COVERING 3-EDGE-COLOURED RANDOM GRAPHS WITH MONOCHROMATIC TREES

YOSHIHARU KOHAYAKAWA, WALNER MENDONÇA, GUILHERME OLIVEIRA MOTA, AND BJARNE SCHÜLKE

Abstract. We investigate the problem of determining how many monochromatic trees are necessary to cover the vertices of an edge-coloured random graph. More precisely, we show that for $p \gg n^{-1/6}(\ln n)^{1/6}$, in any 3-edge-colouring of the random graph $G(n,p)$ we can find three monochromatic trees such that their union covers all vertices. This improves, for three colours, a result of Bucić, Korándi and Sudakov.

§1. Introduction

Given a graph G and a positive integer r, let $\mathrm{tc}_r(G)$ denote the minimum number k such that in any r-edge-colouring of G, there are k monochromatic trees $T_1, \dots, T_k$ such that the union of their vertex sets covers $V(G)$, i.e.,

$$V(G) = V(T_1) \cup \cdots \cup V(T_k).$$

We define $\mathrm{tp}_r(G)$ analogously by requiring the union above to be disjoint.

It is easy to see that $\mathrm{tp}_2(K_n) = 1$ for all $n \geqslant 1$, and Erdős, Gyárfás and Pyber [8] proved that $\mathrm{tp}_3(K_n) = 2$ for all $n \geqslant 1$, and conjectured that $\mathrm{tp}_r(K_n) = r - 1$ for every n and r. Haxell and Kohayakawa [10] showed that $\mathrm{tp}_r(K_n) \leqslant r$ for all sufficiently large $n \geqslant n_0(r)$. We remark that it is easy to see that $\mathrm{tc}_r(K_n) \leqslant r$ (just pick any vertex $v \in V(K_n)$ and let T_i, for $i \in [r]$, be a maximal monochromatic tree of colour i containing v), but it is not even known whether or not $\mathrm{tc}_r(K_n) \leqslant r - 1$ for every n and r (as would be implied by the conjecture of Erdős, Gyárfás and Pyber).

Concerning general graphs instead of complete graphs, Gyárfás [9] noted that a well-known conjecture of Ryser on matchings and transversal sets in hypergraphs is equivalent to the statement that for every graph G and integer $r \geqslant 2$, we have $\mathrm{tc}_r(G) \leqslant (r-1)\alpha(G)$.

1991 *Mathematics Subject Classification.* 05C80 (primary); 05C70 (secondary).

Key words and phrases. Graph partitioning, monochromatic trees, random graphs.

Y. Kohayakawa was partially supported by CNPq (311412/2018–1, 423833/2018–9) and FAPESP (2018/04876–1). G. O. Mota was partially supported by CNPq (304733/2017–2, 428385/2018–4) and FAPESP (2018/04876–1). W. Mendonça was partially supported by CAPES (88882.332408/2010–01). B. Schülke was partially supported by G.I.F. Grant Agreements No. I-1358–304.6/2016. The cooperation of the authors was supported by a joint CAPES/DAAD PROBRAL project (Proj. 430/15, 57350402, 57391197). This study was financed in part by CAPES, Coordenação de Aperfeiçoamento de Pessoal de Nível Superior, Brasil, Finance Code 001. FAPESP is the São Paulo Research Foundation. CNPq is the National Council for Scientific and Technological Development of Brazil.

In particular, Ryser's conjecture, if true, would imply that $\mathrm{tc}_r(K_n) \leqslant r-1$, for every $n \geqslant 1$ and $r \geqslant 2$. Ryser's conjecture was proved in the case $r=3$ by Aharoni [1], but for $r \geqslant 4$ very little is known. For example, Haxell and Scott [11] proved (in the context of Ryser's original conjecture) that there exists $\varepsilon > 0$ such that for $r \in \{4,5\}$, we have $\mathrm{tc}_r(G) \leqslant (r-\varepsilon)\alpha(G)$, for any graph G.

Bal and DeBiasio [2] initiated the study of covering and partitioning random graphs by monochromatic trees. They proved that if $p \ll \left(\frac{\ln n}{n}\right)^{1/r}$, then with high probability[1] we have $\mathrm{tc}_r(G(n,p)) \to \infty$. They conjectured that for any $r \geqslant 2$, this was the correct threshold for the event $\mathrm{tp}_r(G(n,p)) \leqslant r$. Kohayakawa, Mota and Schacht [14] proved that this conjecture holds for $r=2$, while Ebsen, Mota and Schnitzer[2] showed that it does not hold for more than two colours.

Bucić, Korándi and Sudakov [6] proved that if $p \ll \left(\frac{\ln n}{n}\right)^{\sqrt{r}/2^{r-2}}$, then w.h.p. we have $\mathrm{tc}_r(G(n,p)) \geqslant r+1$, which implies that the threshold for the event $\mathrm{tc}_r(G) \leqslant r$ is in fact significantly larger than the one conjectured by Bal and DeBiasio when r is large. Bucić, Korándi and Sudakov also proved that w.h.p. we have $\mathrm{tc}_r(G(n,p)) \leqslant r$ for $p \gg \left(\frac{\ln n}{n}\right)^{1/2^r}$. They were also able to roughly determine the typical behaviour of $\mathrm{tc}_r(G(n,p))$ in terms of the range where p lies in (see [6, Theorem 1.3 and Theorem 1.4]).

Considering colourings with three colours, the results from [6] imply that if $p \gg \left(\frac{\ln n}{n}\right)^{1/8}$, then w.h.p. we have $\mathrm{tc}_3(G(n,p)) \leqslant 3$, and if $\left(\frac{\ln n}{n}\right)^{1/6} \ll p \ll \left(\frac{\ln n}{n}\right)^{1/7}$, then w.h.p. $\mathrm{tc}_3(G(n,p)) \leqslant 88$. Our main result improves these bounds for three colours.

Theorem 1.1. *If $p = p(n)$ satisfies $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$, then with high probability we have*

$$\mathrm{tc}_3\big(G(n,p)\big) \leqslant 3.$$

It can be easily seen that if $1-p \ll n^{-1}$, then w.h.p. there is a 3-edge-colouring of $G(n,p)$ for which 3 monochromatic trees are needed to cover all vertices — it suffices to consider three non-adjacent vertices x_1, x_2 and x_3, and colour the edges incident to x_i with colour i and colour all the remaining edges with any colour. Therefore, the bound for $tc_3(G(n,p))$ in Theorem 1.1 is the best possible as long as p is not too close to 1.

We remark that, from the example described in [14], we know that for $p \ll \left(\frac{\ln n}{n}\right)^{1/4}$, we have w.h.p. $\mathrm{tc}_3(G(n,p)) \geqslant 4$. It would be very interesting to describe the behaviour of $\mathrm{tc}_3(G(n,p))$ when $\left(\frac{\ln n}{n}\right)^{1/4} \ll p \ll \left(\frac{\ln n}{n}\right)^{1/6}$.

This paper is organized as follows. In Section 2 we present some definitions and auxiliary results that we will use in the proof of Theorem 1.1, which is outlined in Section 3. The details of the proof of Theorem 1.1 are given in Section 4.

[1] We will write shortly *w.h.p.* for *with high probability*.

[2] A description of this construction can be found in [14].

§2. Preliminaries

Most of our notation is standard (see [3, 5, 7] and [4, 13]). However, we will mention in the following few definitions regarding hypergraphs that will play a major role in our proofs just for completeness.

We say that a set A of vertices in a hypergraph $\mathcal{H}$ is a *vertex cover* if every hyperedge of $\mathcal{H}$ contains at least one element of A. The *covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the smallest size of a vertex cover in $\mathcal{H}$. A *matching* in $\mathcal{H}$ is a collection of disjoint hyperedges in $\mathcal{H}$. The *matching number* of $\mathcal{H}$, denoted by $\nu(\mathcal{H})$, is the largest size of a matching in $\mathcal{H}$. An immediate relationship between $\tau(\mathcal{H})$ and $\nu(\mathcal{H})$ is the inequality $\nu(\mathcal{H}) \leqslant \tau(\mathcal{H})$. If additionally $\mathcal{H}$ is r-uniform, then we have $\tau(\mathcal{H}) \leqslant r\nu(\mathcal{H})$. A conjecture due to Ryser (which first appeared in the thesis of his Ph.D. student, Henderson [12]) states that for every r-uniform r-partite hypergraph $\mathcal{H}$, we have $\tau(\mathcal{H}) \leqslant (r-1)\nu(\mathcal{H})$. Note that König-Egerváry theorem corresponds to Ryser's conjecture for $r = 2$. Aharoni [1] proved that Ryser's conjecture holds for $r = 3$, but the conjecture remains open for $r \geqslant 4$.

Given a vertex v in a 3-uniform hypergraph $\mathcal{H}$, the *link graph* of $\mathcal{H}$ with respect to v is the graph $L_v = (V, E)$ with vertex set $V = V(\mathcal{H})$ and edge set $E = \{xy : \{x, y, v\} \subseteq \mathcal{H}\}$.

We will use the following theorem due to Erdős, Gyárfás and Pyber [8] in the proof of our main result.

Theorem 2.1 (Erdős, Gyárfás and Pyber)**.** *For any 3-edge-colouring of a complete graph K_n, there exists a partition of $V(K_n)$ into 2 monochromatic trees.*

We will also use the following lemma, which is a simple application of Chernoff's inequality. For a proof of the first item see [15, Lemma 3.8]. The second item is an immediate corollary of [15, Lemma 3.10].

Lemma 2.2. *Let $\varepsilon > 0$. If $p = p(n) \gg \left(\frac{\ln n}{n}\right)^{1/6}$, then w.h.p. $G \in G(n,p)$ has the following properties.*

(*i*) *For any disjoint sets $X, Y \subseteq V(G)$ with $|X|, |Y| \gg \frac{\ln n}{p}$, we have*

$$|E_G(X,Y)| = (1 \pm \varepsilon)p|X||Y|.$$

(*ii*) *Every vertex $v \in V(G)$ has degree $d_G(v) = (1 \pm \varepsilon)pn$ and every set of $i \leqslant 6$ vertices has $(1 \pm \varepsilon)p^i n$ common neighbours.*

§3. A sketch of the proof

In this section we will give an overview of the proof of Theorem 1.1. Let $G = G(n,p)$, with $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$, and let $\varphi : E(G) \to \{\text{red}, \text{ green}, \text{ blue}\}$ be any 3-edge-colouring of G. We consider an auxiliary graph F, with $V(F) = V(G)$ and $ij \in E(F)$ if and only if there is, in the colouring φ, a monochromatic path in G connecting i and j. Then we define a 3-edge-colouring φ' of F with $\varphi'(ij)$ being the color of any monochromatic path in G

connecting i and j. Note that any covering of F with monochromatic trees with respect to the colouring φ' corresponds to a covering of G with monochromatic trees with respect to the colouring φ with the same number of trees.

Next, we consider different cases depending on the value of $\alpha(F)$. If $\alpha(F) = 1$, then F is a complete 3-edge-coloured graph and by a theorem of Erdős, Gyárfás and Pyber (see Theorem 2.1), there exists a partition of $V(F)$ into 2 monochromatic trees. The remaining proof now is divided into the cases $\alpha(F) \geqslant 3$ and $\alpha(F) = 2$.

Case $\alpha(F) \geqslant 3$. From the condition on the independence number of G, there exist three vertices $r, b, g \in V(G)$ that pairwise do not have any monochromatic path connecting them. With high probability, they have a common neighbourhood in G of size at least $np^3/2$. Let X_{rbg} be the largest subset of this common neighbourhood such that for each $i \in \{r, b, g\}$, the edges from i to X_{rbg} in G are all coloured with one colour. Then, since there are no monochromatic paths between any two of r, b, g, we have $|X_{rbg}| \geqslant np^3/12$ and moreover we may assume that all edges between r and X_{rbg} are red, all between b and X_{rbg} are blue and those between g and X_{rbg} are green. Now we notice that all vertices that have a neighbour in X_{rbg} are covered by the union of the spanning trees of the red component of r, the blue component of b and the green component of g.

We are done in the case where every vertex has a neighbour in X_{rbg}, as the vertices in $X_{rbg} \cup N_G(X_{rbg})$ are covered by the red, blue and green component containing r, b and g, respectively. Otherwise, w.h.p. any vertex $y \in V \smallsetminus (X_{rbg} \cup N_G(X_{rbg}))$ has many common neighbours with r, g and b in G that are also neighbours of some vertex in X_{rbg}. An analysis of the possible colourings of the edges between X_{rbg} and the common neighbourhood of the vertices r, b, g and y yields the following: for some $i \in \{r, g, b\}$, let us say $i = r$, every vertex $y \in X_{rbg}$ can be connected to r by a monochromatic path in colour red or either to g or b by a monochromatic path in the colour blue or green, respectively.

This already gives us that all vertices in G can be covered by 5 monochromatic trees, since all the vertices in $N_G(X_{rbg})$ lie in the red component of r, or the green component of g, or in the blue component of b and every vertex in $V \smallsetminus N_G(X_{rbg})$ lies in the red component of r, in the blue component of g or in the green component of b. By analysing the colours of edges to the common neighbourhood of carefully chosen vertices, we are able to show that actually three of those five trees already cover all the vertices of G.

Case $\alpha(F) = 2$. Let us consider a 3-uniform hypergraph $\mathcal{H}$ defined as follows (this definition is inspired by a construction of Gyárfás [9]). The vertices of $\mathcal{H}$ are the monochromatic components of F and three vertices form a hyperedge if the corresponding three components have a vertex in common (in particular, those three monochromatic components must be of different colours). Hence $\mathcal{H}$ is an 3-uniform 3-partite hypergraph.

We observe that if A is a vertex cover of $\mathcal{H}$, then the monochromatic components associated with the vertices in A cover all the vertices of G. This implies that $\mathrm{tc}_3(G) \leqslant \tau(\mathcal{H})$. Also, it is easy to see that $\nu(\mathcal{H}) \leqslant \alpha(F) = 2$. Now, recall that Aharoni's result [1] (which corresponds to Ryser's conjecture for $r = 3$) states that for every 3-uniform 3-partite hypergraph $\mathcal{H}$ we have $\tau(\mathcal{H}) \leqslant 2\nu(\mathcal{H})$. Together with the previous observation, this implies $\mathrm{tc}_3(G) \leqslant 4$. But our goal is to prove that $\mathrm{tc}_3(G) \leqslant 3$. To this aim, we analyze the hypergraph $\mathcal{H}$ more carefully, reducing the situation to a few possible settings of components covering all vertices. In each of those cases, we can again analyse the possible colouring of edges of common neighbours of specific vertices, inferring that indeed there are 3 monochromatic components cover all vertices.

§4. Proof of Theorem 1.1

Instead of analysing the colouring of the graph $G = G(n, p)$, it will be helpful to analyse the following auxiliary graph.

Definition 4.1 (Shortcut graph)**.** *Let G be a graph and φ be a 3-edge-colouring of G. The* shortcut graph *of G (with respect to φ) is the graph $F = F(G, \varphi)$ that has $V(G)$ as the vertex set and the following edge set:*

$$\{uv : u, v \in V(G) \text{ and } u \text{ and } v \text{ are connected in } G \text{ by a path monochromatic under } \varphi\}.$$

We can consider a natural edge colouring φ' of $F(G, \varphi)$ by assigning to an edge $uv \in E(F(G, \varphi))$ the colour of any monochromatic path connecting u and v in G under the colouring φ. We will say that φ' is an *inherited colouring* of $F(G, \varphi)$. Let $\mathrm{tc}(F, \varphi')$ be the minimum number of monochromatic components (under the colouring φ') covering all the vertices of F. Note that any covering of $F(G, \varphi)$ with monochromatic trees under φ' corresponds to a covering of G with monochromatic trees under the colouring φ. In particular, if we show that for every 3-edge-colouring φ of G, we have $\mathrm{tc}(F, \varphi') \leqslant 3$, for every ineherited colouring φ', then we have shown that $\mathrm{tc}_3(G) \leqslant 3$. Therefore, Theorem 1.1 follows from the following lemma.

Lemma 4.2. *Let $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$ and let $G = G(n, p)$. The following holds with high probability. For any 3-edge-colouring φ of G and any inherited colouring φ' of the shortcut graph $F = F(G, \varphi)$, we have $\mathrm{tc}(F, \varphi') \leqslant 3$.*

The proof of Lemma 4.2 is divided into two different cases, depending on the independence number of F. Subsections 4.1 and 4.2 are devoted, respectively, to the proof of Lemma 4.2 when $\alpha(F) \geqslant 3$ and $\alpha(F) \leqslant 2$.

From now on, we fix $\varepsilon > 0$ and assume that $p \gg \left(\frac{\ln n}{n}\right)^{1/6}$ and n is sufficiently large. Then, by Lemma 2.2, we may assume that the following holds w.h.p.:

(1) There is an edge between any two sets of size $\omega\left((\ln n)/p\right)$.
(2) Every vertex $v \in V(G)$ has degree $d_G(v) = (1 \pm \varepsilon)pn$.

(3) Every set of $i \leqslant 6$ vertices has $(1 \pm \varepsilon)p^i n$ common neighbours.

4.1. **Shortcut graphs with independence number at least three.**

Proof of Lemma 4.2 for $\alpha(F) \geqslant 3$. Since $\alpha(F) \geqslant 3$, there exist three vertices $r, b, g \in V(G)$ that pairwise do not have any monochromatic path connecting them in G. In particular, if v is a common neighbour of r, b and g in G, then the edges vr, vb and vg have all different colours. The common neighbourhood of r, b and g in G has size at least $np^3/2$. Let X_{rbg} be the largest subset of this common neighbourhood such that for each $i \in \{r, b, g\}$, the edges between i and the vertices of X_{rbg} are all coloured with the same colour in G. Then $|X_{rbg}| \geqslant np^3/12$. Without loss of generality, assume that all edges between r and the vertices of X_{rbg} are red, between b and the vertices of X_{rbg} are blue and those between g and the vertices of X_{rbg} are green. Let $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ and $C_{\text{green}}(g)$ be respectively the red, blue and green components in G containing r, g and b.

Notice that all vertices of F that have a neighbour in X_{rbg} are covered by $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ or $C_{\text{green}}(g)$. Therefore, the proof would be finished if every vertex had a neighbour in X_{rbg}. If this is not the case, we fix an arbitrary vertex $y \in V \setminus (X_{rbg} \cup N_G(X_{rbg}))$. By our choice of p, there are at least $np^4/2$ common neighbours of y, r, b and g. Let X_{yrbg} be the largest subset of the common neighbourhood of y, r, b and g such that for each $i \in \{r, b, g\}$, the edges between i and X_{yrbg} are all coloured the same. Then $|X_{yrbg}| \geqslant np^4/12$. Note that since $y \notin N_G(X_{rbg})$, the sets X_{yrbg} and X_{rbg} are disjoint. Furthermore, since $|X_{yrbg}|, |X_{rbg}| \gg \frac{\ln n}{p}$, we have

$$|E_G(X_{yrbg}, X_{rbg})| \geqslant 1.$$

We now analyse the colours between r, b, g and the set X_{yrbg}. Again, since there is no monochromatic path connecting any two of r, b and g, all $i \in \{r, b, g\}$ have to connect to X_{yrbg} in different colours. Since X_{yrbg} is disjoint of X_{rbg}, we cannot have r, b and g being simultaneously connected to X_{yrbg} by red, blue and green edges, respectively. Assume first that for each $i \in \{r, b, g\}$, the edges between i and X_{yrbg} have different colours from the edges between i and X_{rbg}. Then let uv be an edge between X_{yrbg} and X_{rbg} and notice that whatever the colour of uv is, we will have a monochromatic path connecting two of the vertices in $\{r, g, b\}$. Therefore, we can assume that for some $i \in \{r, g, b\}$, we have that all the edges between i and X_{rbg} and all the edges between i and X_{yrbg} coloured the same. Without loss of generality, we may say that such i is r. In this case, the edges between b and X_{yrbg} are green and the edges between g and X_{yrbg} are blue. Finally, all the edges between X_{yrbg} and X_{rbg} are red, otherwise we would be able to connect b and g by some monochromatic path. Figure 4.1 shows the colouring of the edges that we have analysed so far.

Let us now consider any further vertex $x \in V \setminus (X_{rbg} \cup N_G(X_{rbg}))$ with $x \neq y$, if such a vertex exists. We define X_{xrbg} analogously to X_{yrbg} and observe that the colour pattern

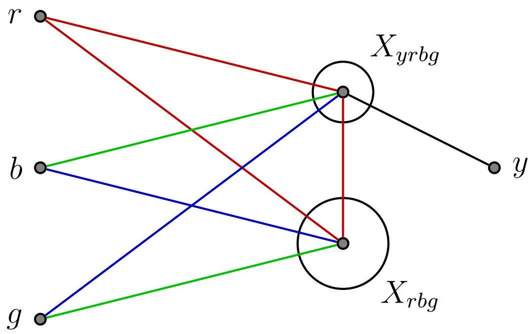

FIGURE 4.1. Analysis of the colouring of the edges incident on X_{rbg} and on X_{yrbg}.

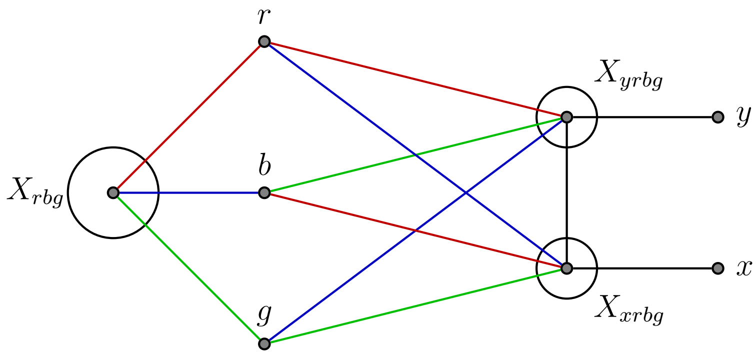

FIGURE 4.2. Analysis of the color of the edges incident on X_{yrbg} and on X_{xrbg}.

from r, b, g to X_{xrbg} must be the same as the one to X_{yrbg}. Indeed, if this is not the case, then a similar analysis of the colours of the edges between $\{r, b, g\}$ and X_{xrbg} yields that for some $i \in \{b, g\}$, we know that the edges connecting i to X_{xrbg} are of the same colour as the edges connecting i to X_{rbg}. Without loss of generality, let us say that i is g. Then the edges between b and X_{xrbg} are red and the edges between r and X_{xrbg} are green, otherwise X_{xrbg} and X_{rbg} would not be disjoints sets. Figure 4.2 shows the colouring of the edges incident to X_{yrbg} and X_{xrbg}. Since $|X_{yrbg}|, |X_{xrbg}| \gg \frac{\ln n}{p}$, we have that there is some edge uv between X_{yrbg} and X_{xrbg}. But then however we colour uv, we will get an monochromatic path connecting two vertices in $\{r, b, g\}$, which is a contradiction. Thus, the colour pattern of edges between $\{r, b, g\}$ and X_{xrbg} is the same as the colour pattern of the edges between $\{r, b, g\}$ and X_{yrbg}.

Therefore, we have that each vertex in $X_{rbg} \cup N_G(X_{rbg})$ belongs to one of the monochromatic components $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$ or $C_{\text{green}}(g)$, while a vertex in $V(G) \setminus (X_{rbg} \cup N_G(X_{rbg}))$ belongs to one of the monochromatic components $C_{\text{red}}(r)$, $C_{\text{green}}(b)$ or $C_{\text{blue}}(g)$ where the latter two are the green component containing b and the blue component containing g, respectively. This gives a covering of G with five monochromatic trees. Next we will show that actually three of those trees already cover all the vertices.

Suppose that at least 4 among the components $C_{\rm red}(r)$, $C_{\rm blue}(b)$, $C_{\rm green}(b)$, $C_{\rm green}(g)$, and $C_{\rm blue}(g)$ are needed to cover all vertices. Since there does not exist any monochromatic path between any two of r, b, g, we know that for each $i \in \{r, b, g\}$, any monochromatic component containing i does not intersect $\{r, g, b\} \smallsetminus \{i\}$. Hence, among those at least 4 components, we have for each $i \in \{r, b, g\}$ one component containing it and, without loss of generality, two containing b. That is, three components of those at least 4 components needed to cover all the vertices are $C_{\rm red}(r)$, $C_{\rm blue}(b)$ and $C_{\rm green}(b)$. Now there are two cases regarding the fourth component: we need $C_{\rm green}(g)$ as the fourth component or we need $C_{\rm blue}(g)$ (those two cases might intersect).

We begin with the first case, where we need the components $C_{\rm red}(r)$, $C_{\rm blue}(b)$, $C_{\rm green}(b)$ and $C_{\rm green}(g)$ to cover all the vertices of G. Let

$$\tilde{b} \in C_{\rm blue}(b) \smallsetminus (C_{\rm red}(r) \cup C_{\rm green}(b) \cup C_{\rm green}(g))$$

and let

$$\tilde{g} \in C_{\rm green}(b) \smallsetminus (C_{\rm red}(r) \cup C_{\rm blue}(b) \cup C_{green}(g)) \,.$$

Then let $X_{\tilde{b}\tilde{g}rbg}$ be the maximum set of common neighbours of $\tilde{b}, \tilde{g}, r, g, b$ such that for each $i \in \{\tilde{b}, \tilde{g}, r, b, g\}$, the edges from i to $X_{\tilde{b}\tilde{g}rbg}$ are all coloured the same. Since $|X_{\tilde{b}\tilde{g}rbg}| \geqslant np^5/240 \gg \frac{\ln n}{p}$, we have

$$|E_G(X_{\tilde{b}\tilde{g}rbg}, X_{yrbg})| \geqslant 1 \quad \text{and} \quad |E_G(X_{\tilde{b}\tilde{g}rbg}, X_{rbg})| \geqslant 1.$$

We will analyse the possible colours of the edges between the specified vertices and $X_{\tilde{b}\tilde{g}rbg}$. If for each of r, b, g, the colour it sends to $X_{\tilde{b}\tilde{g}rbg}$ is different from the colour it sends to X_{rbg}, then any edge between $X_{\tilde{b}\tilde{g}rbg}$ and X_{rbg} ensures a monochromatic path between two of r, b, g (in the colour of that edge). Similarly, it cannot happen that for each of r, b, g, the colour it sends to $X_{\tilde{b}\tilde{g}rbg}$ is different from the colour it sends to X_{yrbg}. Thus, since r sends red to both X_{rbg} and X_{yrbg} while the colours from b (and g) to X_{rbg} and X_{yrbg} are switched, the colour of the edges between r and $X_{\tilde{b}\tilde{g}rbg}$ is red.

Now note that, by the choice of $\tilde{b}$ and $\tilde{g}$, the edges between each of them and $X_{\tilde{b}\tilde{g}rbg}$ can not be red. Further, the choice implies that an edge between $\tilde{b}$ and $X_{\tilde{b}\tilde{g}rbg}$ can not be of the same colour (green or blue) as an edge between $\tilde{g}$ and $X_{\tilde{b}\tilde{g}rbg}$. If g would send blue (and hence b would send green) edges to $X_{\tilde{b}\tilde{g}rbg}$, there would either be a blue path between b and g (if the edges between $\tilde{b}$ and $X_{\tilde{b}\tilde{g}rbg}$ are blue) or $\tilde{b}$ would lie in $C_{\rm green}(b)$ (if the edges between $\tilde{b}$ and $X_{\tilde{b}\tilde{g}rbg}$ are green). Since both those situations would mean a contradiction, we may assume that each of r, b, g sends edges with that colour to $X_{\tilde{b}\tilde{g}rbg}$ as it does to X_{rbg}. But then $X_{\tilde{b}\tilde{g}rbg}$ is actually a subset of X_{rbg} and therefore $\tilde{g}$, having an edge to X_{rbg}, lies in one of $C_{\rm red}(r)$, $C_{\rm blue}(b)$, or $C_{\rm green}(g)$, a contradiction.

In the case where the forth component that we need is $C_{\rm blue}(g)$, we repeat the construction of $X_{\tilde{b}\tilde{g}rbg}$ similarly as before by letting

$$\tilde{b} \in C_{\rm blue}(b) \smallsetminus (C_{\rm red}(r) \cup C_{\rm green}(b) \cup C_{\rm blue}(g))$$

and

$$\tilde{g} \in C_{\text{green}}(b) \setminus (C_{\text{red}}(r) \cup C_{\text{blue}}(b) \cup C_{\text{blue}}(g)).$$

Also as before, we end up with $X_{\tilde{b}\tilde{g}rbg}$ being part of X_{rbg}. From the choice of $\tilde{g}$, the edges it sends to $X_{\tilde{b}\tilde{g}rbg}$ have to be green, since otherwise it would be in $C_{\text{red}}(r)$ or $C_{\text{blue}}(b)$. But that gives a green path between b and g, a contradiction.

Summarising, we infer that three components among $C_{\text{red}}(r)$, $C_{\text{blue}}(b)$, $C_{\text{green}}(b)$, $C_{\text{green}}(g)$ and $C_{\text{blue}}(g)$ cover the vertex set of G. □

4.2. **Shortcut graphs with independence number at most two.**

Proof of Lemma 4.2 for $\alpha(F) \leqslant 2$. We start by noticing that if $\alpha(F) = 1$, then the graph F together with the colouring φ' is a complete 3-coloured graph and therefore, by Theorem 2.1, there exists a partition of $V(F)$ into 2 monochromatic trees. Thus, we may assume that $\alpha(F) = 2$.

Let $\mathcal{H}$ be the 3-uniform hypergraph with $V(\mathcal{H})$ being the collection of all the monochromatic components of F under the colouring φ' and three monochromatic components form a hyperedge in $\mathcal{H}$ if they share a vertex. Notice that $\mathcal{H}$ is 3-partite, since distinct monochromatic components of the same colour do not have a common vertex and therefore they can not belong to the same hyperedge. In other words, the colour of each component give us a 3-partition of the vertex set of $\mathcal{H}$. We denote by V_{red},V_{blue} and V_{green} the set of vertices of $V(\mathcal{H})$ that correspond to, respectively, red, blue and green components. Such construction was inspired by a construction due to Gyárfás [9].

Note that every vertex v of F is contained in a monochromatic component for each one of the colours (a monochromatic component could consist only of v). Therefore, any vertex cover of $\mathcal{H}$ corresponds to a covering of the vertices of F with monochromatic trees. Indeed, if A is a vertex cover of $\mathcal{H}$, then consider the monochromatic components corresponding to each vertex in A. If any vertex v of F is not covered by those components, then the vertices in $\mathcal{H}$ corresponding to the red, green and blue components in F containing v do not belong to A and they form an hyperedge. But this contradicts the fact that A is a vertex cover of $\mathcal{H}$. Therefore,

$$\text{tc}(F, \varphi') \leqslant \tau(\mathcal{H}). \tag{4.1}$$

Let $L = \bigcup_{s \in V_{\text{red}}} L_s$ be the union of the link graphs L_s of all vertices $s \in V_{\text{red}}$. Any vertex cover of this bipartite graph L corresponds to a vertex cover of $\mathcal{H}$ of the same size. Therefore,

$$\tau(\mathcal{H}) \leqslant \tau(L). \tag{4.2}$$

Furthermore, by König's theorem we know that $\tau(L) = \nu(L)$. Thus, if $\nu(L) \leqslant 3$, then by (4.1) and (4.2), we have

$$\text{tc}(F, \varphi') \leqslant \tau(\mathcal{H}) \leqslant \tau(L) = \nu(L) \leqslant 3.$$

Therefore, we may assume that $\nu(L) \geqslant 4$, and fix a matching M_L of size at least 4 in L. Let us say that M_L consists of the edges G_1B_1, G_2B_2, G_3B_3, and G_4B_4, where $\{G_1, G_2, G_3, G_4\} \subseteq V_{\text{green}}$ and $\{B_1, B_2, B_3, B_4\} \subseteq V_{\text{blue}}$.

Now we give an upper bound for $\nu(\mathcal{H})$. Note that any matching $M_{\mathcal{H}}$ in $\mathcal{H}$ gives us an independent set I in F. Indeed, for each hyperedge $e \in M_{\mathcal{H}}$, let $v_e \in V(F)$ be any vertex in the intersection of those monochromatic components associated to the vertices in e and let $I = \{v_e : e \in M_{\mathcal{H}}\}$. We claim that I is an independent set in F. Indeed, if v_e and v_f were adjacent vertices in I, then e and f intersect, as the edge connecting v_e to v_f in F will connect the monochromatic components containing v_e and v_f of that colour that is given to the edge $v_e v_f$. Therefore, since $\alpha(F) = 2$, we have

$$\nu(\mathcal{H}) \leqslant \alpha(F) = 2. \tag{4.3}$$

Now, if there are three different edges in M_L that are edges in the link graphs of three different vertices of V_{red}, then there would be a matching of size 3 in $\mathcal{H}$, contradicting (4.3). Therefore, we may assume that M_L is contained in the union of at most two link graphs, say L_{R_1} and L_{R_2}, of vertices $R_1, R_2 \in V_{\text{red}}$. Now we are left with three cases: (Case 1) two edges of M_L belong to L_{R_1} and two belong to L_{R_2}; (Case 2) three edges of M_L belong to L_{R_1} and one to L_{R_2}; (Case 3) the four edges of M_L belong to L_{R_1}. Without loss of generality, we can describe each of those three cases as follows (see Figures 4.3, 4.4 and 4.5):

Case 1: The edges G_1B_1 and G_2B_2 belong to L_{R_1} and the edges G_3B_3 and G_4B_4 belong to L_{R_2}. That means that all the following four sets are non-empty:

$$\begin{aligned} J_1 &:= R_1 \cap G_1 \cap B_1, \\ J_2 &:= R_1 \cap G_2 \cap B_2, \\ J_3 &:= R_2 \cap G_3 \cap B_3, \\ J_4 &:= R_2 \cap G_4 \cap B_4. \end{aligned}$$

Case 2: The edges G_1B_1, G_2B_2 and G_3B_3 belong to L_{R_1} and the edge G_4B_4 belongs to L_{R_2}. That means that all the following four sets are non-empty:

$$\begin{aligned} J_1 &:= R_1 \cap G_1 \cap B_1, \\ J_2 &:= R_1 \cap G_2 \cap B_2, \\ J_3 &:= R_1 \cap G_3 \cap B_3, \\ J_4 &:= R_2 \cap G_4 \cap B_4. \end{aligned}$$

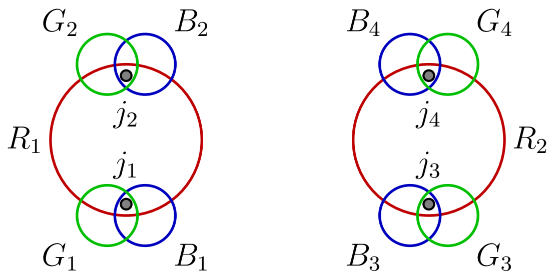

FIGURE 4.3. Case 1

Case 3: The edges G_1B_1, G_2B_2, G_3B_3 and G_4B_4 belong to L_{R_1}. That means that all the following four sets are non-empty:

$$J_1 := R_1 \cap G_1 \cap B_1,$$
$$J_2 := R_1 \cap G_2 \cap B_2,$$
$$J_3 := R_1 \cap G_3 \cap B_3,$$
$$J_4 := R_1 \cap G_4 \cap B_4.$$

In this case, let R_2 be any other red component different from R_1 and let B and G be, respectively, a blue and a green component with $R_2 \cap B \cap G \neq \varnothing$. Suppose that $G \notin \{G_1, G_2, G_3, G_4\}$. Then the three of the edges G_1, B_1, G_2, B_2, G_3, B_3 and G_4, B_4 are not incident to GB (because B must be different of at least three of the sets B_1, B_2, B_3 and B_4) and those three edges together with GB may be analysed just as in Case 2. Therefore, we may suppose that $G \in \{G_1, G_2, G_3, G_4\}$. Let us say, without loss of generality, that $G = G_4$. If $B \notin \{B_1, B_2, B_3\}$, then the edges G_1B_1, G_2B_2 and G_3B_3 belong to L_{R_1}, the edge GB belongs to L_{R_2} and this case may be analysed, again, just as in Case 2. Therefore, we may assume that $B \in \{B_1, B_2, B_3\}$. Let us say, without loss of generality that $B = B_3$. Then let J_5 be the following non-empty set:

$$J_5 := R_2 \cap G_4 \cap B_3. \tag{4.4}$$

Let us further remark that, since $\nu(\mathcal{H}) \leqslant 2$, in each of the three cases above, we have

$$V(F) = R_1 \cup R_2 \cup G_1 \cup G_2 \cup G_3 \cup G_4 \cup B_1 \cup B_2 \cup B_3 \cup B_4.$$

Otherwise, for any uncovered vertex $v \in V(F)$, the hyperedge given by the red, blue and green components containing v together with the hyperedges $R_1B_1G_1$ and $R_2B_3G_3$ (in Cases 1 and 2) or $R_2B_3G_4$ (in Case 3) give a matching of size 3 in $\mathcal{H}$.

Let us start with Case 1.

Proof in Case 1: We will prove that R_1 and R_2 together with possibly one further monochromatic component cover $V(F)$. For each $i \in \{1, 2, 3, 4\}$, let $\tilde{B}_i = B_i \smallsetminus (R_1 \cup R_2)$ and $\tilde{G}_i = G_i \smallsetminus (R_1 \cup R_2)$.

Pick vertices $j_i \in J_i$, with $i \in \{1,2,3,4\}$, arbitrarily. Consider a vertex $o \in \tilde{B}_1$ (if such a vertex exists). Since $\alpha(F) = 2$, there is an edge connecting two of o, j_2, j_3. Because j_2 and j_3 belong to different components of each colour, such an edge must be incident to o. So let us say that such edge is oj_i, for some $i \in \{2,3\}$. Since $o \notin R_1 \cup R_2$, the edge oj_i cannot be red. And since $o \in B_1$, oj_i cannot be blue either, otherwise we would connect the blue components B_1 and B_i. Now assume that o and j_2 are not adjacent. Then oj_3 is a green edge in F. By analogously analysing the edge between o, j_2 and j_4 together with the supposition that oj_2 is not an edge in F, we get that oj_4 must be a green edge in F. But then we have a green path j_3oj_4 connecting j_3 to j_4, a contradiction. Therefore oj_2 is an edge in F and it is green. That implies that $o \in G_2$. Therefore $\tilde{B}_1 \subseteq G_2$. Analogously, we can conclude the following:

$$\begin{aligned} \tilde{B}_1 &\subseteq G_2, & \tilde{G}_1 &\subseteq B_2, \\ \tilde{B}_2 &\subseteq G_1, & \tilde{G}_2 &\subseteq B_1, \\ \tilde{B}_3 &\subseteq G_4, & \tilde{G}_3 &\subseteq B_4, \\ \tilde{B}_4 &\subseteq G_3, & \tilde{G}_4 &\subseteq B_3. \end{aligned} \tag{4.5}$$

Claim 4.3. *We have* $\tilde{B}_1 \cup \tilde{G}_1 \cup \tilde{B}_2 \cup \tilde{G}_2 = \varnothing$ *or* $\tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4 = \varnothing$.

Proof. Suppose for a contradiction that there exist $o_1 \in \tilde{B}_1 \cup \tilde{G}_1 \cup \tilde{B}_2 \cup \tilde{G}_2$ and $o_2 \in \tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4$. Recall that from our choice of p, there is some $z \in N(j_1, j_2, j_3, j_4, o_1, o_2)$. Two of the edges zj_i,for $i \in \{1,2,3,4\}$, have the same colour. Since each j_i belongs to different green and blue components, those two edges are red. Since $\{j_1, j_2\} \in R_1$ and $\{j_3, j_4\} \in R_2$, those two red edges are either zj_1 and zj_2 or zj_3 and zj_4. Let us say that zj_1 and zj_2 are red (the other case is similar). Then one of the edges zj_3 and zj_4 has to be green and the other blue. Now, since $o_1 \notin R_1$, the edge zo_1 is either green or blue. Then one of the paths o_1zj_3 or o_1zj_4 is green or blue. This implies that $o_1 \in B_3 \cup G_3 \cup B_4 \cup G_4$. On the other hand, (4.5) implies that $o_1 \in (B_1 \cup B_2) \cap (G_1 \cup G_2)$. But then we reached a contradiction, since that would mean that o_1 belongs to two different components of the same colour. □

We may assume without loss of generality that $\tilde{B}_3 \cup \tilde{G}_3 \cup \tilde{B}_4 \cup \tilde{G}_4$ is empty. Then, recalling that $\nu(\mathcal{H}) \leqslant 2$ and in view of (4.5), the union of the components R_1, B_1, G_1 and R_2 covers every vertex of F. If we show that $B_1 \subseteq G_1 \cup R_1 \cup R_2$ or that $G_1 \subseteq B_1 \cup R_1 \cup R_2$, then we get three monochromatic components covering the vertices of F. Our next claim states precisely that.

Claim 4.4. *We have* $\tilde{B}_1 \smallsetminus G_1 = \varnothing$ *or* $\tilde{G}_1 \smallsetminus B_1 = \varnothing$.

Proof. Suppose that there exist two distinct vertices $b \in \tilde{B}_1 \smallsetminus G_1$ and $g \in \tilde{G}_1 \smallsetminus B_1$. Let $z \in N(j_1, j_2, j_3, j_4, b, g)$. As before, either zj_1 and zj_2 or zj_3 and zj_4 are red edges. First assume that zj_1 and zj_2 are red. Then one of the edges zj_3 and zj_4 has to be green

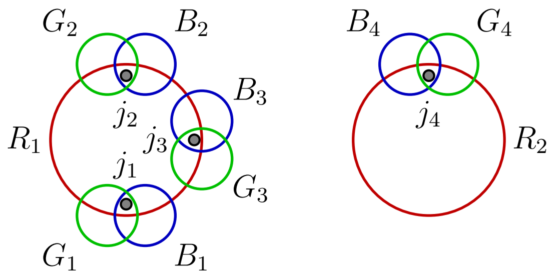

FIGURE 4.4. Case 2

and the other blue. Now, since $b \notin R_1$, the edge zb is either green or blue. Then one of the paths bzj_3 or bzj_4 is green or blue. This implies that $b \in B_3 \cup G_3 \cup B_4 \cup G_4$. On the other hand, (4.5) implies that $b \in B_1 \cap G_2$. Then we reached a contradiction, since that would mean that b belongs to two different components of the same colour.

Therefore, the edges zj_3 and zj_4 are red and one of the edges zj_1 and zj_2 is green and the other is blue. First let us say that zj_1 is green and zj_2 is blue. Since $b \notin (R_1 \cup R_2)$, the edge zb cannot be red. Also the edge zb cannot be blue otherwise the path bzj_2 would connect the components B_1 and B_2. Finally, zb cannot be green, otherwise the path bzj_1 would gives us that $b \in G_1$. Therefore zj_1 is blue and zj_2 is green. But this case analogously leads to a contradiction (with g and G_i instead of b and B_i and green and blue switched). □

We proceed to the proof of Case 2.

Proof in Case 2: As in Case 1, pick vertices $j_i \in J_i$, with $i \in \{1, 2, 3, 4\}$ arbitrarily. We claim that $V(F) \subseteq R_1 \cup R_2 \cup B_4 \cup G_4$. Indeed, let $o \in V(F) \setminus (R_1 \cup R_2)$. Notice that since $\alpha(F) = 2$, there is an edge in each of the following sets of three vertices: $\{o, j_4, j_1\}$, $\{o, j_4, j_2\}$, and $\{o, j_4, j_3\}$. We claim that oj_4 is an edge of F. Indeed, if this was not the case, then since there cannot be an edge between j_4 and j_i for $i = 1, 2, 3$, we would have the edges oj_1, oj_2 and oj_3 and all of them would be coloured green or blue. Thus, two of them would be coloured the same, connecting two distinct components of one colour in this colour, a contradiction. So $oj_4 \in E(F)$ and since oj_4 cannot be red, we conclude that $o \in (B_4 \cup G_4)$. Therefore, R_1, R_2, B_4 and G_4 cover all vertices of F.

If $B_4 \setminus (R_1 \cup R_2 \cup G_4) = \varnothing$ or $G_4 \setminus (R_1 \cup R_2 \cup B_4) = \varnothing$, then we get three monochromatic components covering $V(F)$. So let us assume that there exist $b \in B_4 \setminus (R_1 \cup R_2 \cup G_4)$ and $g \in G_4 \setminus (R_1 \cup R_2 \cup B_4)$. If b and g are not adjacent, then since each of the sets $\{b, g, j_i\}$, for $i = 1, 2, 3$, has to induce at least one edge, there are two edges between b and $\{j_1, j_2, j_3\}$ or two edges between g and $\{j_1, j_2, j_3\}$. However, from the choice of b, we know that all the edges between b and $\{j_1, j_2, j_3\}$ are green, and therefore two of such edges would give us a green connection between two different green components, a contradiction. Similarly, from the choice of g, we know that all the edges between b and $\{j_1, j_2, j_3\}$ are blue, and

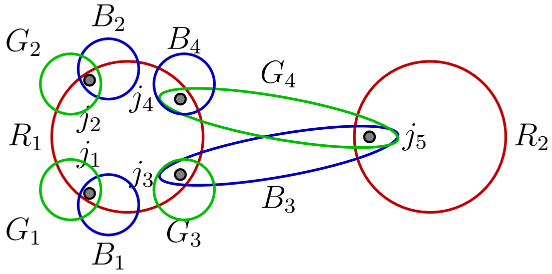

Figure 4.5. Case 3

two of such edges would give us a blue connection between two different blue components, again a contradiction.

Hence, we conclude that $bg \in F$ for any $b \in B_4 \smallsetminus (R_1 \cup R_2 \cup G_4)$ and any $g \in G_4 \smallsetminus (R_1 \cup R_2 \cup B_4)$ and any such edge bg is red. Therefore, there is a red component R_3 covering $(B_4 \triangle G_4) \smallsetminus (R_1 \cup R_2)$, where $B_4 \triangle G_4$ denotes the symmetric difference. If $(B_4 \cap G_4) \smallsetminus (R_1 \cup R_2) = \varnothing$, then R_1, R_2 and R_3 cover $V(F)$ and we are done. Therefore, suppose there is a vertex $x \in (B_4 \cap G_4) \smallsetminus (R_1 \cup R_2)$. If $R_2 \smallsetminus (B_4 \cup G_4) = \varnothing$, then R_1, B_4, G_4 cover $V(F)$ and we are done. Therefore, suppose there is a vertex $y \in R_2 \smallsetminus (B_4 \cup G_4)$. Note that $xy \notin E(F)$, since x and y belong to different components in each of the colours. Also, $xj_i \notin E(F)$, for $i \in \{1, 2, 3\}$, since otherwise two different components of the same colour would be connected in that colour by the edge xj_i. Now $\alpha(F) = 2$ implies that $yj_i \in E(F)$, for $i \in \{1, 2, 3\}$ (otherwise, $\{x, y, j_i\}$ would be an independent set). But these edges must all be green or blue, hence two of them are of the same colour, connecting two different components of one colour in that colour, a contradiction.

We arrived at the last case, Case 3.

Proof in Case 3: Similarly to the previous cases, let us pick vertices $j_i \in J_i$, with $i \in \{1, 2, 3, 4, 5\}$ arbitrarily. We will show first that we can cover all vertices of F with 4 monochromatic components. Let $o_1, o_2 \in V(F) \smallsetminus (R_1 \cup B_3 \cup G_4)$ and let $z \in N(j_1, j_2, j_3, o_1, o_2, j_5)$. At least one of the edges zj_1, zj_2 and zj_3 is red, as otherwise we would connect two distinct components of one colour in that colour. Therefore $z \in R_1$. Since $o_1, o_2, j_5 \notin R_1$, the edges zo_1, zo_2 and zj_5 cannot be red. Furthermore, o_1z and o_2z are coloured with a colour different from the colour of the edge j_5z, as otherwise they would belong to B_3 or G_4. Thus, o_1 and o_2 are connected by a monochromatic path in green or blue. Hence, we showed that any two vertices of $V(F) \smallsetminus (R_1 \cup B_3 \cup G_4)$ are connected by a monochromatic path in green or blue. We infer that there is a green or blue component covering $V(F) \smallsetminus (R_1 \cup B_3 \cup G_4)$. Therefore, R_1, B_3, G_4 and one further blue or green component C cover all vertices of G. Let us assume that C is a green component; the case where C is a blue component is analogous.

We claim that $R_1 \cup B_3 \cup C$, or $R_1 \cup G_4 \cup C$, or $R_1 \cup B_3 \cup G_4$ covers $V(F)$. Indeed, suppose for the sake of contradiction that there exist vertices $g \in G_4 \smallsetminus (R_1 \cup B_3 \cup C)$, $b \in$

$B_3 \smallsetminus (R_1 \cup G_4 \cup C)$ and $c \in C \smallsetminus (R_1 \cup B_3 \cup G_4)$. Let $z \in N(j_1, j_2, j_3, g, b, c)$ and note that one of zj_1, zj_2 and zj_3 is red. Consequently gz, cz and bz are not red. Notice, however, that gz and bz can not be both green and neither both blue. Now let us say cz is green. Since $c \notin G_4$ and $g \in G_4$, we would have gz blue in this case. But then bz must be green and since $c \in C$ and C is a green component, we have $b \in C$, which is a contradiction. Therefore cz must be blue. Then, since $c \notin B_3$ and $b \in B_3$, the edge bz should be green. Thus the edge gz is blue. Since this argument holds for any $g \in G_4 \smallsetminus (R_1 \cup B_3 \cup C)$ and $c \in C \smallsetminus (R_1 \cup B_3 \cup G_4)$, we conclude that $V(F) \smallsetminus (R_1 \cup B_3)$ can be covered by one blue tree. Hence, G can be covered by the three monochromatic trees. This finishes the last case and thereby the proof of Lemma 4.2. □

References

[1] R. Aharoni, *Ryser's conjecture for tripartite* 3*-graphs*, Combinatorica **21** (2001), no. 1, 1–4. ↑1, 2, 3

[2] D. Bal and L. DeBiasio, *Partitioning random graphs into monochromatic components*, Electron. J. Combin. **24** (2017), no. 1, Paper 1.18, 25. MR3609188 ↑1

[3] B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998. MR1633290 ↑2

[4] ———, *Random graphs*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001. MR1864966 ↑2

[5] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008. MR2368647 ↑2

[6] M. Bucić, D. Korándi, and B. Sudakov, *Covering random graphs by monochromatic trees and Helly-type results for hypergraphs*, arXiv (2019), available at arXiv:1902.05055. ↑1

[7] R. Diestel, *Graph theory*, 4th ed., Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010. MR2744811 ↑2

[8] P. Erdős, A. Gyárfás, and L. Pyber, *Vertex coverings by monochromatic cycles and trees*, J. Combin. Theory Ser. B **51** (1991), no. 1, 90–95. ↑1, 2

[9] A. Gyárfás, *Particiófedések és lefogóhalmazok hipergráfokban: Kandidátusi értekezés*, MTA Számítástechnikai és Automatizálási Kutató Intézet, 1977. ↑1, 3, 4.2

[10] P. E. Haxell and Y. Kohayakawa, *Partitioning by monochromatic trees*, J. Combin. Theory Ser. B **68** (1996), no. 2, 218–222. ↑1

[11] P. E. Haxell and A. D. Scott, *On Ryser's conjecture*, Electron. J. Combin. **19** (2012), no. 1, 23. ↑1

[12] J. R. Henderson, *Permutation decomposition of* $(0,1)$*-matrices and decomposition transversals*, Ph.D. Thesis, 1971. ↑2

[13] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. MR1782847 ↑2

[14] Y. Kohayakawa, G. O. Mota, and M. Schacht, *Monochromatic trees in random graphs*, Math. Proc. Cambridge Philos. Soc. **166** (2019), no. 1, 191–208. MR3893310 ↑1, 1, 2

[15] D. Korándi, F. Mousset, R. Nenadov, N. Škorić, and B. Sudakov, *Monochromatic cycle covers in random graphs*, Random Structures Algorithms **53** (2018), no. 4, 667–691. MR3875280 ↑2

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