin{aligned}\mathbb{P}\big((x_2, \dots, x_{k'}) \text{ fails to be } \tfrac{\xi}{2}\text{-leftconnectable in } \Psi_{zu}[S] \mid (x_2, \dots, x_{k'}) \in S^{k'-1}\,,\\ \text{and } (z' \cup \{x_1\}) \subseteq S\big) \leqslant \exp\big(-\Omega(M)\big)\end{aligned}$$

for every $u \in U$, whence

$$\begin{aligned}\mathbb{P}(\mathfrak{A} \mid \vec{x} \in S^{k'} \text{ and } z' \subseteq S) &\leqslant \sum_{u \in U} \mathbb{P}(u \in S) \exp\big(-\Omega(M)\big)\\ &\leqslant M^2 \exp\big(-\Omega(M)\big) \leqslant \exp\big(-\Omega(M)\big)\,.\end{aligned}$$ □

By applying the case $k' = k-1$ of Claim 6.8 to all ξ-leftconnectable $(k-1)$-tuples in Ψ we obtain

$$\begin{aligned}\mathbb{P}\big(\text{Some } \vec{x} \in S^{k-1} \text{ that is } \xi\text{-leftconnectable in } \Psi\\ \text{fails to be } \tfrac{\xi}{2}\text{-leftconnectable in } \Psi[S]\big) \leqslant \exp\big(-\Omega(M)\big)\,.\end{aligned}$$

By symmetry the same holds for rightconnectability as well and, therefore,

$$\mathbb{P}\big(\text{Some } \xi\text{-bridge } \vec{x} \in S^k \text{ fails to be a } \tfrac{\xi}{2}\text{-bridge in } \Psi[S]\big) \leqslant \exp\big(-\Omega(M)\big)\,.$$

In other words, we have thereby proved (6.5) and, hence, Lemma 6.6. □

The next lemma shows how to ascend from $(k-1)$-uniform coverings to k-uniform coverings.

Lemma 6.9. *For every $k \geqslant 4$ the covering principle $\spadesuit_{k-1}$ implies $\heartsuit_k$.*

Proof. Let $\alpha, \beta, \vartheta_\star > 0$, and an odd integer $\ell \geqslant 3$ be given. Without loss of generality we may assume that $\vartheta_\star \ll \alpha, \beta, k^{-1}, \ell^{-1}$. Pick a sufficiently small constant

$$\zeta_{\star\star} \ll \vartheta_\star\,. \tag{6.8}$$

The statement $\spadesuit_{k-1}$ applied to $\frac{\alpha}{2}$, $\frac{\beta}{2}$, ℓ, $\frac{\zeta_{\star\star}}{2}$ here in place of α, β, ℓ, ξ there delivers an infinite arithmetic progression $P \subseteq (k-1)\mathbb{N}$. Choose $M \gg \zeta_{\star\star}^{-1}$ such that $\frac{k-1}{k}(M+1) \in P$ and notice that $M \equiv -1 \pmod{k}$ is clear.

Now let Ψ be a $(\alpha, \beta, \ell, \frac{4\alpha}{17^k})$-constellation on n vertices, where n is sufficiently large. We are to prove that all but at most $\vartheta_\star^2|V(\Psi)|$ vertices of Ψ can be covered by vertex-disjoint M-vertex paths starting end ending with $\zeta_{\star\star}$-connectable $(k-1)$-tuples. Let

$$\mathscr{P} = \big\{P \subseteq H(\Psi)\colon P \text{ is a } k\text{-uniform } M\text{-vertex path} \\ \text{whose first and last } (k-1)\text{-tuple is } \zeta_{\star\star}\text{-connectable}\big\}$$

be the collection of all paths that might occur in such a covering, and let $\mathscr{C} \subseteq \mathscr{P}$ be a maximal subcollection of vertex-disjoint paths from $\mathscr{P}$. Further, let

$$U = V(\Psi) \smallsetminus \bigcup_{P \in \mathscr{C}} V(P)$$

be the set of uncovered vertices. We may assume that

$$|U| > \vartheta_\star^2 |V(\Psi)|\,, \tag{6.9}$$

since otherwise nothing is left to show. Now roughly speaking the strategy is to find a set $S \subseteq V(\Psi)$ of size M^2 meeting at most M paths from $\mathscr{C}$ such that for 'many' vertices $u \in U$ we can apply $\spadesuit_{k-1}$ to the $(k-1)$-uniform constellation $\Psi_u[S]$, thus getting at least $M+1$ vertex-disjoint paths with $\frac{k-1}{k}(M+1)$ vertices. These paths will agree for many vertices $u \in U$ and can then be augmented to k-uniform paths engendering a contradiction to the maximality of $\mathscr{C}$. In the intended application of $\spadesuit_{k-1}$ we are allowed to specify a set of bridges $\mathfrak{B}$ that we potentially would like to see at the ends of the paths we obtain. Since we ultimately aim at generating paths in $\mathscr{P}$ and, hence, paths starting and ending with $\zeta_{\star\star}$-connectable $(k-1)$-tuples, it seems advisable to let $\mathfrak{B}$ be the set of

$\frac{\zeta_{\star\star}}{2}$-bridges in $\Psi_u[S]$ that are $\zeta_{\star\star}$-connectable in Ψ. This choice of $\mathfrak{B}$ is only permissible if $|\mathfrak{B}|$ is sufficiently large (i.e., at least $\frac{\zeta_{\star\star}}{2}|S|^{k-1}$). Our way of ensuring this in sufficiently many cases exploits that for fixed $u \in U$ and a random choice of $S \subseteq V(\Psi)$ Lemma 6.6 tells us that the $\zeta_{\star\star}$-bridges in Ψ_u are likely to be $\frac{\zeta_{\star\star}}{2}$-bridges in $\Psi_u[S]$. Thus it suffices to focus on vertices $u \in U$ which are not in the set

$$U_{\mathrm{bad}} = \left\{u \in U\colon \text{at most } \tfrac{1}{20}n^{k-1} \text{ of the } \zeta_{\star\star}\text{-bridges in } \Psi_u \text{ are } \zeta_{\star\star}\text{-connectable in } \Psi\right\}.$$

The next claim states that this set is indeed small.

Claim 6.10. *We have $|U_{\mathrm{bad}}| \leqslant 40\zeta_{\star\star}n$.*

Proof. Set

$$\Pi = \Big\{(x_1, \dots, x_{k-1}, u) \in V(\Psi)^{k-1} \times U_{\mathrm{bad}}\colon (x_1, \dots, x_{k-1}) \text{ is a } \zeta_{\star\star}\text{-bridge in } \Psi_u \\ \text{but not } \zeta_{\star\star}\text{-connectable in } \Psi\Big\}.$$

For every $u \in U_{\mathrm{bad}}$ Corollary 2.28 tells us that the number of $\zeta_{\star\star}$-bridges $(x_1, \dots, x_{k-1})$ in Ψ_u is at least $\frac{1}{9}(n-1)^{k-1} > \frac{1}{10}n^{k-1}$ and by the definition of U_{bad} at least $\frac{1}{20}n^{k-1}$ among them fail to be $\zeta_{\star\star}$-connectable in Ψ. This proves that

$$|\Pi| \geqslant \frac{1}{20}n^{k-1}|U_{\mathrm{bad}}|.$$

On the other hand, an upper bound on $|\Pi|$ can be obtained as follows. Let Π_{left} be the set of k-tuples in Π for which $(x_1, \dots, x_{k-1})$ fails to be $\zeta_{\star\star}$-leftconnectable and define Π_{right} similarly with respect to rightconnectability. As a $(k-1)$-tuple that is not $\zeta_{\star\star}$-leftconnectable in Ψ can only be a $\zeta_{\star\star}$-bridge in Ψ_u for less than $\zeta_{\star\star}n$ vertices u, we have $|\Pi_{\mathrm{left}}| \leqslant \zeta_{\star\star}n^k$. The same upper bound can be proved for $|\Pi_{\mathrm{right}}|$ and because of $\Pi = \Pi_{\mathrm{left}} \cup \Pi_{\mathrm{right}}$ this yields $|\Pi| \leqslant 2\zeta_{\star\star}n^k$. Combining the two bounds on $|\Pi|$ we obtain indeed $|U_{\mathrm{bad}}| \leqslant 40\zeta_{\star\star}n$. $\square$

Because of our choice of $\zeta_{\star\star}$ in (6.8) this yields $|U_{\mathrm{bad}}| \leqslant \frac{1}{2}\vartheta_\star^2 n$, which combined with (6.9) implies

$$|U \smallsetminus U_{\mathrm{bad}}| \geqslant \frac{1}{2}\vartheta_\star^2 n. \tag{6.10}$$

Next we will partition the vertex set into blocks some of which will later be selected randomly for hosting the augmentation of $\mathscr{C}$. Form a partition

$$V(\Psi) = B_1 \mathbin{\dot\cup} \dots \mathbin{\dot\cup} B_\nu \mathbin{\dot\cup} B', \tag{6.11}$$

with $|B_1| = \dots = |B_\nu| = M > |B'|$, where the first $|\mathscr{C}|$ classes $B_1, \dots, B_{|\mathscr{C}|}$ are the vertex sets of the paths in the collection $\mathscr{C}$, and $B_{|\mathscr{C}|+1}, \dots, B_\nu$ are arbitrary disjoint M-sets making (6.11) true. The sets $B_1, \dots, B_\nu$ are called *blocks*. A *society* is a set of M blocks. We point out that

$$\text{if } \mathscr{S} \text{ is a society and } S = \bigcup \mathscr{S}, \text{ then } |S| = M^2. \tag{6.12}$$

Definition 6.11. *A society $\mathscr{S}$ with $S = \bigcup \mathscr{S}$ is called useful for a vertex $u \in U$ if*

(*1*) $u \notin S$,

(*2*) $\Psi_u[S]$ *is a* $(k-1)$*-uniform* $(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, \frac{\alpha/2}{17^{k-1}})$*-constellation.*

(*3*) *The number of* $(k-1)$*-tuples in* S^{k-1} *that are* $\frac{\zeta_{\star\star}}{2}$*-bridges in* $\Psi_u[S]$ *and* $\zeta_{\star\star}$*-connectable in* Ψ *is at least* $\frac{\zeta_{\star\star}}{2}|S|^{k-1}$.

The next claim explains the naming of useful societies: $\Psi_u[S]$ contains $M+1$ "suitable" paths.

Claim 6.12. *If a society $\mathscr{S}$ is useful for $u \in U$ and $S = \bigcup \mathscr{S}$, then there is a collection $\mathscr{W}$ of mutually disjoint $(k-1)$-uniform paths in $\Psi_u[S]$ with the following properties.*

(*i*) *Every path in* $\mathscr{W}$ *has* $\frac{k-1}{k}(M+1)$ *vertices.*

(*ii*) *Every path in* $\mathscr{W}$ *starts and ends with a* $(k-1)$*-tuple that is* $\zeta_{\star\star}$*-connectable in* Ψ.

(*iii*) $|\mathscr{W}| \geqslant M+1$.

Proof. By Definition 6.11 (*3*) and (6.12) the set

$$\Xi = \left\{\vec{e} \in S^{k-1} \colon \vec{e} \text{ is } \zeta_{\star\star}\text{-connectable in } \Psi \text{ and a } \tfrac{\zeta_{\star\star}}{2}\text{-bridge in } \Psi_u[S]\right\}$$

satisfies $|\Xi| \geqslant \frac{\zeta_{\star\star}}{2}(M^2)^{k-1}$. Now we apply $\spadesuit_{k-1}$ to $\Psi_u[S]$, Ξ, $\frac{\zeta_{\star\star}}{2}$, and $\frac{k-1}{k}(M+1)$ here in place of Ψ, $\mathfrak{B}$, ξ, and M there – which is permissible due to the selection of parameters in the beginning of the proof of Lemma 6.9.

This application of $\spadesuit_{k-1}$ yields a collection $\mathscr{W}$ of mutually disjoint $(k-1)$-uniform paths in $\Psi_u[S]$ that covers all but at most $\frac{\zeta_{\star\star}}{2}|S| + \frac{k-1}{k}(M+1)$ vertices of S such that each path starts and ends with a bridge from Ξ. Since each bridge in Ξ is a $\zeta_{\star\star}$-connectable tuple in Ψ, it remains to check that $|\mathscr{W}| \geqslant M+1$. Because of $M \gg \zeta_{\star\star}^{-1} \gg k$ we have indeed

$$|\mathscr{W}| \geqslant \frac{(1-\zeta_{\star\star}/2)M^2 - \frac{k-1}{k}(M+1)}{\frac{k-1}{k}(M+1)} \geqslant \frac{(1-\zeta_{\star\star})M(M+1)}{(1-\zeta_{\star\star})M} = M+1\,. \qquad \square$$

Lemma 6.6 implies that some society is useful for many vertices.

Claim 6.13. *There exists a society $\mathscr{S}$ that is useful for $\frac{2}{3}|U \smallsetminus U_{bad}|$ vertices in $U \smallsetminus U_{bad}$.*

Proof. By double counting it suffices to establish that for every vertex $u \in U \smallsetminus U_{\mathrm{bad}}$ at least $\frac{2}{3}$ of all societies are useful. Fix an arbitrary such vertex u and suppose first that $u \notin B'$. Without loss of generality we may assume that $u \in B_\nu$. We plan to apply

Lemma 6.6 with $(k-1, \frac{\alpha}{4\cdot 17^{k-1}}, \zeta_{\star\star})$ here in place of (k, μ, ξ) there to the $(k-1)$-uniform constellation Ψ_u, the partition

$$V(\Psi_u) = B_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} B_{\nu-1} \mathbin{\dot\cup} (B_\nu \cup B' \smallsetminus \{u\})\,,$$

and the set

$$\mathfrak{B}_u = \left\{\vec{x} \in V(\Psi_u)^{k-1} \colon \vec{x} \text{ is } \zeta_{\star\star}\text{-connectable in } \Psi \text{ and a } \zeta_{\star\star}\text{-bridge in } \Psi_u\right\}.$$

Notice that Fact 2.35 tell us that Ψ_u is indeed an $(\alpha, \beta, \ell, \frac{\alpha}{4\cdot 17^{k-1}})$-constellation. Moreover, $u \notin U_{\mathrm{bad}}$ implies $|\mathfrak{B}_u| \geqslant \frac{1}{20}n^{k-1} > \zeta_{\star\star}|V(\Psi_u)|^{k-1}$. So all assumptions of Lemma 6.6 hold and we conclude that at least $\frac{3}{4}\binom{\nu-1}{M} > \frac{2}{3}\binom{\nu}{M}$ societies are useful for u. The case $u \in B'$ is similar. $\square$

For the remainder of this proof we fix a society $\mathscr{S}$ that is useful for at least $\frac{2}{3}|U \smallsetminus U_{\mathrm{bad}}|$ vertices in $U \smallsetminus U_{\mathrm{bad}}$ and set $S = \bigcup \mathscr{S}$. Claim 6.12 informs us that for every $u \in U$, for which $\mathscr{S}$ is useful, there is a collection $\mathscr{W}_u$ of $M+1$ mutually vertex disjoint $(k-1)$-uniform paths in $\Psi_u[S]$ consisting of $\frac{k-1}{k}(M+1)$ vertices each, which start and end with $\zeta_{\star\star}$-connectable $(k-1)$-tuples.

Since there are at most $(M^2)!$ possibilities to order the vertices in S, there has to exist a subset $U' \subseteq U \smallsetminus U_{\mathrm{bad}}$ such that $\mathscr{W}_u = \mathscr{W}$ is the same for every $u \in U'$ and

$$|U'| \geqslant \frac{\frac{2}{3}|U \smallsetminus U_{\mathrm{bad}}|}{(M^2)!} \overset{(6.10)}{\geqslant} \frac{\vartheta_\star^2 n}{3(M^2)!} \geqslant \frac{(M-(k-1))(M+1)}{k}\,.$$

Now, for every path in $\mathscr{W}$ put $\frac{M-(k-1)}{k}$ distinct vertices from U' aside and insert them at every k-th position into the path from $\mathscr{W}$ (see Figure 6.2).

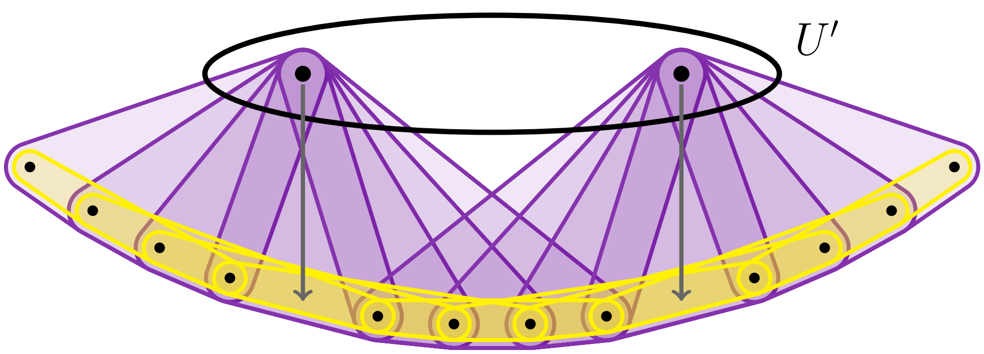


FIGURE 6.2. Augmenting a yellow $\frac{4}{5}(M+1)$-vertex path to a lila M-vertex path.

Since the starting and ending $(k-1)$-tuples of every path in $\mathscr{W}$ are $\zeta_{\star\star}$-connectable in Ψ and the insertion of the additional vertices increases their length to $\frac{k-1}{k}(M+1) + \frac{M-(k-1)}{k} = M$, the resulting $M+1$ paths are elements of $\mathscr{P}$. Hence, the collection $\mathscr{C}$ can be augmented by removing the at most M paths whose blocks lie in $\mathscr{S}$ and adding the $M+1$ newly constructed paths instead. As this contradicts the maximality of $\mathscr{C}$, the assumption (6.9) must have been false. This concludes the proof of Lemma 6.9. □

Finally, we arrive at the main result of this section.

Proposition 6.14. *For every $k \geqslant 3$ the statement $\heartsuit_k$ holds.*

Proof. We argue by induction on k, the base case being provided by Fact 6.2. The Lemmata 6.4 and 6.9 show that $\heartsuit_{k-1} \Rightarrow \spadesuit_{k-1} \Rightarrow \heartsuit_k$, which is the induction step. □

§7. The proof of Theorem 1.2

The results in the foregoing sections routinely imply Theorem 1.2, but for the sake of completeness we provide the details.

Proof of Theorem 1.2. Given $k \geqslant 3$ and $\alpha > 0$ we choose some auxiliary constants fitting into the hierarchy

$$\alpha, k^{-1} \gg \mu \gg \beta, \ell^{-1} \gg \zeta_\star \gg \vartheta_\star \gg \zeta_{\star\star} \gg \vartheta_{\star\star} \gg M^{-1} \gg n_0^{-1}\,, \tag{7.1}$$

where $\ell \geqslant 3$ is an odd integer and $M \equiv -1 \pmod{k}$.

Now let $H = (V, E)$ be a k-uniform hypergraph on $n \geqslant n_0$ vertices satisfying the minimum $(k-2)$-degree condition $\delta_{k-2}(H) \geqslant (\frac{5}{9} + \alpha)\frac{n^2}{2}$. By Fact 2.34 and $\alpha \gg \mu \gg \beta, \ell^{-1}$ there exists an $(\alpha, \beta, \ell, \mu)$-constellation Ψ with underlying hypergraph H.

Stage A. We set aside a reservoir set $\mathcal{R}$ of size $|\mathcal{R}| \leqslant \vartheta_\star^2 n$ provided by Proposition 4.1. Let us recall that by Corollary 4.2 and $\vartheta_{\star\star} \ll \vartheta_\star, k^{-1}, \ell^{-1}$

(*1*) for every set $\mathcal{R}' \subseteq \mathcal{R}$ of at most $\vartheta_{\star\star}^2 n$ "forbidden" vertices, every $\zeta_{\star\star}$-leftconnectable $(k-1)$-tuple $\vec{a}$, every $\zeta_{\star\star}$-rightconnectable $(k-1)$-tuple $\vec{b}$ that is disjoint to $\vec{a}$, and every $i \in [0, k)$, there is an $\vec{a}$-$\vec{b}$-path through $\mathcal{R} \smallsetminus \mathcal{R}'$ with $f(k, i, \ell)$ inner vertices.

Stage B. Next, we choose an absorbing path avoiding $\mathcal{R}$. More precisely, Proposition 5.1 yields a path $P_A \subseteq H - \mathcal{R}$ with the properties that

(*2*) $|V(P_A)| \leqslant \vartheta_\star n$,

(*3*) the starting and ending $(k-1)$-tuple of P_A are $\zeta_{\star\star}$-connectable,

(*4*) and for every subset $Z \subseteq V \smallsetminus V(P_A)$ with $|Z| \leqslant 2\vartheta_\star^2 n$ and $|Z| \equiv 0 \pmod{k}$, there is a path $Q \subseteq H$ with $V(Q) = V(P_A) \cup Z$ having the same end-$(k-1)$-tuples as P_A.

Stage C. We proceed by covering almost all vertices belonging neither to $\mathcal{R}$ nor to P_A by long paths. To this end we set $X = \mathcal{R} \cup V(P_A)$ and consider the constellation $\Psi' = \Psi - X$. Since $|X| \leqslant \vartheta_\star^2 n + \vartheta_\star n \leqslant 2\vartheta_\star n$, Lemma 2.36 tells us that Ψ' is an $\big(\frac{\alpha}{2}, \frac{\beta}{2}, \ell, 2\mu\big)$-constellation. So the covering principle $\heartsuit_k$ defined in Definition 6.1 and proved in Proposition 6.14 applies to Ψ', $2\zeta_{\star\star}$ here in place of Ψ, $\zeta_{\star\star}$ there. In other words, in Ψ' there exists a collection $\mathscr{C}$ of mutually disjoint M-vertex paths whose end-tuples are $(2\zeta_{\star\star})$-connectable in Ψ' such that

$$\Big|V(\Psi') \smallsetminus \bigcup_{P\in\mathscr{C}} V(P)\Big| \leqslant \vartheta_\star^2 n\,.$$

Due to Fact 2.18, the end-tuples of the paths in $\mathscr{C}$ are $\zeta_{\star\star}$-connectable in Ψ.

Stage D. Now we want to connect the paths in $\mathscr{C}$ and P_A, thus obtaining one long path T with $\zeta_{\star\star}$-connectable end-tuples. This is to be done by means of $|\mathscr{C}|$ connections through the reservoir, iteratively using *(1)* with $i = 0$. Altogether these connections require

$$|\mathscr{C}| f(k, 0, \ell) \leqslant \frac{4^k \ell k n}{M} \leqslant \vartheta_{\star\star}^2 n$$

vertices from the reservoir. So $|\mathscr{C}|$ successive applications of *(1)* indeed allow us to construct this long path T (see Figure 7.1).

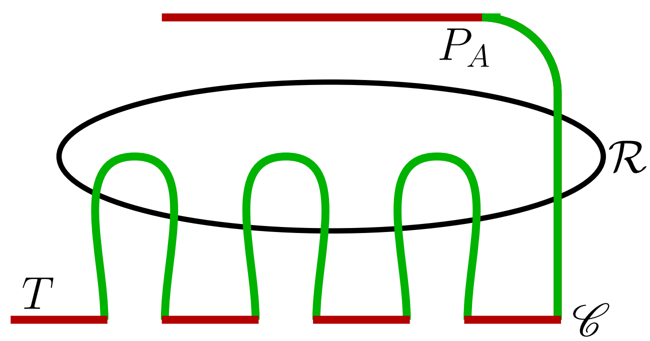


FIGURE 7.1. The situation after Stage D.

Stage E. Moreover, we can still use *(1)* one more time in order to connect the end-tuples of T, thus creating one long cycle C. For this last connection we use $f(k, i, \ell)$ inner vertices, where $i \in [0, k)$ is determined by the congruence $i \equiv n - |V(T)| \pmod k$. The current situation is depicted in Figure 7.2.

Our choice of i guarantees that the set $Z = V(\Psi) \smallsetminus V(C)$ satisfies

$$|Z| \equiv n - |V(T)| - f(k, i, \ell) \equiv 0 \pmod k\,.$$

Furthermore, Z has at most the size

$$|Z| \leqslant |\mathcal{R}| + \Big|V(\Psi') \smallsetminus \bigcup_{P\in\mathscr{C}} V(P)\Big| \leqslant 2\vartheta_\star^2 n\,.$$

Stage F. Taken together, the last two displayed formulae and *(4)* show that Z can be absorbed by P_A, i.e., that there exists a path Q with $V(Q) = V(P_A) \cup Z$ having the same end-tuples as P_A. Upon replacing the subpath P_A of C by Q we obtain the desired Hamiltonian cycle in H (see Figure 7.3). $\square$

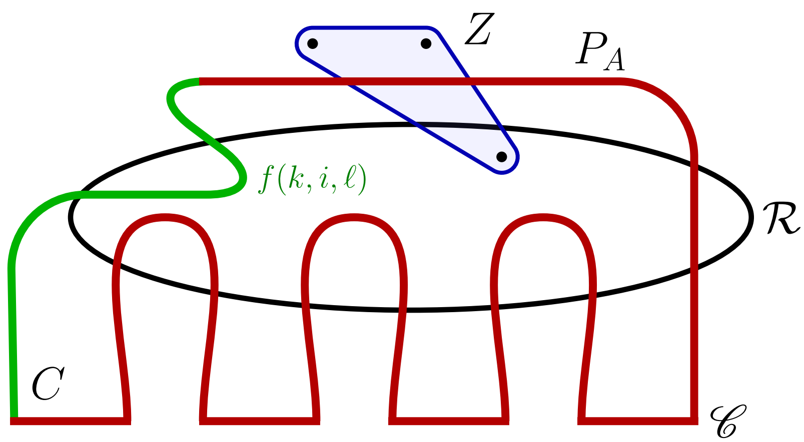


FIGURE 7.2. The situation after Stage E. The dots in Z represent sets of k vertices each.

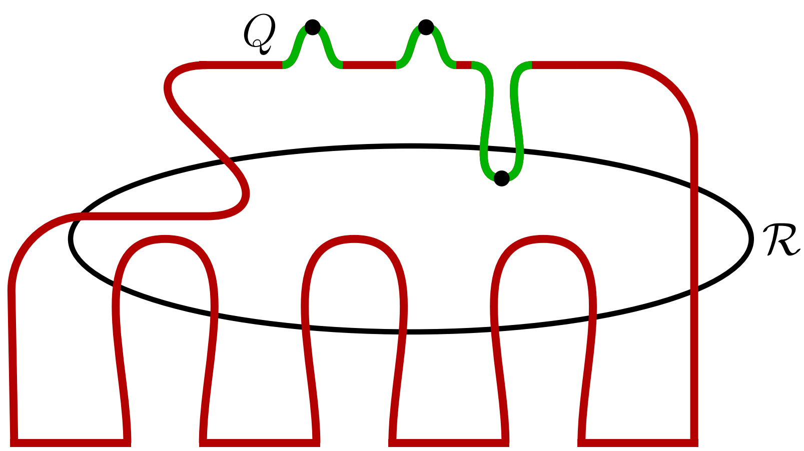


FIGURE 7.3. The situation after Stage F.

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