On the Regularity Method for Hypergraphs

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Abstract

We present alternative proofs of density versions of some combinatorial partition theorems originally obtained by E. Szemerédi, H. Furstenberg and Y. Katznelson. The proofs presented here are based on an extension of the *Regularity Method* from graphs to k-uniform hypergraphs.

Szemerédi's Regularity Lemma for graphs asserts that every graph can be decomposed into relatively few random-like subgraphs. This random-like behavior enables one to find and enumerate subgraphs of a given isomorphism type. This observation is called Counting Lemma. The interplay of Szemerédi's Regularity Lemma and the Counting Lemma, referred to as the Regularity Method for graphs, has many applications in the area of extremal graph theory.

Recently, V. Rödl and J. Skokan (based on earlier work of P. Frankl and V. Rödl) generalized Szemerédi's Regularity Lemma from graphs to k-uniform hypergraphs for arbitrary $k \geq 2$. In the main part of this dissertation we prove a Counting Lemma accompanying the Rödl–Skokan Hypergraph Regularity Lemma. Both lemmas together establish a generalization of the Regularity Method from graphs to k-uniform hypergraphs.

A similar extension of the Regularity Method was independently and alternatively obtained by W. T. Gowers. His results can also be used to derive alternative proofs of the density theorems mentioned above.

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dedicated to my family and my friends

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Chapter 1

Preface

In 1975 Szemerédi [54] confirmed a long standing conjecture of Erdős and Turán [13] concerning the upper density of sets containing no arithmetic progression of fixed finite length. More precisely Szemerédi showed the following celebrated density theorem.

Theorem 1.1 (Szemerédi's Density Theorem)**.** *For every positive integer t and every positive real δ, there exist an integer $N_0 = N_0(t, \delta)$ such that for $N \geq N_0$ any subset $Z \subseteq \{1, \ldots, N\}$ with $|Z| \geq \delta N$ contains an arithmetic progression of length t.*

Shortly after Szemerédi's combinatorial proof appeared, in 1977 Furstenberg [20] gave an alternative proof of Theorem 1.1 based on ergodic theory. Refining the techniques of that proof Furstenberg and Katznelson were later able to derive several other density versions of combinatorial partition theorems. The following, which can be viewed as a density version of Gallai–Witt's theorem, is one of them. We denote by $[-N; N]$ the set $\{-N, -N+1, \ldots, N\}$.

Theorem 1.2 (Furstenberg & Katznelson [21])**.** *Let T be a finite subset of $\mathbb{R}^d$, and let $\delta > 0$. Then there exists a finite subset C of $\mathbb{R}^d$ such that any subset $Z \subset C$ with $|Z| > \delta|C|$ contains a homothetic copy of T, i.e., a set of the form $\boldsymbol{z} + \lambda T$ for some $\boldsymbol{z} \in \mathbb{R}^d$ and some positive real λ.*

Moreover, if $T \subset [-t;t]^d$ for some positive integer t, then $C = [-N;N]^d$ has the above property for every sufficiently large $N \geq N_0(t,d,\delta)$.

Note that the special case of Theorem 1.2 for $d = 1$ implies Theorem 1.1. More generally, for a fixed d, the special case of the above result allows us to find a homothetic copy of a full-dimensional cube $[-t;t]^d$ in a dense subset of a sufficiently large cube $[-N;N]^d$. Two other results in a similar vein, also due to Furstenberg and Katznelson, address the complementary case when the dimension is allowed to grow.

Theorem 1.3 (Furstenberg & Katznelson [22])**.** *Let $\mathbb{F}_q$ be the finite field with q elements. Then for every positive integer d and every $\delta > 0$, there exists $M_0 = M_0(q,d,\delta)$ such that, for $M \geq M_0$, any subset $Z \subset \mathbb{F}_q^M$ with $|Z| > \delta|\mathbb{F}_q^M| = \delta q^M$ contains a d-dimensional affine subspace.*

Theorem 1.4 (Furstenberg & Katznelson [22])**.** *Let G be a finite abelian group, and let $\delta > 0$. Then there exists $M_0 = M_0(G,\delta)$ such that if $M \geq M_0$ and Z is a subset of G^M with $|Z| > \delta|G|^M$, then Z contains a coset of a subgroup of G^M isomorphic to G.*

The techniques introduced by Furstenberg and Katznelson have been extended to prove other generalizations of Theorem 1.1–1.4, among which are a density version of the Hales–Jewett Theorem [27], again due to Furstenberg and Katznelson [23], and polynomial extensions of Szemerédi's Theorem, due to Bergelson and Leibman [3] and Bergelson and McCutcheon [4].

Another area of investigation concerns estimates on N_0 and M_0 in Theorem 1.1–1.4. Szemerédi's original proof of Theorem 1.1 used the Regularity Lemma for graphs [55], Theorem 2.1, which forces (see [25]) the upper bound on $N_0 = N_0(t,\delta)$ to be bigger than a tower-type function where the height of the tower is given by a polynomial in $1/\delta$. (The proof also used van der Waerden's Theorem [56] which at that time, before Shelah's proof in [50], was not known to be primitive recursive.) Gowers gave an alternative proof of Theorem 1.1 in [26] which significantly improved the upper bound on N_0. He proved that

$$N_0 = \exp\left(\exp\left((1/\delta)^{2^{2^{t+9}}}\right)\right)$$

suffices. The proofs by Furstenberg and Katznelson of Theorem 1.1–1.4 and its generalizations rely on ergodic theory and do not yield any upper bounds on N_0 and M_0.

In this dissertation we present new proofs of Theorem 1.2–1.4 (consequently we prove Theorem 1.1, as well). The proofs presented here are purely combinatorial and based on an extension of the *Regularity Method* from graphs to k-uniform hypergraphs. They can be used to give the first quantitaive bounds on N_0 and M_0. We make no attempt, however, to give any bounds.

We also want to mention that in [24] Gowers also developed independently similar techniques, which also allow quantitative proofs of Theorem 1.2–1.4.

Organization of this dissertation. This dissertation splits into two parts. In Part I we focus on the Regularity Method for k-uniform hypergraphs. Roughly speaking, the aforementioned Regularity Lemma for graphs (or 2-uniform hypergraphs) of Szemerédi, Theorem 2.1, asserts that every graph can be decomposed into relatively few random-like subgraphs. The Counting Lemma, Fact 2.2, then states that this random-like behavior enables one to find and enumerate subgraphs of fixed size. The interplay of Szemerédi's Regularity Lemma and the Counting Lemma, referred to as the Regularity Method for graphs, has many applications in the area of extremal graph theory. In [48] Rödl and Skokan (based on the work of Frankl and Rödl [16]) generalized Szemerédi's Regularity Lemma from graphs to k-uniform hypergraphs for arbitrary $k \geq 2$ (see Theorem 4.11 in Section 4.2).

The main objective of Part I of this dissertation is to prove a Counting Lemma accompanying the Rödl–Skokan Hypergraph Regularity Lemma. We defer the precise statement of the Counting Lemma, Theorem 3.6, to Section 3.3.

In Part II we focus on applications of the Rödl–Skokan Hypergraph Regularity Lemma and the Counting Lemma derived in Part I. In particular we present a proof of the following extremal hypergraph result.

Theorem 1.5. *For all fixed integers $\ell \geq k \geq 2$ and every positive real ε there exist a positive real $\delta = \delta(\ell, k, \varepsilon)$ and some integer $n_0 = n_0(\ell, k, \varepsilon)$ so that the following holds.*

Suppose $\mathcal{F}^{(k)}$ is a fixed k-uniform hypergraph on ℓ vertices and $\mathcal{H}^{(k)}$ is a k-uniform hypergraph on $n \geq n_0$ vertices. If $\mathcal{H}^{(k)}$ contains at most δn^ℓ copies of $\mathcal{F}^{(k)}$, then one can delete εn^k edges of $\mathcal{H}^{(k)}$ to make it $\mathcal{F}^{(k)}$-free.

Theorem 1.5 was first conjectured by Erdős, Frankl and Rödl [12] (see also [16] for similar problems) and it then was essentially proved by Rödl and Skokan in [46] based on the Hypergraph Regularity Lemma and the Counting Lemma mentioned above. We, however, feel it is appropriate to reproduce their short argument in the context of this dissertation and we do so in Chapter 11. Finally, in Chapter 12 we present the proofs of Theorem 1.2–1.4, which are solely based on Theorem 1.5. We survey some other applications of the Regularity Method for Hypergraphs in Chapter 13.

Part I of this dissertation is joint work with Brendan Nagle and my advisor Vojtěch Rödl. It is mainly drawn from [38]. As mentioned earlier the proof of Theorem 1.5 presented in Chapter 11 was originally obtained by Vojtěch Rödl and Jozef Skokan in [46] and we only present it here for completeness. The reductions of Theorem 1.2–1.4 to Theorem 1.5 presented in the remainder of Part II were obtained in collaboration with Vojtěch Rödl, Eduardo Tengan, and Norihide Tokushige in [45].

Part I

The Regularity Method

Chapter 2

Introduction

The results presented in this part of this dissertation were obtained in collaboration with Brendan Nagle and Vojtěch Rödl [38].

Extremal problems are among the most central and most extensively studied in combinatorics. Many of these problems concern thresholds for properties concerning deterministic structures and have proven to be difficult as well as interesting. An important recent trend in combinatorics has been to consider the analogous problems for random structures. Tools are then sometimes afforded for determining with what probability a random structure possesses certain properties.

The study of *quasi-random structures*, pioneered by the work of Szemerédi [54], merges features of deterministic and random settings. Roughly speaking, a quasi-random structure is one which, while deterministic, mimics the behavior of random structures in certain important points of view. The (quasi-random) combinatorial structures we consider in this dissertation are *set systems* or *hypergraphs*. We begin our discussion with graphs.

2.1 Szemerédi's Regularity Lemma

In the course of proving his celebrated Density Theorem concerning arithmetic progressions, Szemerédi established a lemma which decomposes the edge set of any graph into constantly many "blocks", almost all of which are quasi-random (cf. [32, 33, 55]). In what follows, we give a precise account of Szemerédi's lemma.

For a graph $G = (V, E)$ and two disjoint sets A, $B \subset V$, let $E(A, B)$ denote the set of edges $\{a, b\} \in E$ with $a \in A$ and $b \in B$ and set $e(A, B) = |E(A, B)|$. We also set $d(A, B) = d(G_{AB}) = e(A, B)/|A||B|$ for the *density* of the bipartite graph $G_{AB} = (A \cup B, E(A, B))$.

The concept central to Szemerédi's lemma is that of an *ε-regular pair*. Let $\varepsilon > 0$ be given. We say that the pair A, B is ε-regular if $|d(A, B) - d(A', B')| < \varepsilon$ holds whenever $A' \subset A$, $B' \subset B$, and $|A'| > \varepsilon|A|$, $|B'| > \varepsilon|B|$.

We call a partition $V = V_1 \cup \cdots \cup V_t$ an *equitable partition* if it satisfies $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$; we call an equitable partition *ε-regular* if all but $\varepsilon\binom{t}{2}$ pairs V_i, V_j are ε-regular. Szemerédi's lemma may then be stated as follows.

Theorem 2.1 (Szemerédi's Regularity Lemma)**.** *Let $\varepsilon > 0$ be given and let t_0 be a positive integer. There exist positive integers $n_0 = n_0(\varepsilon, t_0)$ and $T_0 = T_0(\varepsilon, t_0)$ such that any graph $G = (V, E)$ with $|V| = n \geq n_0$ vertices admits an ε-regular equitable partition $V = V_1 \cup \cdots \cup V_t$ with t satisfying $t_0 \leq t \leq T_0$.*

Szemerédi's Regularity Lemma is a powerful tool in the area of extremal graph theory. One of its most important consequences is that, in appropriate circumstances, it can be used to imply a given graph contains a fixed subgraph. Suppose that a (large) graph is given along with an ε-regular partition $V = V_1 \cup \ldots \cup V_t$ and let H be a fixed graph. If an appropriate collection of pairs $I_H \subseteq \binom{[t]}{2}$ have each $\{V_i, V_j\}$, $\{i, j\} \in I_H$, ε-regular and sufficiently dense (with respect to ε), one is guaranteed a copy of H within this collection of bipartite graphs $E(V_i, V_j)$, $\{i, j\} \in I_H$. This observation is due to the following well-known fact which may be appropriately called the Counting Lemma.

Fact 2.2 (Counting Lemma)**.** *For every integer ℓ and positive reals d and γ there exists $\delta > 0$ so that the following holds. Let $G = \bigcup_{1 \le i < j \le \ell} G^{ij}$ be an ℓ-partite graph with ℓ-partition $V_1 \cup \ldots \cup V_\ell$ where $G^{ij} = G[V_i, V_j]$, $1 \le i < j \le \ell$, and $|V_1| = \ldots = |V_\ell| = n$. Suppose further all graphs G^{ij} are ε-regular with density d. Then the number of copies of the ℓ-clique K_ℓ in G is within the interval $(1 \pm \gamma) d^{\binom{\ell}{2}} n^\ell$.*

Unlike Szemerédi's Regularity Lemma, Fact 2.2 is fairly easy to prove.

2.2 Extensions of Szemerédi's Lemma to hypergraphs

Several hypergraph regularity lemmas were considered, in part, by various authors [6, 9, 15, 19, 41]. None of these regularity lemmas seemed to admit a companion counting result (i.e. a corresponding generalization of Fact 2.2). The first attempt of developing a Hypergraph Regularity Lemma together with a corresponding Counting Lemma was undertaken in [16]. In that paper, Frankl and Rödl established an extension of Szemerédi's Regularity Lemma to 3-graphs, hereafter called the FR-Lemma (see [10, 11] for an algorithmic version).

Analogously to the feature that Szemerédi's Regularity Lemma decomposes a given graph into an ε-regular partition, the FR-Lemma decomposes the edge set of given a 3-graph into constantly many "blocks", almost all of which are, in a specific sense, "quasi-random". The concept of 3-graph regularity which plays the analogous role of the ε-regular pair is, unfortunately, considerably more technical than its graph counterpart. As well, it is not necessary at this time to know this precise definition in order to understand the current Introduction. We therefore postpone precise discussion until later.

Just as Fact 2.2, the Counting Lemma, is an important companion statement to Szemerédi's Regularity Lemma, most applications of the FR-Lemma require a similar companion lemma - the "3-graph Counting Lemma". Analogously to Fact 2.2, the 3-graph Counting Lemma estimates the number of copies of the clique $K_\ell^{(3)}$ (i.e., the complete 3-graph on ℓ vertices) contained in an appropriate collection of "dense and regular blocks" within a regular partition provided by the FR-Lemma. This 3-graph Counting Lemma was established in [16] for the special case $K_4^{(3)}$ and subsequently fully established by Nagle and Rödl in [36]. Unlike the case for graphs, the proof of the 3-graph Counting Lemma in [36] was technical and rather lengthy, suggesting that the effort to fully develop Hypergraph Regularity Methods may not be straightforward.

Recently, Rödl and Skokan [48] established a generalization of the FR-Lemma to k-graphs for $k \geq 3$ (see Section 4.2). We will refer to this lemma as the RS-Lemma. In [47], they also succeeded to prove a companion Counting Lemma in the special case of $K_5^{(4)}$. In this part of the dissertation,

we prove the k-graph Counting Lemma corresponding to the RS-Lemma.

Our Counting Lemma, the main objective of Part I, requires some notation. Therefore, we defer its precise statement to Section 3.3 (see Theorem 3.6).

Last, but not least, we mention that a Regularity Lemma as well as a corresponding Counting Lemma for k-graphs was recently independently proved by Gowers [24]. The approach taken by Gowers is different from the one taken in [48] and this dissertation.

2.3 Quasi-random hypergraphs

A related line of research is the study of quasi-random hypergraphs, some topics of which play a crucial role in our proof. We feel a few words on quasi-random hypergraphs at this time are appropriate.

Haviland and Thomason [28] and Chung and Graham [7, 8] were the first to investigate systematic properties of quasi-random hypergraphs. In particular, Chung and Graham considered several quite disparate looking properties of random-like hypergraphs of density 1/2 and proved that they are, in fact, equivalent. An important concept in their work is the *deviation* of a hypergraph. It is proved in [7, 8] that for fixed integers $\ell \geq k$, a k-graph of density 1/2 with small deviation contains asymptotically the same number of copies of the clique $K_\ell^{(k)}$ as the random hypergraph of the same density. This result can be viewed as a Counting Lemma for that notion of quasi-randomness.

This research was continued by Kohayakawa, Rödl, and Skokan [31] whose approach was based on the concept of *discrepancy* of a hypergraph. Discrepancy is more compatible with respect to the type of regularity a typical "block" exhibits in a partition obtained from the RS-Lemma. One particularly relevant result in [31] is a 'dense Counting Lemma' for hypergraphs with small discrepancy (cf. Theorem 4.1 in Section 4.1). Unfortunately, the counting needed to match the RS-Lemma deals with a 'sparse' and more difficult environment. However, the 'dense' ancestor of our result plays an important role in this dissertation.

Our attempt for proving the Counting Lemma (corresponding to the RS-Lemma) is to reduce, in an appropriate sense, the harder sparse case to the easier dense case. Our 'reduction' employs the RS-Lemma itself.

Finally, we discuss a way to bridge the methods of this dissertation with the RS-Lemma to produce a new variant of the regularity lemma for k-uniform hypergraphs in Chapter 10. For a more thorough discussion we refer to [44].

Chapter 3

The Counting Lemma

3.1 Basic notation

We denote by $[\ell]$ the set $\{1, \ldots, \ell\}$. For a set V and an integer $k \geq 1$, let $\binom{V}{k}$ be the set of all k-element subsets of V. A subset $\mathcal{G}^{(k)} \subseteq \binom{V}{k}$ is a *k-uniform hypergraph* on the vertex set V. We identify hypergraphs with their edge sets. For a given k-uniform hypergraph $\mathcal{G}^{(k)}$, we denote by $V\left(\mathcal{G}^{(k)}\right)$ and $E\left(\mathcal{G}^{(k)}\right)$ its vertex and edge set, respectively. For $U \subseteq V\left(\mathcal{G}^{(k)}\right)$, we denote by $\mathcal{G}^{(k)}[U]$ the subhypergraph of $\mathcal{G}^{(k)}$ induced on U (i.e. $\mathcal{G}^{(k)}[U] = \mathcal{G}^{(k)} \cap \binom{U}{k}$). A k-uniform *clique* of *order* j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \geq k$ vertices consisting of all $\binom{j}{k}$ many k-tuples (i.e., $K_j^{(k)}$ is isomorphic to $\binom{[j]}{k}$).

The central objects of Part I of this dissertation are ℓ-partite hypergraphs. The underlying vertex partition $V = V_1 \cup \cdots \cup V_\ell$, $|V_1| = \cdots = |V_\ell| = n$, is fixed. The vertex set itself can be seen as a 1-uniform hypergraph and, hence, we will frequently refer to the underlying fixed vertex set as $\mathcal{G}^{(1)}$. For integers $\ell \geq k \geq 1$ and vertex partition $V_1 \cup \cdots \cup V_\ell$, we denote by $K_\ell^{(k)}(V_1, \ldots, V_\ell)$ the *complete* ℓ-partite, k-uniform hypergraph (i.e. the family of all k-element subsets $K \subseteq \bigcup_{i \in [\ell]} V_i$ satisfying $|V_i \cap K| \leq 1$ for every $i \in [\ell]$). Then, an *(n, ℓ, k)-cylinder* $\mathcal{G}^{(k)}$ is any subset of $K_\ell^{(k)}(V_1, \ldots, V_\ell)$. Observe, that $\left|V\left(\mathcal{G}^{(k)}\right)\right| = \ell \times n$ for an (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$. Observe that the vertex partition $V_1 \cup \cdots \cup V_\ell$ is an $(n, \ell, 1)$-cylinder $\mathcal{G}^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $k \le j \le \ell$ and set $\Lambda_j \in \binom{[\ell]}{j}$, we denote by $\mathcal{G}^{(k)}[\Lambda_j] = \mathcal{G}^{(k)}\left[\bigcup_{\lambda \in \Lambda_j} V_\lambda\right]$ the subhypergraph of the (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ induced on $\bigcup_{\lambda \in \Lambda_j} V_\lambda$.

For an (n, ℓ, j)-cylinder $\mathcal{G}^{(j)}$ and an integer i, $j \le i \le \ell$, we denote by $\mathcal{K}_i^{(j)}\left(\mathcal{G}^{(j)}\right)$ the family of all i-element subsets of $V\left(\mathcal{G}^{(j)}\right)$ which span complete subhypergraphs in $\mathcal{G}^{(j)}$ of order i. Note that $\left|\mathcal{K}_i^{(j)}\left(\mathcal{G}^{(j)}\right)\right|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{G}^{(j)}$.

Given an $(n, \ell, j-1)$-cylinder $\mathcal{G}^{(j-1)}$ and an (n, ℓ, j)-cylinder $\mathcal{G}^{(j)}$, we say an edge J of $\mathcal{G}^{(j)}$ *belongs to* $\mathcal{G}^{(j-1)}$ if $J \in \mathcal{K}_j^{(j-1)}(\mathcal{G}^{(j-1)})$, i.e., J corresponds to a clique of order j in $\mathcal{G}^{(j-1)}$. Moreover, $\mathcal{G}^{(j-1)}$ *underlies* $\mathcal{G}^{(j)}$ if $\mathcal{G}^{(j)} \subseteq \mathcal{K}_j^{(j-1)}\left(\mathcal{G}^{(j-1)}\right)$, i.e., every edge of $\mathcal{G}^{(j)}$ belongs to $\mathcal{G}^{(j-1)}$. This brings us to one of the main concepts of this dissertation, the notion of a *complex*.

Definition 3.1 (**(n, ℓ, k)-complex**)**.** *Let $n \ge 1$ and $\ell \ge k \ge 1$ be integers. An (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ is a collection of (n, ℓ, j)-cylinders $\{\mathcal{G}^{(j)}\}_{j=1}^k$ such that*

- *(a) $\mathcal{G}^{(1)}$ is an $(n, \ell, 1)$-cylinder, i.e., $\mathcal{G}^{(1)} = V_1 \cup \cdots \cup V_\ell$ with $|V_i| = n$ for $i \in [\ell]$,*
- *(b) $\mathcal{G}^{(j-1)}$ underlies $\mathcal{G}^{(j)}$ for $2 \le j \le k$.*

3.2 Regular complexes

We begin with a notion of density of an (n, ℓ, j)-cylinder with respect to a family of $(n, \ell, j-1)$-cylinders.

Definition 3.2 (**density**)**.** *Let $\mathcal{G}^{(j)}$ be an (n, ℓ, j)-cylinder and let $\boldsymbol{\mathcal{Q}}^{(j-1)} = \{\mathcal{Q}_1^{(j-1)}, \ldots, \mathcal{Q}_r^{(j-1)}\}$ be a family of $(n, \ell, j-1)$-cylinders. We define the den-*

sity of $\mathcal{G}^{(j)}$ *w.r.t. the family* $\boldsymbol{\mathcal{Q}}^{(j-1)}$ *as*

$$d\left(\mathcal{G}^{(j)}\middle|\boldsymbol{\mathcal{Q}}^{(j-1)}\right)=\begin{cases}\frac{\left|\mathcal{G}^{(j)}\cap\bigcup_{s\in[r]}\mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|}{\left|\bigcup_{s\in[r]}\mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|} & \text{if} \quad \left|\bigcup_{s\in[r]}\mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|>0\\ 0 & \text{otherwise}.\end{cases}$$

We now define a notion of regularity of an (n,j,j)-cylinder with respect to an $(n,j,j-1)$-cylinder.

Definition 3.3. *Let positive reals* δ_j *and* d_j *and a positive integer* r *be given along with an* (n,j,j)*-cylinder* $\mathcal{G}^{(j)}$ *and an underlying* $(n,j,j-1)$*-cylinder* $\mathcal{G}^{(j-1)}$*. We say* $\mathcal{G}^{(j)}$ *is* (δ_j,d_j,r)*-regular* w.r.t. $\mathcal{G}^{(j-1)}$ *if whenever* $\boldsymbol{\mathcal{Q}}^{(j-1)}=\{\mathcal{Q}_1^{(j-1)},\dots,\mathcal{Q}_r^{(j-1)}\}$, $\mathcal{Q}_s^{(j-1)}\subseteq\mathcal{G}^{(j-1)}$, $s\in[r]$, *satisfies*

$$\left|\bigcup_{s\in[r]}\mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|\geq\delta_j\left|\mathcal{K}_j^{(j-1)}\left(\mathcal{G}^{(j-1)}\right)\right|,$$

then

$$d\left(\mathcal{G}^{(j)}\middle|\boldsymbol{\mathcal{Q}}^{(j-1)}\right)=d_j\pm\delta_j\,.$$

We now extend the notion of (δ_j,d_j,r)-regularity from (n,j,j)-cylinders to (n,ℓ,j)-cylinders $\mathcal{G}^{(j)}$.

Definition 3.4 ((δ_j,d_j,r)**-regular**)**.** *Let positive reals* δ_j *and* d_j *and a positive integer* r *be given along with an* (n,ℓ,j)*-cylinder* $\mathcal{G}^{(j)}$ *and an underlying* $(n,\ell,j-1)$*-cylinder* $\mathcal{G}^{(j-1)}$*. We say* $\mathcal{G}^{(j)}$ *is* (δ_j,d_j,r)*-regular* w.r.t. $\mathcal{G}^{(j-1)}$ *if for every* $\Lambda_j\in\binom{[\ell]}{j}$ *the restriction* $\mathcal{G}^{(j)}[\Lambda_j]=\mathcal{G}^{(j)}\left[\bigcup_{\lambda\in\Lambda_j}V_\lambda\right]$ *is* (δ_j,d_j,r)*-regular w.r.t. to the restriction* $\mathcal{G}^{(j-1)}[\Lambda_j]=\mathcal{G}^{(j-1)}\left[\bigcup_{\lambda\in\Lambda_j}V_\lambda\right]$.

We sometimes write (δ_j,r)-regular *to mean* $\left(\delta_j,d\left(\mathcal{G}^{(j)}\middle|\mathcal{G}^{(j-1)}\right),r\right)$*-regular for cylinders* $\mathcal{G}^{(j)}$ *and* $\mathcal{G}^{(j-1)}$.

Finally, we close this section of basic definitions with the central notion of a regular complex.

Definition 3.5 (**($\boldsymbol{\delta}, \boldsymbol{d}, r$)-regular complex**)**.** *Let vectors* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ *and* $\boldsymbol{d} = (d_2, \dots, d_k)$ *of positive reals be given and let* r *be a positive integer. We say an* (n, ℓ, k)*-complex* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ *is* $(\boldsymbol{\delta}, \boldsymbol{d}, r)$*-regular if:*

(*a*) $\mathcal{G}^{(2)}$ *is* $(\delta_2, d_2, 1)$*-regular w.r.t.* $\mathcal{G}^{(1)}$ *and*

(*b*) $\mathcal{G}^{(j)}$ *is* (δ_j, d_j, r)*-regular w.r.t.* $\mathcal{G}^{(j-1)}$ *for* $3 \leq j \leq k$*.*

3.3 Statement of the Counting Lemma

The following assertion is the main theorem of this dissertation.

Theorem 3.6 (Counting Lemma)**.** *For all integers* $2 \leq k \leq \ell$ *the following is true:* $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ *and there are integers* r *and* n_0 *so that, with* $\boldsymbol{d} = (d_2, \dots, d_k)$ *and* $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ *and* $n \geq n_0$*, whenever* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ *is a* $(\boldsymbol{\delta}, \boldsymbol{d}, r)$*-regular* (n, ℓ, k)*-complex, then*

$$\left|\mathcal{K}_{\ell}^{(k)}\big(\mathcal{G}^{(k)}\big)\right| = (1 \pm \gamma) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^{\ell}\,.$$

For given integers k and ℓ we shall refer to this theorem by $\mathbf{CL}_{k,\ell}$.

Observe from the quantification $\forall \gamma, d_k\ \exists \delta_k\ \forall d_{k-1}\ \exists \delta_{k-1}\ \dots\ \forall d_2\ \exists \delta_2$, the constants of Theorem 3.6 can satisfy $\delta_h \gg d_{h-1}$ for any $3 \leq h \leq k$. In particular, the hypothesis of Theorem 3.6 allows for the possibility that

$$\gamma, d_k \gg \delta_k \gg d_{k-1} \gg \dots \gg d_h \gg \delta_h \gg d_{h-1} \gg \dots \gg d_2 \gg \delta_2. \tag{3.1}$$

Consequently, the Counting Lemma includes the case when complexes $\boldsymbol{\mathcal{G}}$ consists of fairly sparse hypergraphs. It seems that this is the main difficulty in proving Theorem 3.6.

3.4 Generalizations of the Counting Lemma

The main result of Part I of this dissertation, Theorem 3.6, allows us to count complete hypergraphs of fixed order within a sufficiently regular complex. For some applications, it is more useful to consider slightly more general lemmas.

The first generalization enables us to estimate the number of copies of an arbitrary hypergraph $\mathcal{F}^{(k)}$ with vertices $\{1, \ldots, \ell\}$ in an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ satisfying that $\mathcal{G}^{(j)}[\Lambda_j]$ is regular w.r.t. $\mathcal{G}^{(j-1)}[\Lambda_j]$ whenever $\Lambda_j \subseteq K$ for some edge K of $\mathcal{F}^{(k)}$. Rather than counting copies of $K_\ell^{(k)}$ in an "everywhere" regular complex, this lemma counts copies of $\mathcal{F}^{(k)}$ in the complex $\boldsymbol{\mathcal{G}}$ satisfying the less restrictive assumptions above. We introduce some more notation before we give the precise statement below (see Corollary 3.9).

For a fixed k-uniform hypergraph $\mathcal{F}^{(k)}$, we define the j-th shadow for $j \in [k]$ by

$$\Delta_j(\mathcal{F}^{(k)}) = \{J\colon\ J \subseteq K \text{ for some } K \in \mathcal{F}^{(k)} \text{ and } |J| = j\}\,.$$

We extend the notion of a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex to $(\boldsymbol{\delta}, \geq\!\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex.

Definition 3.7 **($(\boldsymbol{\delta}, \geq \boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex).** *Let $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$ and $\boldsymbol{d} = (d_2, \ldots, d_k)$ be vectors of positive reals and let r be a positive integer. Let $\mathcal{F}^{(k)}$ be a k-uniform hypergraph on ℓ vertices $\{1, \ldots, \ell\}$. We say an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ with $\mathcal{G}^{(1)} = V_1 \cup \cdots \cup V_\ell$ is $\big(\boldsymbol{\delta}, \geq\!\boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular if:*

(a) for every $\Lambda_2 \in \Delta_2(\mathcal{F}^{(k)})$, the $(n, 2, 2)$-cylinder $\mathcal{G}^{(2)}[\Lambda_2]$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{G}^{(1)}[\Lambda_2]$,

(b) for every $\Lambda_j \in \Delta_j(\mathcal{F}^{(k)})$, the (n, j, j)-cylinder $\mathcal{G}^{(j)}[\Lambda_j]$ is (δ_j, d_j, r)-regular w.r.t. $\mathcal{G}^{(j-1)}$ for $3 \leq j < k$, and

(*c*) *for every* $\Lambda_k \in \mathcal{F}^{(k)}$, *the* (n,k,k)-*cylinder* $\mathcal{G}^{(k)}[\Lambda_k]$ *is* $(\delta_k, d_{\Lambda_k}, r)$-*regular w.r.t.* $\mathcal{G}^{(k-1)}$ *with* $d_{\Lambda_k} \geq d_k$.

The '$\geq$' in a $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex indicates that we only enforce a lower bound on the densities in the k-th layer of $\boldsymbol{\mathcal{G}}$ (cf. part (*c*) of the definition). This is the environment which usually appears in applications. We also observe that the Definition 3.7 imposes only a regular structure on those (m,k,k)-subcomplexes of $\boldsymbol{\mathcal{G}}$ which naturally correspond to edges of $\mathcal{F}^{(k)}$ (i.e., on a subcomplex induced on $V_{\lambda_1}, \dots, V_{\lambda_k}$, where $\{\lambda_1, \dots, \lambda_k\}$ forms an edge in $\mathcal{F}^{(k)}$). We need one more definition before we can state the corollary.

Definition 3.8 (**partite isomorphic**). *Suppose* $\mathcal{F}^{(k)}$ *is a* k-*uniform hypergraph with* $V(\mathcal{F}^{(k)}) = [\ell]$ *and* $\mathcal{G}^{(k)}$ *is an* (n,ℓ,k)-*cylinder with vertex partition* $V(\mathcal{G}^{(k)}) = V_1 \cup \dots \cup V_\ell$. *We say a copy* $\mathcal{F}_0^{(k)}$ *of* $\mathcal{F}^{(k)}$ *in* $\mathcal{G}^{(k)}$ *is partite isomorphic to* $\mathcal{F}^{(k)}$ *if there is a labeling of* $V(\mathcal{F}_0^{(k)}) = \{v_1, \dots, v_\ell\}$ *such that*

(*i*) $v_\alpha \in V_\alpha$ *for every* $\alpha \in [\ell]$, *and*

(*ii*) $v_\alpha \mapsto \alpha$ *is a hypergraph isomorphism (edge preserving bijection of the vertex sets) between* $\mathcal{F}_0^{(k)}$ *and* $\mathcal{F}^{(k)}$.

Corollary 3.9. *For all integers* $2 \leq k \leq \ell$ *and* $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \dots\ \forall d_2 > 0\ \exists \delta_2 > 0$ *and there are integers* r *and* n_0 *so that the following holds for* $\boldsymbol{d} = (d_2, \dots, d_k)$, $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$, *and* $n \geq n_0$. *If* $\mathcal{F}^{(k)}$ *is a* k-*uniform hypergraph on* ℓ-*vertices and* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ *is a* $\big(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-*regular* (n,ℓ,k)-*complex with* $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$, *then the number of partite isomorphic copies of* $\mathcal{F}^{(k)}$ *in* $\mathcal{G}^{(k)}$ *is at least*

$$(1-\gamma) \prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} d_{\Lambda_k} \times n^\ell \geq (1-\gamma) \prod_{h=2}^{k} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times n^\ell \,.$$

Corollary 3.9 can be derived from Theorem 3.6 and we present the technical details in Section 9.1. Below we briefly outline that proof.

The idea of the proof consists of two basic parts. For $2 \le j \le k$, for each $\Lambda_j = \{\lambda_1, \dots, \lambda_j\} \notin \Delta_j(\mathcal{F}^{(k)})$, we replace the (n, j, j)-cylinder $\mathcal{G}^{(j)}[\Lambda_j]$ with the complete j-partite j-uniform system $K_j^{(j)}(V_{\lambda_1}, \dots, V_{\lambda_j})$. Doing so over all $2 \le j \le k$ and all $\Lambda_j \notin \Delta_j(\mathcal{F}^{(k)})$ clearly results in an "everywhere" regular complex, let us call it $\boldsymbol{\mathcal{H}}$, whose cliques $K_\ell^{(k)}$ correspond to copies of $\mathcal{F}^{(k)}$ in $\boldsymbol{\mathcal{G}}$.

One now wishes to apply the Counting Lemma, Theorem 3.6, to the complex $\boldsymbol{\mathcal{H}}$ to finish the job. The only minor technicality in doing so is that, unlike the hypothesis of Theorem 3.6, the complex $\boldsymbol{\mathcal{H}}$ potentially has, for each $2 \le j \le k$, (n, j, j)-cylinders $\mathcal{H}^{(j)}[\Lambda_j]$, $\Lambda_j \in \binom{[\ell]}{j}$, of differing densities. This is handled, however, by "randomly slicing" the (n, j, j)-cylinders $\mathcal{H}^{(j)}[\Lambda_j]$, $\Lambda_j \in \binom{[\ell]}{j}$, into appropriately many pieces of the same density as formally required in Theorem 3.6. Consequently, we create a series of pairwise $K_\ell^{(k)}$-disjoint complexes $\boldsymbol{\mathcal{H}}_1, \boldsymbol{\mathcal{H}}_2, \dots$, each of which satisfies the hypothesis of the Counting Lemma. Theorem 3.6 applies to each of the newly created complexes $\boldsymbol{\mathcal{H}}_i$, $i \ge 1$, and so we add the resulting number of cliques to finish the proof of Corollary 3.9.

Later in Chapter 11 it will be convenient for us to consider the following (slightly weakened) rephrasing of Corollary 3.9.

Corollary 3.9′. *For all integers $2 \le k \le \ell$ and every positive real γ there are functions*

$$\delta_j(D_j, \dots, D_k) \quad \textit{for} \quad j = 2, \dots, k\,,$$
$$r(D_2, \dots, D_k)\,, \quad \textit{and} \quad n_0(D_2, \dots, D_k)$$

in variables $D_2, \dots, D_k$ so that for every $\boldsymbol{d} = (d_2, \dots, d_k) \in (0,1]^{k-1}$ and $n \ge n_0(\boldsymbol{d})$ the following holds for $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ with $\delta_j = \delta_j(d_j, \dots, d_k)$. If $\mathcal{F}^{(k)}$ is a k-uniform ℓ-vertex hypergraph and $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ is a $\big(\boldsymbol{\delta}, \ge$

$\boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$*-regular* (n, ℓ, k)*-complex, then the number of copies of* $\mathcal{F}^{(k)}$ *in* $\mathcal{G}^{(k)}$ *is at least*

$$(1-\gamma)\prod_{h=2}^{k} d_h^{\binom{\ell}{k}} \times n^{\ell}\,.$$

The second extension of the Counting Lemma allows us to estimate the number of "non-crossing" copies of a fixed hypergraph $\mathcal{F}^{(k)}$. For that we recall the notion of a homomorphic image of a hypergraph.

Definition 3.10 (**hypergraph homomorphism**). *Suppose* $\mathcal{F}^{(k)}$ *and* $\tilde{\mathcal{F}}^{(k)}$ *are* k*-uniform hypergraphs. We say* $\tilde{\mathcal{F}}^{(k)}$ *is an homomorphic image of* $\mathcal{F}^{(k)}$ *if there exist a surjective map* $\vartheta\colon V(\mathcal{F}^{(k)}) \twoheadrightarrow V(\tilde{\mathcal{F}}^{(k)})$ *such that for every edge* $K \in E(\tilde{\mathcal{F}}^{(k)})$ *we have* $\vartheta(K) = \bigcup_{v\in K}\vartheta(v) \in E(\tilde{\mathcal{F}}^{(k)})$*. We say* ϑ *is a homomorphism from* $\mathcal{F}^{(k)}$ *to* $\tilde{\mathcal{F}}^{(k)}$*.*

In other words, ϑ *is a homomorphism from* $\mathcal{F}^{(k)}$ *to* $\tilde{\mathcal{F}}^{(k)}$ *if it is an edge-preserving map between the vertex sets of* $\mathcal{F}^{(k)}$ *and* $\tilde{\mathcal{F}}^{(k)}$*.*

Definition 3.11 (**$\boldsymbol{\vartheta}$-partite isomorphic**). *Suppose* $\mathcal{F}^{(k)}$ *is a* k*-uniform hypergraph with* $V(\mathcal{F}^{(k)}) = [\ell]$*,* $\tilde{\mathcal{F}}^{(k)}$ *with* $V(\tilde{\mathcal{F}}^{(k)}) = [\tilde{\ell}]$ *is a homomorphic image under* $\vartheta\colon\ [\ell] \twoheadrightarrow [\tilde{\ell}]$ *and* $\tilde{\mathcal{G}}^{(k)}$ *is an* $(n, \tilde{\ell}, k)$*-cylinder with vertex partition* $V(\tilde{\mathcal{G}}^{(k)}) = \tilde{V}_1 \cup \dots \cup \tilde{V}_{\tilde{\ell}}$*. We say a copy* $\mathcal{F}_0^{(k)}$ *of* $\mathcal{F}^{(k)}$ *in* $\tilde{\mathcal{G}}^{(k)}$ *is* ϑ*-partite isomorphic to* $\mathcal{F}^{(k)}$ *if there is a labeling of* $V(\mathcal{F}_0^{(k)}) = \{v_1, \dots, v_\ell\}$ *such that*

- (i) $v_\alpha \in \tilde{V}_{\vartheta(\alpha)}$ *for every* $\alpha \in [\ell]$*, and*
- (ii) $v_\alpha \mapsto \alpha$ *is a hypergraph isomorphism between* $\mathcal{F}_0^{(k)}$ *and* $\mathcal{F}^{(k)}$*.*

We now state the second extension of Therorem 3.6 considered here.

Corollary 3.12. *For all integers* $2 \le k \le \ell$ *and* $\forall\gamma > 0$ $\forall d_k > 0$ $\exists\delta_k > 0$ $\forall d_{k-1} > 0$ $\exists\delta_{k-1} > 0$ $\dots$ $\forall d_2 > 0$ $\exists\delta_2 > 0$ *and there are integers* r *and* n_0 *so that the following holds for* $\boldsymbol{d} = (d_2, \dots, d_k)$ *and* $\boldsymbol{\delta} = \big(\delta_2, \dots, \delta_k\big)$ *and* $n \ge n_0$*.*

Suppose $\mathcal{F}^{(k)}$ is a k-uniform ℓ-vertex hypergraph and $\tilde{\mathcal{F}}^{(k)}$ is a homomorphic image with $\big|V\big(\tilde{\mathcal{F}}^{(k)}\big)\big| = \tilde{\ell}$ under the homomorphism ϑ. If $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^{k}$ is a $\big(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \tilde{\mathcal{F}}^{(k)}\big)$-regular $(n, \tilde{\ell}, k)$-complex with $\tilde{\mathcal{G}}^{(1)} = \tilde{V}_1 \cup \cdots \cup \tilde{V}_{\tilde{\ell}}$, then the number of ϑ-partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$ is at least

$$(1-\gamma) \prod_{\beta \in [\tilde{\ell}]} \frac{1}{|\vartheta^{-1}(\beta)|!} \times \prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} d_{\Lambda_k} \times n^{\ell} \times$$
$$\geq (1-\gamma) \prod_{\beta \in [\tilde{\ell}]} \frac{1}{|\vartheta^{-1}(\beta)|!} \times \prod_{h=2}^{k} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times n^{\ell}\,.$$

Corollary 3.12 easily follows from Corollary 3.9. However, the proof is somewhat technical and we defer it to Section 9.2.

The following is a (slightly weakened) rephrasing of Corollary 3.12.

Corollary 3.12′. *For all integers $2 \leq k \leq \ell$ and every positive real γ there are functions*

$$\delta_j(D_j, \dots, D_k) \quad \textit{for} \quad j = 2, \dots, k\,,$$
$$r(D_2, \dots, D_k)\,, \quad \textit{and} \quad n_0(D_2, \dots, D_k)$$

in variables $D_2, \dots, D_k$ so that for every $\boldsymbol{d} = (d_2, \dots, d_k) \in (0,1]^{k-1}$ and $n \geq n_0(\boldsymbol{d})$ the following holds for $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ with $\delta_j = \delta_j(d_j, \dots, d_k)$.

Suppose $\mathcal{F}^{(k)}$ is a k-uniform ℓ-vertex hypergraph and $\tilde{\mathcal{F}}^{(k)}$ is a homomorphic image with $\big|V\big(\tilde{\mathcal{F}}^{(k)}\big)\big| = \tilde{\ell}$. If $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^{k}$ is a $\big(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \tilde{\mathcal{F}}^{(k)}\big)$-regular $(n, \tilde{\ell}, k)$-complex, then the number of copies of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$ is at least

$$\frac{(1-\gamma)}{\ell!} \prod_{h=2}^{k} d_h^{\binom{\ell}{k}} \times n^{\ell}\,.$$

Chapter 4

Auxiliary results

In this chapter we review a few results that are essential for our proof of Theorem 3.6 in Chapter 5.

4.1 The Dense Counting Lemma

We recall that Theorem 3.6 is formulated under the involved quantification $\forall d_k\ \exists \delta_k\ \forall d_{k-1}\ \exists \delta_{k-1} \ldots \forall d_2\ \exists \delta_2$ and that the Counting Lemma owes its difficulty in proof to the sparseness arising from this quantification. If the quantification can be simplified so that

$$\min_{2\le j\le k} d_j \gg \max_{2\le j\le k} \delta_j \tag{4.1}$$

is ensured, then the so-called Dense Counting Lemma (see Theorem 4.1 below) is known to be true. This was proved by Kohayakawa, Rödl, and Skokan (see Theorem 6.5 in [31]). Observe that (4.1) represents the 'dense case' in contrast to the 'sparse case' (3.1), since all densities are bigger then the measure of regularity $\max \delta_j$.

Theorem 4.1 (Dense Counting Lemma)**.** *For all integers $2 \le k \le \ell$ and any positive constants $d_2, \ldots, d_k$, there exist $\varepsilon > 0$ and integer m_0 so that, with $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $\boldsymbol{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^{k-1}$ and $m \ge m_0$, whenever $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ is a $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$-regular (m, ℓ, k)-complex, then*

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big)\right| = (1 \pm g_{k,\ell}(\varepsilon)) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times m^\ell$$

where $g_{k,\ell}(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

While the quantification of the main Theorem, Theorem 3.6, does not allow us to assume (4.1), Peng, Rödl, and Skokan in [40] used Theorem 4.1 to prove Theorem 3.6 for $k = 3$ by reducing the harder 'sparse case' to the easier 'dense case'. This is also the idea of our current proof. The reduction scheme used here, which is entirely different, is somewhat simpler and allows an extension for arbitrary k.

4.2 The Regularity Lemma

One of the major tools we use in our proof of Theorem 3.6 is the recently developed regularity lemma of Rödl and Skokan [48] for k-uniform hypergraphs. Our plan is to apply the regularity lemma to the (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ in the (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ from the hypothesis of Theorem 3.6. Since $\mathcal{G}^{(k)}$ is ℓ-partite with ℓ-partition $\mathcal{G}^{(1)} = V_1 \cup \ldots \cup V_\ell$, the regularity lemma below is formulated for ℓ-partite hypergraphs.

The regularity lemma of Rödl and Skokan provides well-structured partitions of all complete (n, ℓ, j)-cylinders $K_\ell^{(j)}(V_1, \ldots, V_\ell)$ for $j \in [k-1]$. We later refer to the family of these partitions by $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{k-1})$ is a family of functions which describes the partitions of $\mathscr{R}$ and $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ describes the image sets of $\boldsymbol{\varphi}$. In what follows, we use the language of [48] to give the precise definitions of these concepts.

4.2.1 Partitions

Let $V_1 \cup \cdots \cup V_\ell$ be a partition of V with $|V_\lambda| = n$ for every $\lambda \in [\ell]$. Let k be an integer and for every $j \in [k-1]$, let $a_j \in \mathbb{N}$ and let φ_j be a function such

that
$$\varphi_j\colon\ K_\ell^{(j)}(V_1,\dots,V_\ell)\to[a_j]\,.$$
Note, for every $\lambda\in[\ell]$, mapping φ_1 defines a partition $V_\lambda=V_{\lambda,1}\cup\dots\cup V_{\lambda,a_1}$, where $V_{\lambda,\alpha}=\varphi_1^{-1}(\alpha)\cap V_\lambda$ for all $\alpha\in[a_1]$. Here, we only consider functions φ_1 such that
$$\big||\varphi_1^{-1}(\alpha)\cap V_\lambda|-|\varphi_1^{-1}(\alpha')\cap V_\lambda|\big|=\big||V_{\lambda,\alpha}|-|V_{\lambda,\alpha'}|\big|\le 1 \tag{4.2}$$
for every $\lambda\in[\ell]$ and α, $\alpha'\in[a_1]$. Consequently, we have $\lfloor n/a_1\rfloor\le|V_{\lambda,\alpha}|\le\lceil n/a_1\rceil$.

Remark 4.2. *For convenience, we delete all floors and ceilings and simply write $|V_{\lambda,\alpha}|=n/a_1$ for every $\lambda\in[\ell]$ and $\alpha\in[a_1]$.*

Let $\binom{[\ell]}{j}_<=\{(\lambda_1,\dots,\lambda_j)\in[\ell]^j\colon\ \lambda_1<\dots<\lambda_j\}$ be the set of vectors that naturally correspond to the totally ordered j-element subsets of $[\ell]$. More generally, for a totally ordered set Π of cardinality at least j, let $\binom{\Pi}{j}_<$ be the family of totally ordered j-element subsets of Π. For $j\in[k-1]$ we consider the projection π_j of $K_\ell^{(j)}(V_1,\dots,V_\ell)$ onto $[\ell]$;
$$\pi_j\colon\ K_\ell^{(j)}(V_1,\dots,V_\ell)\to\binom{[\ell]}{j}_<\,,$$
mapping every $J\in K_\ell^{(j)}(V_1,\dots,V_\ell)$ to the totally ordered set $\pi_j(J)=(\lambda_1,\dots,\lambda_j)\in\binom{[\ell]}{j}_<$ satisfying $|J\cap V_{\lambda_h}|=1$ for every $h\in[j]$. Moreover, for every $h\in[|J|]$, let $\Phi_h(J)=(x_{\pi_h(H)}=\varphi_h(H))_{H\in\binom{J}{h}}$. In other words, $\Phi_h(J)$ is a vector of length $\binom{|J|}{h}$ and its entries are indexed by elements from $\binom{\pi_j(J)}{h}_<$. For our purposes it will be convenient to assume that the entries of $\Phi_h(J)$ are ordered lexicographically w.r.t. their indices. Observe that for $h>0$
$$\Phi_h(J)\in[a_h]\times\dots\times[a_h]=[a_h]^{\binom{j}{h}}\,.$$
We define
$$\mathbf{\Phi}^{(j)}(J)=(\pi_j(J),\Phi_1(J),\dots,\Phi_j(J)).$$

Note that $\mathbf{\Phi}^{(j)}(J)$ is a vector with $j+2^j-1$ entries. Observe that if we set $\boldsymbol{a}=(a_1,a_2,\dots,a_{k-1})$ and

$$A(j,\boldsymbol{a})=\binom{[\ell]}{j}_{<}\times\prod_{h=1}^{j}[a_h]^{\binom{j}{h}},$$

then $\mathbf{\Phi}^{(j)}(J)\in A(j,\boldsymbol{a})$ for every set $J\in K_\ell^{(j)}(V_1,\dots,V_\ell)$. In other words, to each edge J of cardinality j we assign $\pi_j(J)$ and a vector $(x_{\pi_h(H)})_{H\subset J}$ with each entry $x_{\pi_h(H)}$ corresponding to a non-empty subset H of J such that $x_{\pi_h(H)}=\varphi_h(H)$, where $h=|H|$.

For two edges $J_1,J_2\in K_\ell^{(j)}(V_1,\dots,V_\ell)$, the equality $\mathbf{\Phi}^{(j)}(J_1)=\mathbf{\Phi}^{(j)}(J_2)$ defines an equivalence relation on $K_\ell^{(j)}(V_1,\dots,V_\ell)$ into at most

$$|A(j,\boldsymbol{a})|\leq\binom{\ell}{j}\times\prod_{h=1}^{j}a_h^{\binom{j}{h}}\tag{4.3}$$

parts. Now we describe these parts explicitly using $(j+2^j-1)$-dimensional vectors from $A(j,\boldsymbol{a})$.

For each $j<k$ we define a partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1,\dots,V_\ell)$ with partition classes corresponding to the equivalence relation defined above. This way each partition class in $\mathscr{R}^{(j)}$ has a unique address $\boldsymbol{x}^{(j)}\in A(j,\boldsymbol{a})$. While $\boldsymbol{x}^{(j)}$ is a $(j+2^j-1)$-dimensional vector, we will frequently view it as a $j+1$-dimensional vector $(\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_j)$, where $\boldsymbol{x}_0=(x_1,\dots,x_j)\in\binom{[\ell]}{j}_<$ is a totally ordered set and $\boldsymbol{x}_h=(x_\Xi)_{\Xi\in\binom{\boldsymbol{x}_0}{h}_<}\in[a_h]^{\binom{j}{h}}$ for $1\leq h\leq j$. For each address $\boldsymbol{x}^{(j)}\in A(j,\boldsymbol{a})$ we denote its corresponding partition class from $\mathscr{R}^{(j)}$ by

$$\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})=\left\{R\in K_\ell^{(j)}(V_1,\dots,V_\ell)\colon\ \mathbf{\Phi}^{(j)}(R)=\boldsymbol{x}^{(j)}\right\}.$$

This way we ensure some structure between the classes from $\mathscr{R}^{(j)}$ and $\mathscr{R}^{(j-1)}$. More formally, for each partition class $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})\in\mathscr{R}^{(j)}$ there exist j partition classes $\mathcal{R}_1^{(j-1)},\dots,\mathcal{R}_j^{(j-1)}\in\mathscr{R}^{(j-1)}$ such that for $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})=\bigcup_{h\in[j]}\mathcal{R}_h^{(j-1)}$ we have $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})\subseteq\mathcal{K}_j^{(j-1)}\big(\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})\big)$. In other words $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ forms an underlying $(j-1)$-uniform j-partite hypergraph of $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)})$ consisting of $\binom{j}{j-1}$ classes from $\mathscr{R}^{(j-1)}$. Given $\boldsymbol{x}^{(j)} \in A(j,\boldsymbol{a})$ (and the corresponding $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \in \mathscr{R}^{(j)}$) we give a formal definition of $\mathcal{R}^{(j-1)}(\boldsymbol{x}^{(j)})$ below. In fact, for $h < j$ we introduce a notation for the h-uniform j-partite hypergraph $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ which consists of $\binom{j}{h}$ partition classes of $\mathscr{R}^{(h)}$ and satisfies $\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \subseteq \mathcal{K}_j^{(h)}\big(\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})\big)$.

To that end, we need the following notation. Let $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_0,\dots,\boldsymbol{x}_j) \in A(j,\boldsymbol{a})$ with $\boldsymbol{x}_u = (x_\Upsilon)_{\Upsilon \in \binom{\boldsymbol{x}_0}{u}_<} \in [a_u]^{\binom{j}{u}}$ for $1 \le u \le j$. Given an ordered subset $\Xi \subseteq \boldsymbol{x}_0$ (recall $\boldsymbol{x}_0 \in \binom{[\ell]}{j}_<$) where $|\Xi| = h \le j$. For $1 \le u \le h = |\Xi| \le j$ let

$$\boldsymbol{x}_u^{\Xi} = (x_\Upsilon)_{\Upsilon \in \binom{\Xi}{u}_<} .$$

be the $\binom{h}{u}$-dimensional vector consisting of those entries of $\boldsymbol{x}_u$ which are labeled with the ordered u-element subsets of Ξ. For every vector $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_j) \in A(j,\boldsymbol{a})$ and for each $h \in [j]$, we then set

$$\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)}) = \bigcup_{\Xi \in \binom{\boldsymbol{x}_0}{h}_<} \Big\{ R \in K_\ell^{(h)}(V_1,\dots,V_\ell) \colon \boldsymbol{\Phi}^{(h)}(R) = (\Xi, \boldsymbol{x}_1^{\Xi},\dots,\boldsymbol{x}_h^{\Xi}) \Big\}. \tag{4.4}$$

Using the language above, the following claim holds.

Claim 4.3. *For every $j \in [k-1]$ and every $\boldsymbol{x}^{(j)} = (\boldsymbol{x}_0,\boldsymbol{x}_1,\dots,\boldsymbol{x}_j) \in A(j,\boldsymbol{a})$, the following is true.*

(a) For all $h \in [j]$, $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ is an $(n/a_1, j, h)$-cylinder;

(b) $\boldsymbol{\mathcal{R}}(\boldsymbol{x}^{(j)}) = \big\{\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})\big\}_{h=1}^{j}$ is an $(n/a_1, j, j)$-complex.

To give a precise description of the family of partitions of $K_\ell^{(j)}(V_1,\dots,V_\ell)$, we summarize the notation above in the following Setup in which we work.

Setup 4.4. *Let $k \le \ell$ and n be fixed positive integers, let $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ be a $(n,\ell,1)$-cylinder, and $\boldsymbol{a} = \boldsymbol{a}_{\mathscr{R}} = (a_1, a_2, \dots, a_{k-1})$ be a vector of positive* *integers. Let*

$$A(j, \boldsymbol{a}) = \binom{[\ell]}{j}_{<} \times \prod_{h=1}^{j} [a_h]^{\binom{j}{h}},$$

and for every $j \in [k-1]$ let $\varphi_j \colon K_\ell^{(j)}(V_1, \dots, V_\ell) \to [a_j]$ be a mapping . Moreover, suppose that φ_1 satisfies (4.2) for every $\lambda \in [\ell]$ and α, $\alpha' \in [a_1]$. Set $\boldsymbol{\varphi} = \{\varphi_j \colon j \in [k-1]\}$.

We now define the family of partitions of $K_\ell^{(j)}(V_1, \dots, V_\ell)$.

Definition 4.5 (Partition). *Given Setup 4.4, for every $j \in [k-1]$, we define a partition $\mathscr{R}^{(j)}$ of $K_\ell^{(j)}(V_1, \dots, V_\ell)$ by*

$$\mathscr{R}^{(j)} = \left\{\mathcal{R}^{(j)}(\boldsymbol{x}^{(j)}) \colon \boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})\right\}.$$

We also define the family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ and the rank of $\mathscr{R}$ by

$$\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})|.$$

4.2.2 Polyads

The ε-regular pair played a central role in the definition of a regular partition for graphs (cf. Szemerédi's regularity lemma, Theorem 2.1). In order to define a regular partition $\mathscr{R}$ for a k-uniform hypergraph, this concept was extended in [48] by introducing *polyads*. Given Setup 4.4, let $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be the family of partitions as defined in Definition 4.5. Polyads are $(n/a_1, j, j-1)$-cylinders consisting of selected j members of $\mathscr{R}^{(j)}$ for $j = 2, \dots, k$. The precise definition of a polyad (which we give below) requires some notation.

Recall that for each edge $J \in K_\ell^{(j)}(V_1, \dots, V_\ell)$ and $h \in [j-1]$, we defined $\Phi_h(J)$ as the $\binom{j}{h}$-dimensional vector $\Phi_h(J) = (\varphi_h(H))_{H \in \binom{J}{h}}$. We also defined $\pi_j(J)$ to be the totally ordered set $(\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_{<}$ such that $|J \cap V_{\lambda_h}| = 1$ for every $h \in [j]$. We set

$$\hat{\boldsymbol{\Phi}}^{(j-1)}(J) = (\pi_j(J), \Phi_1(J), \dots, \Phi_{j-1}(J)).$$

Note that $\hat{\mathbf{\Phi}}^{(j-1)}(J)$ is just the projection of $\mathbf{\Phi}^{(j)}(J)$ onto its first j vector coordinates. As such, $\hat{\mathbf{\Phi}}^{(j-1)}(J)$ is a vector having $j+\sum_{h=1}^{j-1}\binom{j}{h} = j+2^j-2$ entries.

We define the set $\hat{A}(j-1,\boldsymbol{a})$ of $(j+2^j-2)$-dimensional vectors for $j\in[k-1]$ by

$$\hat{A}(j-1,\boldsymbol{a}) = \binom{[\ell]}{j}_< \times \prod_{h=1}^{j-1}[a_h]^{\binom{j}{h}}.$$

Observe that then $\hat{\mathbf{\Phi}}^{(j-1)}(J)\in\hat{A}(j-1,\boldsymbol{a})$ for every edge $J\in K_\ell^{(j)}(V_1,\dots,V_\ell)$.

Let $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{A}(j-1,\boldsymbol{a})$. We write the vector $\hat{\boldsymbol{x}}^{(j-1)}=(\hat{\boldsymbol{x}}_0,\hat{\boldsymbol{x}}_1,\dots,\hat{\boldsymbol{x}}_{j-1})$, where $\hat{\boldsymbol{x}}_0\in\binom{[\ell]}{j}_<$ is an ordered set and $\hat{\boldsymbol{x}}_u=(\hat{x}_\Upsilon)_{\Upsilon\in\binom{\hat{\boldsymbol{x}}_0}{u}_<}\in[a_u]^{\binom{j}{u}}$ for $1\le u\le j-1$. Given an ordered set $\Xi\subseteq\hat{\boldsymbol{x}}_0$ with $1\le h=|\Xi|<j$, we set for $1\le u\le h$

$$\hat{\boldsymbol{x}}_u^{\Xi} = (\hat{x}_\Upsilon)_{\Upsilon\in\binom{\Xi}{u}_<}. \tag{4.5}$$

For each $j\in[k]$, $\hat{\boldsymbol{x}}^{(j-1)}=(\hat{\boldsymbol{x}}_0,\hat{\boldsymbol{x}}_1,\dots,\hat{\boldsymbol{x}}_{j-1})\in\hat{A}(j-1,\boldsymbol{a})$, and $h\in[j-1]$ we define $\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})$ by

$$\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)}) = \bigcup_{\Xi\in\binom{\hat{\boldsymbol{x}}_0}{h}_<}\Big\{R\in K_\ell^{(h)}(V_1,\dots,V_\ell):\mathbf{\Phi}^{(h)}(R)=(\Xi,\hat{\boldsymbol{x}}_1^{\Xi},\dots,\hat{\boldsymbol{x}}_h^{\Xi})\Big\}. \tag{4.6}$$

Note that if $\boldsymbol{x}^{(j)}=\big((\hat{\boldsymbol{x}}^{(j-1)},\alpha)\big)$ for some $\alpha\in[a_j]$ and $1\le h<j$, then $\mathcal{R}^{(h)}(\boldsymbol{x}^{(j)})$ defined in (4.4) and $\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})$ defined in (4.6) are identical.

We also set $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(j-1)})=\big\{\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{h=1}^{j-1}$. Similarly to Claim 4.3, we can prove the following.

Claim 4.6. *For every vector $\hat{\boldsymbol{x}}^{(j-1)}=(\hat{\boldsymbol{x}}_0,\hat{\boldsymbol{x}}_1,\dots,\hat{\boldsymbol{x}}_{j-1})\in\hat{A}(j-1,\boldsymbol{a})$, the following statements are true.*

(a) For all $h\in[j-1]$, $\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})$ is an $(n/a_1,j,h)$-cylinder;

(b) $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(j-1)})=\big\{\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{h=1}^{j-1}$ is an $(n/a_1,j,j-1)$-complex.

In this dissertation, $(n/a_1, j, j-1)$-cylinders $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ will play a special role for $1 < j \le k$ and we will call them polyads.

Definition 4.7 (**Polyad**). *Given the Setup 4.4, let $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a family of partitions as defined in Definition 4.5. Then, for each $1 < j \le k$ and each vector $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$, we refer to the $(n/a_1, j, j-1)$-cylinder $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ as polyad.*

For every polyad $\hat{\mathcal{R}}^{(j-1)}$ there exists a unique vector $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$ such that $\hat{\mathcal{R}}^{(j-1)} = \hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Hence, each polyad $\hat{\mathcal{R}}^{(j-1)}$ uniquely defines an $(n/a_1, j, j-1)$-complex $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(j-1)}) = \left\{\hat{\mathcal{R}}^{(h)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{h=1}^{j-1}$ such that $\hat{\mathcal{R}}^{(j-1)} = \hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

Remark 4.8. *For $j = 2$ the set $\hat{A}(1, \boldsymbol{a})$ consists of 2-dimensional vectors $\hat{\boldsymbol{x}}^{(1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1)$, where $\hat{\boldsymbol{x}}_0 = (\lambda_1, \lambda_2) \in \binom{[\ell]}{2}_<$ and $\hat{\boldsymbol{x}}_1 = (\alpha_1, \alpha_2) \in [a_1]^2$. Consequently, a polyad $\hat{\mathcal{R}}^{(1)}(\hat{\boldsymbol{x}}^{(1)})$ is the bipartition $V_{\lambda_1,\alpha_1} \cup V_{\lambda_2,\alpha_2}$*

Every polyad $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ is an $(n/a_1, j, j-1)$-cylinder that is the union of j appropriately chosen partition classes of $\mathscr{R}^{(j-1)}$. We describe these elements using the vectors $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$.

Let $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{A}(j-1, \boldsymbol{a})$ be given. Then, for every $u \in [j-1]$, vector $\hat{\boldsymbol{x}}_u$ can be written as $\hat{\boldsymbol{x}}_u = (x_\Upsilon \colon \Upsilon \in \binom{\hat{\boldsymbol{x}}_0}{u}_<)$, i.e. its entries are labeled by the ordered u-element subsets of the ordered set $\hat{\boldsymbol{x}}_0 \in \binom{[\ell]}{j}_<$ in lexicographic order w.r.t. the indices. For every $\iota \in [j]$, we set

$$\partial_\iota \hat{\boldsymbol{x}}_u = \left(x_\Upsilon \colon \Upsilon \in \binom{\hat{\boldsymbol{x}}_0 \setminus \iota}{u}_<\right). \tag{4.7}$$

In other words, vector $\partial_\iota \hat{\boldsymbol{x}}_u$ contains precisely those entries of $\hat{\boldsymbol{x}}_u$ which are labeled by the u-element subsets of $\hat{\boldsymbol{x}}_0$ not containing ι. Note that, in view of (4.5), $\partial_\iota \hat{\boldsymbol{x}}_u = \hat{\boldsymbol{x}}_u^{\Xi}$ with $\Xi = \hat{\boldsymbol{x}}_0 \setminus \iota$. Clearly, $\partial_\iota \hat{\boldsymbol{x}}_u$ has $\binom{j-1}{u}$ entries from $[a_u]$. Furthermore, we set

$$\partial_\iota \hat{\boldsymbol{x}}^{(j-1)} = (\partial_\iota \hat{\boldsymbol{x}}_1, \partial_\iota \hat{\boldsymbol{x}}_2, \dots, \partial_\iota \hat{\boldsymbol{x}}_{j-1})$$

and observe that $\partial_\iota \hat{\boldsymbol{x}}^{(j-1)}$ is a $(j-1+2^{j-1}-1)$-dimensional vector belonging to $A(j-1, \boldsymbol{a})$.

4.2.3 Regular Partitions and the Regularity Lemma

The regularity lemma of Rödl and Skokan provides a family of partitions $\mathscr{R}$ with nice certain properties. Loosely speaking, for "most" $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$ the $(n/a_1, k, k-1)$-complex $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(k-1)})$ is regular (cf. Definition 3.5).

In the two definitions, below we introduce two concepts central to regularity. We use the notation δ', d' and r' to be consistent with the context in which we apply the Regularity Lemma (Theorem 4.11 below).

Definition 4.9 (**$(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable**). *Let μ be a number in the interval $(0, 1]$, let $\boldsymbol{\delta'} = (\delta'_2, \ldots, \delta'_k)$ and $\boldsymbol{d'} = (d'_2, \ldots, d'_k)$ be two arbitrary but fixed vectors of real numbers between 0 and 1 and let r' be a positive integer. We say that a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ (as defined in Definition 4.5) is $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable if all but μn^k edges of $K_\ell^{(k)}(V_1, \ldots, V_\ell)$ belong to $(\boldsymbol{\delta'}, \boldsymbol{d'}, r')$-regular complexes $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(k-1)}) = \left\{\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(k-1)})\right\}_{j=1}^{k-1}$ where $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$.*

Before finally stating the regularity lemma, we define regular partitions.

Definition 4.10 (**regular partition**). *Let $\mathcal{G}^{(k)}$ be a (n, ℓ, k)-cylinder and let $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ be a $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable family of partitions.*

We say $\mathscr{R}$ is a (δ'_k, r')-regular w.r.t. $\mathcal{G}^{(k)}$ if all but at most $\delta'_k n^k$ edges of $K_\ell^{(k)}(V_1, \ldots, V_\ell)$ belong to polyads $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ such that $\mathcal{G}^{(k)}$ is (δ'_k, r')-regular with respect to $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

We now state the Regularity Lemma of Rödl and Skokan. In what follows, $\boldsymbol{D} = (D_2, \ldots, D_{k-1})$ is a vector of positive real variables.

Theorem 4.11 (Regularity Lemma (ℓ-partite version)). *For all integers $\ell \geq$*

$k \geq 2$ and all positive reals δ'_k and μ and any positive functions

$$\begin{aligned}\boldsymbol{\delta'}(\boldsymbol{D}) &= \left(\delta'_{k-1}(D_{k-1}), \dots, \delta'_2(D_2, \dots, D_{k-1})\right),\\ r'(A_1, \boldsymbol{D}) &= r'(A_1, D_2, \dots, D_{k-1}),\end{aligned}$$

there exist integers n_k and L_k such that the following holds.

For every (n, ℓ, k)-cylinder $\mathcal{G}^{(k)}$ satisfying $n \geq n_k$ there exists a vector $\boldsymbol{d'} = (d'_2, \dots, d'_{k-1})$ of positive reals and a $\left(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r'(a_1, \boldsymbol{d'})\right)$-equitable $\left(\delta'_k, r'(a_1, \boldsymbol{d'})\right)$-regular family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ such that

$$\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})| \leq L_k .$$

In the upcoming Corollary 4.13, we state an easy modification of Theorem 4.11 whose formulation is convenient for us in our proof of Theorem 3.6. Before stating Corollary 4.13, we outline the main differences between Theorem 4.11 and its corollary below. For that we need the following definition.

Definition 4.12 (refinement). *Let $\{\mathcal{G}^{(h)}\}_{h=1}^{j}$ be an (n, ℓ, j)-complex and let $\mathscr{R} = \mathscr{R}(j, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(h)}\}_{h=1}^{j}$ be a family of partitions of $K_\ell^{(h)}(V_1, \dots, V_\ell)$ for $h \in [j]$. We say $\mathscr{R}$ refines $\{\mathcal{G}^{(h)}\}_{h=1}^{j}$ if for every $h \in [j]$ and every $\boldsymbol{x}^{(h)} \in A(h, \boldsymbol{a})$ either $\mathcal{R}^{(h)}(\boldsymbol{x}^{(h)}) \subseteq \mathcal{G}^{(h)}$ or $\mathcal{R}^{(h)}(\boldsymbol{x}^{(h)}) \cap \mathcal{G}^{(h)} = \varnothing$.*

Moreover, adding an additional layer $\mathcal{G}^{(j+1)} \subseteq \mathcal{K}_{j+1}^{(j)}(\mathcal{G}^{(j)})$ to $\{\mathcal{G}^{(h)}\}_{h=1}^{j}$, we will also say that $\mathscr{R} = \{\mathscr{R}^{(h)}\}_{h=1}^{j}$ refines the $(n, \ell, j+1)$-complex $\{\mathcal{G}^{(h)}\}_{h=1}^{j} \cup \mathcal{G}^{(j+1)}$ if $\mathscr{R}$ refines $\{\mathcal{G}^{(h)}\}_{h=1}^{j}$.

It is a well known fact that the proof of Szemerédi's regularity lemma not only yields the existence of a regular partition for any graph $G = \mathcal{G}^{(2)}$, but also shows that **any** given intitial partition $\mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ of the vertex set $V = V\left(\mathcal{G}^{(2)}\right)$ has a regular refinement. Similarily, the proof of Theorem 4.11 (which is proved by induction on k) yields immediately the existence of a regular and equitable partition $\mathscr{R}$ which refines a given partition of the underlying structure. In particular, regularizing the k-th layer $\mathcal{G}^{(k)}$ of a given (n, ℓ, k)-

complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$, one can obtain a partition $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ satisfying the following property: *for any* $1 \leq j \leq k-1$ *and every* $\boldsymbol{x}^{(j)} \in A(j, \boldsymbol{a})$, *either* $\mathcal{R}^{(j)}\left(\boldsymbol{x}^{(j)}\right) \subseteq \mathcal{G}^{(j)}$ *or* $\mathcal{R}^{(j)}\left(\boldsymbol{x}^{(j)}\right) \cap \mathcal{G}^{(j)} = \varnothing$. In other words, $\mathscr{R}$ refines the partitions given by $\mathcal{G}^{(j)} \cup \overline{\mathcal{G}^{(j)}} = K^{(j)}(V_1, \ldots, V_\ell)$ for every $j \in [k-1]$.

One can maintain yet another property of the $(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r')$-equitable family of partitions $\mathscr{R}$ with density vector $\boldsymbol{d'}$. In the proof of Theorem 4.11 (cf. [48]), the $\boldsymbol{d'}$ are chosen explicitly and there is a large freedom to choose them (more precisely there is no necessary lower bound on each d'_j, $2 \leq j \leq k-1$). Hence, we shall assume, without loss of generality, that for any given *fixed* $\sigma_2, \ldots, \sigma_{k-1}$, we may arrange the constants d'_j, $2 \leq j \leq k-1$, so that the quotients σ_j / d'_j, $2 \leq j \leq k-1$, are integers.

Summarizing the discussion above we arrive at the following corollary of Theorem 4.11, Corollary 4.13 (stated below) The full proof of Corollary 4.13 is identical to the proof of Theorem 4.11 with the two minor adjustments indicated above.

Corollary 4.13. *For all integers* $\ell \geq k \geq 2$ *and all positive real constants* $\sigma_2, \ldots, \sigma_{k-1}$, δ'_k, *and* μ *and all positive functions*

$$\begin{aligned} \boldsymbol{\delta'}(\boldsymbol{D}) &= \left(\delta'_{k-1}(D_{k-1}), \ldots, \delta'_2(D_2, \ldots, D_{k-1})\right), \\ r'(A_1, \boldsymbol{D}) &= r'(A_1, D_2, \ldots, D_{k-1}) \ , \end{aligned}$$

there exist integers n_k *and* L_k *such that the following holds.*

For every (n, ℓ, k)*-complex* $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ *satisfying* $n \geq n_k$ *there exists a* $\left(\mu, \boldsymbol{\delta'}(\boldsymbol{d'}), \boldsymbol{d'}, r'(a_1, \boldsymbol{d'})\right)$*-equitable* $\left(\delta'_k, r'(a_1, \boldsymbol{d'})\right)$*-regular (w.r.t.* $\mathcal{G}^{(k)}$*) family of partitions* $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$, $\boldsymbol{d'} = (d'_2, \ldots, d'_{k-1})$, *such that*

(*i*) $\mathscr{R}$ *refines* $\boldsymbol{\mathcal{G}}$,

(*ii*) σ_j / d'_j *is an integer for* $j = 2, \ldots, k-1$, *and*

(*iii*) $\operatorname{rank} \mathscr{R} = |A(k-1, \boldsymbol{a})| \leq L_k$.

4.2.4 Statement of Cleaning Phase I

The proof of the main theorem, Theorem 3.6, presented in Chapter 5 uses the following lemma, Lemma 4.15, which follows from Corollary 4.13 and the induction assumption on Theorem 3.6.

We use Lemma 4.15 in the proof of Theorem 3.6 instead of Corollary 4.13 since it allows a simpler presentation of the later arguments. For $k = 2$, Lemma 4.15 is a straightforward reformulation of Szemerédi's theorem and reduces to the statement that for any graph $\mathcal{G}^{(2)} = (V, E)$, there is a graph $\tilde{\mathcal{G}}^{(2)}$ for which $\left|\mathcal{G}^{(2)} \triangle \tilde{\mathcal{G}}^{(2)}\right|$ is small and where $\tilde{\mathcal{G}}^{(2)} = (V, \tilde{E})$ admits a "perfectly equitable" partition, i.e., $V = V_1 \cup \cdots \cup V_t$ with $|V_1| = \cdots = |V_t|$ and all pairs (V_i, V_j) are ε-regular for $1 \le i < j \le t$. Lemma 4.15 will generalize this concept for $\mathcal{G}^{(k)}$ with $k > 2$.

The following definition reflects the 'almost' ideal situation when, for each $2 \le j < k$, there is just one uncontrolable (but very small) 'garbage partition class'. Similarly to Definition 4.9 and Definition 4.10, we use tilde-notation in the next definition to be consistent with the context in which it is used.

Definition 4.14 (almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family). *Let $\tilde{r} > 0$ be an integer, $\tilde{\boldsymbol{\delta}} = \left(\tilde{\delta}_2, \ldots, \tilde{\delta}_{k-1}\right)$ and $\tilde{\boldsymbol{d}} = \left(\tilde{d}_2, \ldots, \tilde{d}_{k-1}\right)$ be vectors of positive reals, and $\boldsymbol{b} = (b_1, \ldots, b_{k-1})$ be a vector of positive integers. Set*

$$\bar{\boldsymbol{b}} = (b_1, b_2 + 1, b_3, \ldots, b_{k-1}) \ .$$

We say that a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ (as defined in Definition 4.5) is an almost perfect $\left(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b}\right)$-family of partitions if the following holds:

(i) $\tilde{d}_j b_j = 1 \pm \tilde{\delta}_j / \tilde{d}_j$ *for every* $2 \le j \le k-1$, *and*

(ii) *for every* $2 \le j \le k-1$, *for every* $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\,(j-1, \boldsymbol{b})$ *and for every* $\beta \in [b_j]$, *the* $(n/b_1, j, j)$*-cylinder* $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big)$ *is* $\left(\tilde{\delta}_j, \tilde{d}_j, \tilde{r}\right)$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big)$.

An almost perfect family of partitions $\mathscr{P}$ has the property that for every $\boldsymbol{x}^{(k-1)} \in A(k-1, \boldsymbol{b})$ the $(n/b_1, k-1, k-1)$-complex $\boldsymbol{\mathcal{P}}^{(k-1)}(\boldsymbol{x}^{(k-1)}) = \{\mathcal{P}^{(j)}(\boldsymbol{x}^{(k-1)})\}_{j=1}^{k-1}$ (cf. Claim 4.3) is 'garbage free', i.e., $\boldsymbol{\mathcal{P}}^{(k-1)}(\boldsymbol{x}^{(k-1)})$ is $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{k-1}), (\tilde{d}_2, \ldots, \tilde{d}_{k-1}), \tilde{r}\big)$-regular. In other words, all addresses $\boldsymbol{x}^{(k-1)}$ which give rise to irregular parts of the family of partitions $\mathscr{P}$ are in

$$A(k-1, \bar{\boldsymbol{b}}) \setminus A(k-1, \boldsymbol{b}) \,.$$

Lemma 4.15 (Cleaning Phase I)**.** *For every vector $\boldsymbol{d} = (d_2, \ldots, d_k)$ of positive reals, for every choice of $\delta_3, \ldots, \delta_k$, for any positive real $\tilde{\delta}_k$ and all positive functions*

$$\begin{aligned} \tilde{\boldsymbol{\delta}}\big(\boldsymbol{D}\big) &= \Big(\tilde{\delta}_{k-1}\big(D_{k-1}\big), \ldots, \tilde{\delta}_2\big(D_2, \ldots, D_{k-1}\big)\Big) \ , \\ \tilde{r}\big(B_1, \boldsymbol{D}\big) &= \tilde{r}\big(B_1, D_2, \ldots, D_{k-1}\big) \,, \end{aligned}$$

there exist integers $\tilde{n}_k$, $\tilde{L}_k$, a vector of positive reals $\tilde{\boldsymbol{c}} = (\tilde{c}_2, \ldots, \tilde{c}_{k-1})$ and a positive constant δ_2 so that the following holds:

For every $(\boldsymbol{d}, \boldsymbol{\delta} = (\delta_2, \ldots, \delta_k), 1)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$ with $n \geq \tilde{n}_k$ there exist an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, a positive real vector $\tilde{\boldsymbol{d}} = \big(\tilde{d}_2, \ldots, \tilde{d}_{k-1}\big)$ componentwise bigger than $\tilde{\boldsymbol{c}}$, and an almost perfect $\big(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(b_1, \tilde{\boldsymbol{d}}), \boldsymbol{b}\big)$-family of partitions $\mathscr{P} = \mathscr{P}\left(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi}\right) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ refining $\tilde{\boldsymbol{\mathcal{G}}}$ so that:

- *(i)* *$\tilde{\mathcal{G}}^{(k)}$ is $\big(\tilde{\delta}_k, \tilde{r}(b_1, \tilde{\boldsymbol{d}})\big)$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\,(k-1, \boldsymbol{b})$,*
- *(ii)* *$\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$, $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$, and $\tilde{\mathcal{G}}^{(j)} \subseteq \mathcal{G}^{(j)}$ for every $3 \leq j \leq k$,*
- *(iii)* *for every $3 \leq j \leq k$ and every $j \leq i \leq \ell$, the following holds:*

$$\big|\mathcal{K}_i^{(j)}(\mathcal{G}^{(j)}) \triangle \, \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\big| = \big|\mathcal{K}_i^{(j)}(\mathcal{G}^{(j)}) \setminus \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\big| \leq \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i \,,$$

(*iv*) *for any* $\hat{\boldsymbol{x}}^{(1)} = ((\lambda_1, \lambda_2), (\beta_1, \beta_2)) \in \hat{A}(1, \boldsymbol{b})$*, the graph* $\tilde{\mathcal{G}}^{(2)}[V_{\lambda_1,\beta_1}, V_{\lambda_2,\beta_2}]$ *is* $(\tilde{L}_k^2\delta_2, d_2, 1)$*-regular w.r.t.* $\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)}) = V_{\lambda_1,\beta_1} \cup V_{\lambda_2,\beta_2}$*, and*

(*v*) $\operatorname{rank} \mathscr{P} \leq \tilde{L}_k$ *and consequently* $|\hat{A}(k-1, \boldsymbol{b})| \leq (\tilde{L}_k)^k$*.*

In the proof of Lemma 4.15 (see Chapter 6) we construct an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}}$ admitting an almost perfect family of partitions $\mathscr{P}$. Moreover, $\tilde{\boldsymbol{\mathcal{G}}}$ is almost identical to a given (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}}$ (see (*ii*) and (*iii*) of Lemma 4.15). In particular, $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$, while we allow a small difference between $\tilde{\mathcal{G}}^{(j)}$ and $\mathcal{G}^{(j)}$ for $j \geq 3$. For the special role of $\tilde{\mathcal{G}}^{(j)}$ we have to allow 'garbage classes' in $\mathscr{P}^{(2)}$. These 'garbage classes' are contained in $\psi_2^{-1}(b_2 + 1)$. Note that the integer vector $\bar{\boldsymbol{b}}$ differs from $\boldsymbol{b}$ only in the second coordinate.

On the other hand, note that the partition $\mathscr{P}$ given by Lemma 4.15 is perfect in the sense that for $2 \leq j \leq k$ every j-tuple of $\tilde{\mathcal{G}}^{(j)}$ belongs to a regular polyad of $\mathscr{P}$. This feature will later give us a significant notational advantage. On the other hand, Lemma 4.15 (*iii*) ensures that the two complexes $\boldsymbol{\mathcal{G}}$ and $\tilde{\boldsymbol{\mathcal{G}}}$ differ by few cliques only.

4.2.5 The Slicing Lemma

The following lemma whose proof is based on the fact that *randomly chosen subcylinders of a regular cylinder are regular* was proved in [48]. We will find it useful in this dissertation as well.

Lemma 4.16 (Slicing Lemma)**.** *Suppose* ϱ*,* δ *are two real numbers such that* $0 < \delta/2 < \varrho \leq 1$*. There is an* $m_0 = m_0(\varrho, \delta)$ *such that the following holds. Let* $\hat{\mathcal{P}}^{(j-1)}$ *be a* $(m, j, j-1)$*-cylinder satisfying* $\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\right| \geq m^j/\ln m$ *and let* $\mathcal{F}^{(j)} \subseteq \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)$ *be an* (m, j, j)*-cylinder which is* $(\delta, \varrho, r_{\mathrm{SL}})$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}$*. Then for every* $0 < p < 1$*, where* $3\delta < p\varrho$ *and* $u = \lfloor 1/p \rfloor$*, there exists a decomposition of* $\mathcal{F}^{(j)} = \mathcal{F}_0^{(j)} \cup \mathcal{F}_1^{(j)} \cup \cdots \cup \mathcal{F}_u^{(j)}$ *such that* $\mathcal{F}_i^{(j)}$ *is* $(3\delta, p\varrho, r_{\mathrm{SL}})$*-regular w.r.t.* $\hat{\mathcal{P}}^{(j-1)}$ *for* $1 \leq i \leq u$*.*

Moreover if $1/p$ *is an integer then* $\mathcal{F}_0^{(j)} = \varnothing$.

Chapter 5

Proof of the Counting Lemma

The proof of the Counting Lemma, Theorem 3.6, is based on upcoming Theorem 5.3 stated in the Section 5.2. In Section 5.2, we prove Theorem 3.6 follows from Theorem 5.3. The remainder of the dissertation is devoted to the proof of Theorem 5.3, an outline of which is given in Section 5.3.

The proof of Theorem 5.3 splits into four lemmas, Lemmas 4.15, 5.6, 5.7 and 5.9. In Section 5.4, we show how Theorem 5.3 follows from these four lemmas. We defer the proofs of these lemmas to Chapter 6–8.

The structure of the proof of the Counting Lemma outlined here will be summarized in Figure 5.1. Some further consequences of Theorem 5.3 are discussed in Chapter 10.

5.1 Induction assumption on the Counting Lemma

We prove the Counting Lemma, Theorem 3.6, by induction on k. For $k = 2$, the Counting Lemma, i.e, Fact 2.2, is a well known fact (see, e.g., [32, 33]) and for $k = 3$ it was proved by Nagle and Rödl in [36]. Therefore, from now on let $k \geq 4$ be a fixed integer.

Induction Hypothesis. *We assume that*

$$\mathbf{CL}_{j,i} \text{ \textit{holds for} } 2 \leq j \leq k-1 \text{ and } i \geq j\,. \tag{5.1}$$

We prove $\mathbf{CL}_{k,\ell}$ holds for all integers $\ell \geq k$.

Rather than quoting various forms of our induction hypothesis $\mathbf{CL}_{j,i}$ (for varying $2 \leq j \leq k-1$ and $i \geq j$) involving different δ's and γ's, we summarize all such statements in one. The following statement which we denote by $\mathbf{IHC}_{k-1,\ell}$ is a reformulation of our Induction Hypothesis.

Statement 5.1 (Induction Hypothesis on Counting)**.** *For every integer $\ell \geq k-1$ the following is true: $\forall \eta > 0$ $\forall d_{k-1} > 0$ $\exists \delta_{k-1} > 0$ $\forall d_{k-2} > 0$ $\exists \delta_{k-2} > 0$ $\ldots$ $\forall d_2 > 0$ $\exists \delta_2 > 0$ and there are integers r and $m_{k-1,\ell}$ so that for all integers j and i with $2 \leq j \leq k-1$ and $j \leq i \leq \ell$ the following holds.*

If $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ is a $\big((\delta_2, \ldots, \delta_j), (d_2, \ldots, d_j), r\big)$-regular (m, i, j)-complex with $m \geq m_{k-1,\ell}$, then

$$\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}\big)\right| = (1 \pm \eta) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times m^i \,.$$

The following fact confirms that Statement 5.1 is an easy consequence of our Induction Hypothesis.

Fact 5.2. $\quad \mathbf{CL}_{j,i} \ \forall\, 2 \leq j \leq k-1\,,\ j \leq i \leq \ell \Longrightarrow \mathbf{IHC}_{k-1,\ell}\,.$

Note that Fact 5.2 is easy to prove and only requires confirming the constants may be chosen appropriately; when given $\eta, d_{k-1}, \delta_{k-1}, \ldots, \delta_{j+1}$ and d_j, choose δ_j to be the minimum of all δ_j's from the statements $\mathbf{CL}_{h,i}$ with $j \leq h \leq k-1$ and $h \leq i \leq \ell$, where δ_j appears.

Proof of Fact 5.2. For given integers k and ℓ we shall refer to this theorem by $\mathbf{IHC}_{k-1,\ell}$. In what follows we will show how $\mathbf{CL}_{j,i}$ applied with $2 \leq j \leq k-1$ and $j \leq i \leq \ell$ implies $\mathbf{IHC}_{k-1,\ell}$.

Indeed, let η and d_{k-1} be given. For all j and i, $2 \leq j \leq k-1$, $j \leq i \leq \ell$ set

$$\gamma_{j,i} = \eta \,.$$

We fix

$$\delta_{k-1} = \min_{k-1 \le i \le \ell} \left\{ \delta_{k-1}\big(\, \mathbf{CL}_{k-1,i}(\gamma_{k-1,i}, d_{k-1})\big) \right\} \;,$$

where for $k-1 \le i \le \ell$ the constant $\delta_{k-1}\big(\, \mathbf{CL}_{k-1,i}(\gamma_{k-1,i}, d_{k-1})\big)$ has the value of δ_{k-1} given by $\mathbf{CL}_{k-1,i}$ for $\gamma_{k-1,i}$ and d_{k-1}. After d_{k-2} is given we set

$$\begin{aligned}\delta_{k-2} = \min \Big\{ &\min_{k-2 \le i \le \ell} \left\{ \delta_{k-2}\big(\, \mathbf{CL}_{k-2,i}(\gamma_{k-2,i}, d_{k-2})\big) \right\}, \\ &\qquad\qquad \min_{k-1 \le i \le \ell} \left\{ \delta_{k-2}\big(\, \mathbf{CL}_{k-1,i}(\gamma_{k-1,i}, d_{k-1}, \delta_{k-1}, d_{k-2})\big) \right\} \Big\} \,.\end{aligned}$$

In general, after $d_{k-1}, \delta_{k-1}, d_{k-2}, \delta_{k-2}, \ldots, d_j$ were determined we set

$$\delta_j = \min_{j \le h \le k-1} \min_{h \le i \le \ell} \left\{ \delta_j\big(\, \mathbf{CL}_{h,i}(\gamma_{h,i}, d_h, \delta_h, \ldots, d_j)\big) \right\} .$$

With all constants $\gamma_{j,i}$ for $2 \le j < k$ and $j \le i \le \ell$ and $d_{k-1}, \delta_{k-1}, \ldots, d_2, \delta_2$ disclosed we set the promised

$$\begin{aligned} r &= \max_{2 \le h \le k-1} \max_{h \le i \le \ell} \left\{ r\big(\, \mathbf{CL}_{h,i}(\gamma_{h,i}, d_h, \delta_h, \ldots, d_2, \delta_2)\big) \right\} , \\ m_{k-1,\ell} &= \max_{2 \le h \le k-1} \max_{h \le i \le \ell} \left\{ n_0\big(\, \mathbf{CL}_{h,i}(\gamma_{h,i}, d_h, \delta_h, \ldots, d_2, \delta_2)\big) \right\} . \end{aligned}$$

Let $2 \le j \le k-1$ and $j \le i \le \ell$ be arbitrary, and suppose $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ is a $\big((\delta_2, \ldots, \delta_j), (d_2, \ldots, d_j), r\big)$-regular (m, i, j)-complex with $m \ge m_{k-1,\ell}$. Then by $\mathbf{CL}_{j,i}$ and the choice of $\gamma_{j,i}$ we infer that

$$\left| \mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}\big) \right| = (1 \pm \gamma_{j,i}) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i = (1 \pm \eta) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i \,,$$

which establishes the promised implication. $\square$

As a consequence of the induction assumption stated in (5.1) and Fact 5.2, we may assume for the remainder of this dissertation that

$$\mathbf{IHC}_{k-1,\ell} \text{ holds for } \ell \ge k-1\,. \tag{5.2}$$

5.2 Proof of Theorem 3.6

The proof of the Counting Lemma, Theorem 3.6, consists of two main parts. The first part is Theorem 5.3, stated below, which receives input an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ from the hypothesis of the Counting Lemma, Theorem 3.6. Theorem 5.3 then guarantees the existence of output an (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(h)}\}_{h=1}^{k}$ having the following properties.

(a) The complex $\boldsymbol{\mathcal{F}}$ differs only slightly from $\boldsymbol{\mathcal{G}}$. In particular, the number of ℓ-cliques in $\mathcal{G}^{(k)}$ and $\mathcal{F}^{(k)}$ are essentially the same (see property (iii) of Theorem 5.3).

(b) The complex $\boldsymbol{\mathcal{F}}$ is "ready" for an application of the Dense Counting Lemma, Theorem 4.1.

The proof of the following Theorem 5.3 is based on the induction hypothesis, $\mathbf{IHC}_{k-1,\ell}$ (cf. Statement 5.1). In the formulation below, the integer k is already fixed (cf. Section 5.1) according to our induction hypothesis.

Theorem 5.3. *Let $\ell \geq k$ be a fixed integer. The following is true: $\forall \gamma > 0\ \forall d_k > 0\ \exists \delta_k > 0\ \forall d_{k-1} > 0\ \exists \delta_{k-1} > 0\ \ldots\ \forall d_2 > 0, \varepsilon > 0\ \exists \delta_2 > 0$ and there are integers r and n_0 so that, with $\boldsymbol{d} = (d_2, \ldots, d_k)$ and $\boldsymbol{\delta} = (\delta_2, \ldots, \delta_k)$ and $n \geq n_0$, whenever $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^{k}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex, then there exists an (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(h)}\}_{h=1}^{k}$ such that*

(i) *$\boldsymbol{\mathcal{F}}$ is $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$-regular, with $\boldsymbol{\varepsilon} = (\varepsilon, \ldots, \varepsilon) \in \mathbb{R}^{(k-1)}$,*

(ii) *$\mathcal{F}^{(1)} = \mathcal{G}^{(1)}$ and $\mathcal{F}^{(2)} = \mathcal{G}^{(2)}$, and*

(iii) $\left|\mathcal{K}_{\ell}^{(k)}\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_{\ell}^{(k)}\big(\mathcal{F}^{(k)}\big)\right| \leq (\gamma/2) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^{\ell}$.

We mention that Theorem 5.3 has some interesting implications of its own which we discuss in Chapter 10.

We present a proof of Theorem 5.3 in Section 5.4. In the immediate sequel, we give the proof of the Inductive Step for the Counting Lemma based on Theorem 5.3 and Theorem 4.1. We note that this proof of $\mathbf{CL}_{k,\ell}$ does not directly use the induction hypothesis $\mathbf{IHC}_{k-1,\ell}$, but we will use $\mathbf{IHC}_{k-1,\ell}$ in the proof of Theorem 5.3 (see Figure 5.1).

Proof of Theorem 3.6. We begin by describing the constants involved. With the exception of ε, Theorem 3.6 and Theorem 5.3 involve the same constants under the same quantification. Hence, given γ and d_k from Theorem 3.6, we let δ_k be the $\delta_k\big(\text{Thm.5.3}(\gamma, d_k)\big)$ from Theorem 5.3. In general, given d_j, $j = k, \dots, 3$, we set

$$\delta_j = \delta_j\big(\text{Thm.5.3}(\gamma, d_k, \delta_k, d_{k-1}, \dots, \delta_{j+1}, d_j)\big) .$$

Having fixed $\gamma, d_k, \delta_k, d_{k-1}, \dots, \delta_4, d_3, \delta_3$, now let d_2 be given by Theorem 3.6. Next, we fix ε for Theorem 5.3 so that

$$\varepsilon \leq \varepsilon\big(\text{Thm.4.1}(d_2, \dots, d_k)\big) \quad \text{and} \quad g_{k,\ell}(\varepsilon) \leq \frac{\gamma}{2} , \tag{5.3}$$

where $g_{k,\ell}$ is given by the Dense Counting Lemma, Theorem 4.1. Moreover, let $m_0\big(\text{Thm.4.1}(d_2, \dots, d_k)\big)$ be the lower bound on the number of vertices given by Theorem 4.1 applied to $d_2, \dots, d_k$.

Then, Theorem 5.3 yields

$$\begin{gathered}\delta_2\big(\text{Thm.5.3}(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big) , \quad r\big(\text{Thm.5.3}(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big) , \\ \text{and} \qquad n_0\big(\text{Thm.5.3}(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big) . \end{gathered} \tag{5.4}$$

Finally, we set δ_2 and r for Theorem 3.6 to its corresponding constants given in (5.4). Also, we set n_0 for Theorem 3.6 to

$$n_0 = \max\big\{n_0\big(\text{Thm.5.3}(\gamma, d_k, \delta_k, \dots, \delta_3, d_2, \varepsilon)\big), m_0\big(\text{Thm.4.1}(d_2, \dots, d_k)\big)\big\} .$$

Now, let $\boldsymbol{\mathcal{G}}$ be a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular (n, ℓ, k)-complex satisfying $n \geq n_0$. Then, Theorem 5.3 yields an $(\boldsymbol{\varepsilon}, \boldsymbol{d}, 1)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}}$ satisfying (i)–(iii)

of Theorem 5.3. Consequently, by (5.3) and (i), we may apply the Dense Counting Lemma to $\boldsymbol{\mathcal{F}}$. Therefore,

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{F}^{(k)}\big)\right| = \Big(1 \pm \frac{\gamma}{2}\Big) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell$$

and Theorem 3.6 follows from (iii) of Theorem 5.3. □

We note that the proof of Theorem 3.6 did not use the full strength of Theorem 5.3. In particular, we made no use of (ii) here. However, (ii) is important with respect to further consequences of Theorem 5.3 discussed in Chapter 10.

5.3 Outline of the proof of Theorem 5.3

Given an (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$, Theorem 5.3 ensures the existence of an appropriate (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$. This complex is constructed successively in three phases outlined below.

The first phase we call Cleaning Phase I and is a variant of the RS-Lemma (see Theorem 4.11). The lemma corresponding to Cleaning Phase I, Lemma 4.15, was already stated in Section 4.2.4. Given a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular input complex $\boldsymbol{\mathcal{G}}$ with $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$, we fix

$$\tilde{\delta}_k \ll \varepsilon' \ll \min\{\varepsilon, d_2, \dots, d_k\}. \tag{5.5}$$

Lemma 4.15 alters $\boldsymbol{\mathcal{G}}$ slightly (by a measure of $\tilde{\delta}_k$) to obtain an (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$ together with an almost perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(\tilde{\boldsymbol{d}}), \boldsymbol{b})$-family of partitions which is $(\tilde{\delta}_k, \tilde{r}(\tilde{\boldsymbol{d}}))$-regular w.r.t. $\tilde{\mathcal{G}}^{(k)}$ (cf. Lemma 4.15 and Figure 5.2 in Section 5.4.2). Importantly, Lemma 4.15 will ensure that

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{G}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big)\right| \quad \text{is ``small''}. \tag{5.6}$$

Cleaning Phase I enables us to work with a complex $\tilde{\boldsymbol{\mathcal{G}}}$ admitting a partition with almost no irregular polyads. These details are done largely for convenience to help ease subsequent parts of the proof.

We next proceed to Cleaning Phase II with the newly-acquired complex $\tilde{\boldsymbol{\mathcal{G}}}$ and an almost perfect $(\mu, \tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, \tilde{r}(\tilde{\boldsymbol{d}}), \boldsymbol{b})$-family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$, rank $\mathscr{P} \leq \tilde{L}_k$ (cf. Lemma 4.15). Since $\tilde{\boldsymbol{\mathcal{G}}}$ differs from $\boldsymbol{\mathcal{G}}$ only slightly (by a measure of $\tilde{\delta}_k$), it follows from the choice of constants (argued in Fact 7.3) that $\tilde{\boldsymbol{\mathcal{G}}}$ inherits $(2\boldsymbol{\delta}, \boldsymbol{d}, r)$-regularity from $\boldsymbol{\mathcal{G}}$. Moreover, the choice of r ensuring $r \geq \tilde{L}_k$ (cf. (5.22)) implies that the density $d\big(\tilde{\mathcal{G}}^{(j)}|\hat{\mathcal{P}}^{(j-1)}\big)$ is close to what it "should be", namely, d_j, w.r.t. to "most" polyads $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)} \subseteq \tilde{\mathcal{G}}^{(j-1)}$.

The goal in Cleaning Phase II is to perfect the small number of polyads having aberrant density. More specifically, Cleaning Phase II constructs "unidense" (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$ where $d\big(\mathcal{F}^{(j)}|\hat{\mathcal{P}}^{(j-1)}\big)$ is the same and equal to d_j for every polyad $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)} \subseteq \mathcal{F}^{(j-1)}$. The importance of "unidensity" is that it allows us to apply the Union Lemma, Lemma 5.9. Then the "final product" of the Union Lemma, the (n, ℓ, k)-complex $\boldsymbol{\mathcal{F}}$, will satisfy (i) of Theorem 5.3. We now further examine the details of Cleaning Phase II.

Cleaning Phase II splits into two parts. In Part 1 of Cleaning Phase II (cf. Lemma 5.6), we correct the first $k-1$ layers of possible imperfections of $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, $j < k$, by constructing a "unidense" complex $(n, \ell, k-1)$-complex $\boldsymbol{\mathcal{H}}^{\boldsymbol{(k-1)}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$. For the construction of $\boldsymbol{\mathcal{H}}^{\boldsymbol{(k-1)}}$, we need to count cliques in such a complex and use the powerful tool $\mathbf{IHC}_{k-1,\ell}$ which we have available by the induction assumption ($\mathbf{CL}_{j,\ell}$ for $2 \leq j \leq k-1$).

We next remedy imperfections on the k^{th} layer $\tilde{\mathcal{G}}^{(k)}$. However, in the absence of our Induction Assumption herein, we have to proceed more carefully.

We first construct a still "somewhat imperfect" (n, ℓ, k)-cylinder $\mathcal{H}^{(k)}$ so that $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k}$ is an (n, ℓ, k)-complex and $d\big(\mathcal{H}^{(k)}|\hat{\mathcal{P}}^{(k-1)}\big) = d_k \pm \sqrt{\delta_k}$ for every polyad $\hat{\mathcal{P}}^{(j-1)}$ from $\mathscr{P}$ with $\hat{\mathcal{P}}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$. While $\tilde{\mathcal{G}}^{(k)}$ satisfies that for "most" $\hat{\mathcal{P}}^{(k-1)} \subseteq \tilde{\mathcal{G}}^{(k-1)}$, its density is close to d_k, the new $\mathcal{H}^{(k)}$ has density close to d_k for "all" $\hat{\mathcal{P}}^{(k-1)} \subseteq \mathcal{H}^{(k-1)}$. Moreover, we can construct $\mathcal{H}^{(k)}$ in such a way that

$$\left|\mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big)\right| \quad \text{is "small"}. \tag{5.7}$$

Part 2 of Cleaning Phase II deals with $\mathcal{H}^{(k)}$, the k-th layer of the complex $\boldsymbol{\mathcal{H}}$. In this part, we construct unidense (w.r.t. $\mathcal{H}^{(k-1)}$ and $\mathscr{P}^{(k-1)}$) (n, ℓ, k)-cylinders $\mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)}$ (corresponding to the above $d_k - \sqrt{\delta_k}$ and $d_k + \sqrt{\delta_k}$, respectively) and $\mathcal{F}^{(k)}$ where each of these cylinders, together with $\boldsymbol{\mathcal{H}^{(k-1)}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$, forms an (n, ℓ, k)-complex. In the construction, we will also ensure that

$$\mathcal{H}_-^{(k)} \subseteq \mathcal{H}^{(k)} \subseteq \mathcal{H}_+^{(k)} \qquad\qquad \mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}\,. \tag{5.8}$$

We then set $\mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $j < k$ and $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$.

We now discuss how we infer (i) and (iii) of Theorem 5.3 for $\boldsymbol{\mathcal{F}}$ (property (ii) is somewhat technical and we omit it from the outline given here). For part (i) of Theorem 5.3, we need to show that $\mathcal{F}^{(j)}$ is $(\varepsilon, d_j, 1)$-regular w.r.t. $\mathcal{F}^{(j-1)}$, $2 \leq j \leq k$. To this end, we take the union of all "partition blocks" from $\mathscr{P}^{(j)}$ (which are subhypergraphs of $\mathcal{F}^{(j)}$). Note that all these blocks are very regular (w.r.t. their underlying polyads (which are subhypergraphs of $\mathcal{F}^{(j-1)}$)) and have the same relative density (due to the unidensity). In fact, these blocks will be so regular that their union is $(\varepsilon', d_j, 1)$-regular and therefore also $(\varepsilon, d_j, 1)$-regular (cf. (5.5)). Consequently, as proved in the Union Lemma, Lemma 5.9, we obtain that $\boldsymbol{\mathcal{F}}$ is $\big((\varepsilon, \dots, \varepsilon), \boldsymbol{d}, 1\big)$-regular which proves (i) of Theorem 5.3.

We now outline the proof of part (iii). Observe that

$$\begin{aligned}\left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)})\triangle\mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq \left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)})\triangle\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right| + \\ + \left|\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\triangle\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)})\right| + \\ + \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)})\triangle\mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right|.\end{aligned} \tag{5.9}$$

The first two terms of the right-hand side are small as mentioned earlier (see (5.6) and (5.7)). Let us say a few words on how to bound the third quantity.

Due to (5.8), we have

$$\begin{aligned}\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)})\triangle\mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| &\leq \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_+^{(k)})\triangle\mathcal{K}_\ell^{(k)}(\mathcal{H}_-^{(k)})\right| \\ &= \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_+^{(k)})\right| - \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_-^{(k)})\right|.\end{aligned} \tag{5.10}$$

Since both complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ were constructed to be unidense, we can show that $\boldsymbol{\mathcal{H}}_*$ is $\big((\varepsilon',\dots,\varepsilon'),\boldsymbol{d}_*,1\big)$-regular for $*\in\{+,-\}$ where $\boldsymbol{d}_* = (d_2,\dots,d_{k-1},d_k^*)$ and $d_k^* = d_k+\sqrt{\delta_k}$ for $*=+$ and $d_k^* = d_k-\sqrt{\delta_k}$ for $*=-$. Similarly to the proof of (i) where the Union Lemma was applied to $\boldsymbol{\mathcal{F}}$, we can use it here for $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$. Consequently, due to (5.5), we can apply the Dense Counting Lemma, Theorem 4.1 to bound the right-hand side of (5.10) and thus the right-hand side of (5.9). This yields part (iii) of Theorem 5.3.

The flowchart in Figure 5.1 gives a sketch of the connection of theorems and lemmas involved in the proof of $\mathbf{CL}_{k,\ell}$, Theorem 3.6. Each box represents a theorem or lemma containing a reference for its proof. Vertical arcs indicate which statements are needed to prove the statement to which the arc points. The horizontal arcs indicate the alteration of the involved complexes outlined above.

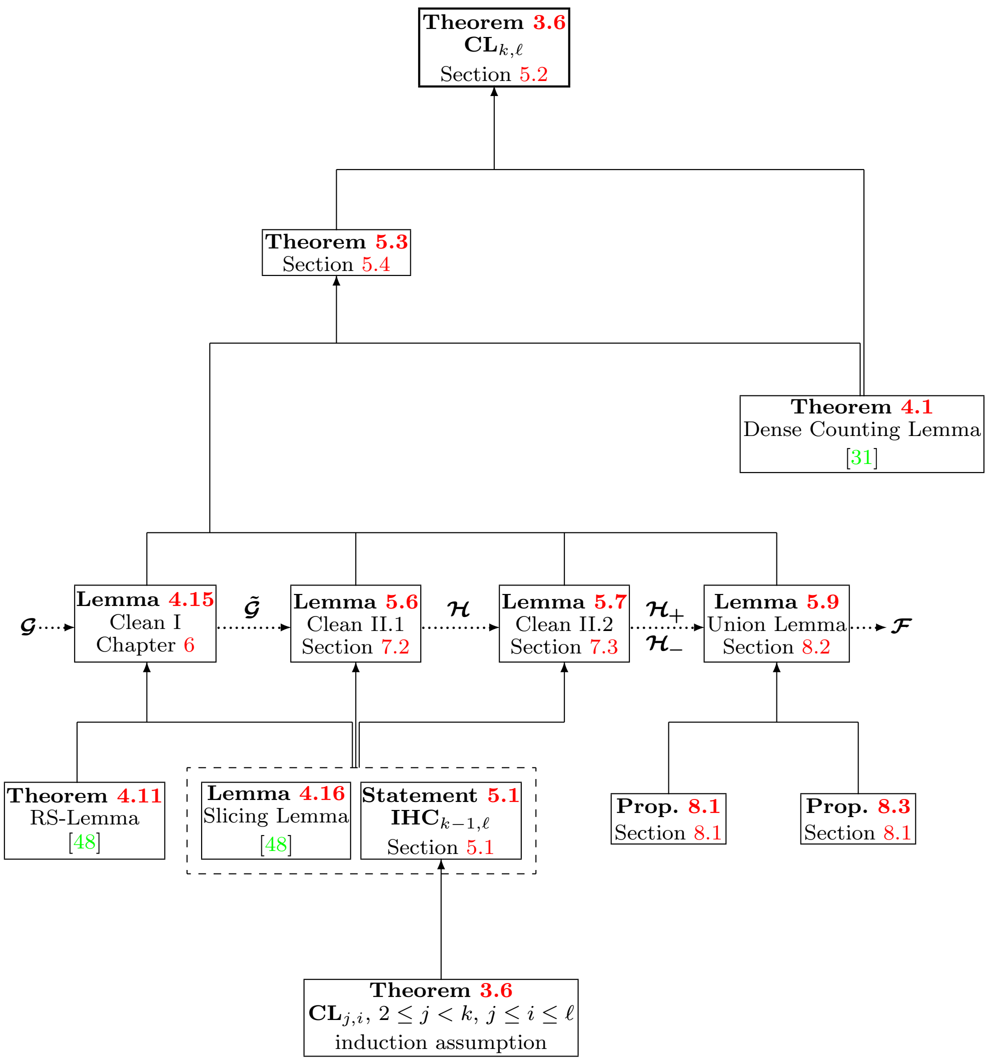

Figure 5.1: Structure of the proof of Theorem 3.6

5.4 Proof of Theorem 5.3

In this section, we give all details of the proof of Theorem 5.3 outlined in the last section. The proof of Theorem 5.3 splits into four parts. We separate these parts across Sections 5.4.1–5.4.4.

5.4.1 Constants

The hierarchy of the involved constants plays an important role in our proof. The choice of the constants breaks into two steps.

Step 1. Let an integer ℓ be given. We first recall the quantification of Theorem 5.3:

$$\forall \gamma, d_k \ \exists \delta_k \ \forall d_{k-1} \dots \exists \delta_3 \ \forall d_2, \varepsilon \ \exists \delta_2, r, n_0 \,.$$

Given γ and d_k we choose δ_k such that

$$\delta_k \ll \min\{\gamma, d_k\} \tag{5.11}$$

holds. Now, let d_{k-1} be given, we set

$$\eta = 1/4 \,. \tag{5.12}$$

(Our proof is not too sensitive to the choice of η, representing the multiplicative error for $\mathbf{IHC}_{k-1,\ell}$.) We choose δ_{k-1} so that $\delta_{k-1} \ll \min\{\delta_k, d_{k-1}\}$ and $\delta_{k-1} \le \delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1})\big)$ where $\delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1})\big)$ is the value of δ_{k-1} given by Statement 5.1 for η and d_{k-1}. We then proceed and define δ_j for $j = k-2, \dots, 3$ in the similar way. Summarizing the above, for $j = k-1, \dots, 3$ we choose δ_j such that

$$\begin{aligned} &\delta_j \ll \min\{\delta_{j+1}, d_j\} \qquad \text{and} \\ &\delta_j \le \delta_j\big(\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \dots, \delta_{j+1}, d_j)\big) \,. \end{aligned} \tag{5.13}$$

We mention that after d_2 is revealed we pause before defining δ_2.

Indeed, next we choose the auxiliary constant ε' so that

$$
\begin{aligned}
\varepsilon' &\le \min\Big\{\varepsilon\big(\text{Thm.}4.1(d_2,\ldots,d_{k-1},d_k-\delta_k^{1/2})\big),\\
&\qquad\qquad\qquad \varepsilon\big(\text{Thm.}4.1(d_2,\ldots,d_{k-1},d_k+\delta_k^{1/2})\big)\Big\}\,, \qquad (5.14)\\
\varepsilon' &\ll \min\big\{\delta_3,d_2,\varepsilon\big\}\,, \qquad \text{and} \qquad g_{k,\ell}(\varepsilon')\ll\delta_k\,,
\end{aligned}
$$

where $g_{k,\ell}$ is given by the Dense Counting Lemma, Theorem 4.1. We then fix $\tilde\eta$, ν, and $\tilde\delta_k$ to satisfy

$$
\varepsilon' \gg \tilde\eta \gg \nu \gg \tilde\delta_k \qquad \text{and} \qquad \tilde\delta_k \le 1/8\,. \tag{5.15}
$$

This completes Step 1 of the choice of the constants. We summarize the choices above in the following flowchart:

$$
\begin{array}{ccccccccccccccc}
 & & d_k & & \ldots & & d_3 & & d_2,\varepsilon & & & & & & \\
 & & \begin{matrix}\vee\\[-6pt]\vee\end{matrix} & & \ldots & & \begin{matrix}\vee\\[-6pt]\vee\end{matrix} & & \begin{matrix}\vee\\[-6pt]\vee\end{matrix} & & & & & & \\
\gamma & \gg & \delta_k & \gg & \cdots & \gg & \delta_3 & \gg & \varepsilon' & \gg & \tilde\eta & \gg & \nu & \gg & \tilde\delta_k
\end{array}
\tag{5.16}
$$

Step 2. The definition of the constants determined here is more subtle. Our goal is to extend (5.16) with the additional constants $\tilde d_j$, $\tilde\delta_j$ (for $j = k-1,\ldots,2$), $\tilde r$, $\tilde L_k^2\delta_2$ and r so that:

$$
\begin{array}{ccccccccccc}
 & & \tilde d_{k-1} & & \ldots & & \tilde d_3 & & \tilde d_2 & & \\
 & & \begin{matrix}\vee\\[-6pt]\vee\end{matrix} & & \ldots & & \begin{matrix}\vee\\[-6pt]\vee\end{matrix} & & \begin{matrix}\vee\\[-6pt]\vee\end{matrix} & & \\
\tilde\delta_k & \gg & \tilde\delta_{k-1} & \gg & \cdots & \gg & \tilde\delta_3 & \gg & \tilde\delta_2,\frac{1}{\tilde r} & \gg & \tilde L_k^2\delta_2,\frac{1}{r}
\end{array}
$$

In our proof, we apply Lemma 4.15 to the (n,ℓ,k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$. Lemma 4.15 has positive functions $\tilde\delta_j(D_j,\ldots,D_{k-1})$ for $j = 2,\ldots,k-1$ and $\tilde r(D_2,\ldots,D_{k-1})$ in variables $D_2,\ldots,D_{k-1}$ as part of its input. The application of Lemma 4.15 results in an almost perfect $\big(\tilde{\boldsymbol\delta}(\tilde{\boldsymbol d}),\tilde{\boldsymbol d},\tilde r(\tilde{\boldsymbol d}),\boldsymbol b\big)$-family of partitions $\mathscr{P}$ with some $\tilde{\boldsymbol d} = (\tilde d_2,\ldots,\tilde d_{k-1})$. We want to be able to count cliques within the polyads of the family of regular partitions $\mathscr{P}$ by applying $\mathbf{IHC}_{k-1,\ell}$. Therefore, we choose the functions $\tilde{\delta}_j(D_j,\dots,D_{k-1})$ for Lemma 4.15 in such a way that they comply with the quantification of $\mathbf{IHC}_{k-1,\ell}$, Statement 5.1.

To this end let $\tilde{\delta}_{k-1}(D_{k-1})$ be a function so that

$$\tilde{\delta}_{k-1}(D_{k-1}) \ll \min\{\tilde{\delta}_k, D_{k-1}\}\,, \qquad \tilde{\delta}_{k-1}(D_{k-1}) \le \delta_{k-1}\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1})\big)\,,$$
$$\text{and} \qquad (1+\tilde{\delta}_{k-1}(D_{k-1})/D_{k-1})^{\binom{\ell}{k-1}} < (1+\nu)^{1/k-2}\,.$$

We next choose the function $\tilde{\delta}_{k-2}(D_{k-2},D_{k-1})$ in a similar way, making sure that

$$\tilde{\delta}_{k-2}(D_{k-2},D_{k-1}) \ll \min\big\{\tilde{\delta}_{k-1}(D_{k-1}), D_{k-2}\big\}\,,$$
$$\tilde{\delta}_{k-2}(D_{k-2},D_{k-1}) \le \delta_{k-2}\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), D_{k-2})\big)\,,$$

and

$$\left(1+\frac{\tilde{\delta}_{k-2}(D_{k-2},D_{k-1})}{D_{k-2}}\right)^{\binom{\ell}{k-2}} < (1+\nu)^{1/k-2}\,.$$

(Since $\tilde{\delta}_{k-1}(D_{k-1})$ is a function of D_{k-1} and $\tilde{\eta}$ was fixed in (5.15) already, we indeed also have that the right-hand sides of the first two inequalities above depend on the variables D_{k-2} and D_{k-1} only.) In general for $j=k-1,\dots,2$ we choose $\tilde{\delta}_j(D_j,\dots,D_{k-1})$ so that

$$\tilde{\delta}_j(D_j,\dots,D_{k-1}) \ll \min\big\{D_j, \tilde{\delta}_{j+1}(D_{j+1},\dots,D_{k-1})\big\}\,,$$
$$\tilde{\delta}_j(D_j,\dots,D_{k-1}) \le \delta_j\big(\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), D_{k-2},\dots$$
$$\dots, \tilde{\delta}_{j+1}(D_{j+1},\dots,D_{k-1}), D_j)\big) \tag{5.17}$$

and

$$\left(1+\frac{\tilde{\delta}_j(D_j,\dots,D_{k-1})}{D_j}\right)^{\binom{\ell}{j}} < (1+\nu)^{1/k-2}\,.$$

We may assume, without any loss of generality, that the functions defined in (5.17) are componentwise monotone decreasing. Since for every $h \geq j+1$ the $\tilde{\delta}_h$ was constructed as a function of $D_h, \dots, D_{k-1}$ only, as before, we may view the right-hand sides of the first two inequalities of (5.17) as a function of $D_j, \dots, D_{k-1}$ only. Consequently, $\tilde{\delta}_j$ is a function of $D_j, \dots, D_{k-1}$, as promised. Furthermore, we set $\tilde{r}(D_2, \dots, D_{k-1})$ to be a componentwise monotone increasing function such that

$$\begin{aligned} &\tilde{r}(D_2, \dots, D_{k-1}) \gg \max\left\{1/D_2, 1/\tilde{\delta}_3(D_3, \dots, D_{k-1})\right\} \qquad \text{and} \\ &\tilde{r}(D_2, \dots, D_{k-1}) \geq r\big(\,\mathbf{IHC}_{k-1,\ell}(\tilde{\eta}, D_{k-1}, \tilde{\delta}_{k-1}(D_{k-1}), \dots D_2)\big)\,. \end{aligned} \tag{5.18}$$

As a result of Lemma 4.15 applied to $\boldsymbol{d} = (d_2, \dots, d_k)$, $\delta_3, \dots, \delta_k$, $\tilde{\delta}_k$, and the functions $\tilde{\delta}_{k-1}(D_{k-1}), \dots, \tilde{\delta}_2(D_2, \dots, D_{k-1})$ and $\tilde{r}(D_2, \dots, D_{k-1})$ we obtain integers $\tilde{n}_k$, $\tilde{L}_k$, a vector of positive reals $\tilde{\boldsymbol{c}} = (\tilde{c}_2, \dots, \tilde{c}_{k-1})$ and a constant $\delta_2^{\text{Lem.4.15}}$. (Here we did not use the variable B_1 for the function $\tilde{r}(D_2, \dots, D_{k-1})$.) Next, we disclose δ_2 and r promised by Theorem 3.6. For that we apply the functions $\tilde{\delta}_2(D_2, \dots, D_{k-1})$ and $\tilde{r}(D_2, \dots, D_{k-1})$, defined in (5.17) and (5.18), to $\tilde{\boldsymbol{c}}$. We set δ_2 and r so that

$$\begin{aligned} \big(\tilde{L}_k^2\delta_2\big) \ll \min\left\{\tilde{\delta}_2(\tilde{\boldsymbol{c}}), 1/\tilde{r}(\tilde{\boldsymbol{c}})\right\} \\ \delta_2 \leq \delta_2\big(\,\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \dots, \delta_3, d_2)\big)\,, \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} r &\gg \max\left\{1/\tilde{\delta}_2(\tilde{\boldsymbol{c}}), \tilde{r}(\tilde{\boldsymbol{c}}), 2^\ell \tilde{L}_k^k\right\} \\ r &\geq r\big(\,\mathbf{IHC}_{k-1,\ell}(\eta, d_{k-1}, \delta_{k-1}, d_{k-2}, \dots, \delta_3, d_2)\big)\,. \end{aligned} \tag{5.20}$$

Finally, we set n_0 so that

$$n_0 \gg \max\left\{\tilde{n}_k,\ 1/\delta_2,\ r,\ m_0,\ \tilde{L}_k m_{k-1,\ell},\ \tilde{L}_k \tilde{m}_{k-1,\ell}\right\}, \tag{5.21}$$

where

$$m_0 = \max\left\{m_0\big(\text{Thm.4.1}(d_2, \dots, d_k - \delta_k^{1/2})\big), m_0\big(\text{Thm.4.1}(d_2, \dots, d_k + \delta_k^{1/2})\big)\right\}$$

is given by Theorem 4.1 applied to $d_2,\dots,d_{k-1}$ and $d_k-\delta_k^{1/2}$ and $d_k+\delta_k^{1/2}$, respectively, and similarly

$$\begin{aligned}m_{k-1,\ell} &= m_{k-1,\ell}\big(\mathbf{IHC}(\eta,d_{k-1},\delta_{k-1},\dots,d_2,\delta_2)\big) \quad \text{and}\\ \tilde m_{k-1,\ell} &= m_{k-1,\ell}\big(\mathbf{IHC}(\tilde\eta,\tilde c_{k-1},\tilde\delta_{k-1}(\tilde c_{k-1}),\dots,\tilde c_2,\tilde\delta_2(\tilde{\boldsymbol{c}}))\big)\end{aligned}$$

come from Statement 5.1.

We now defined all constants involved in the statement of Theorem 5.3. Moreover, we defined the functions and constants needed for Lemma 4.15. This brings us to the next part of the proof, Cleaning Phase I.

5.4.2 Cleaning Phase I

Let $\boldsymbol{\mathcal{G}}=\{\mathcal{G}^{(j)}\}_{j=1}^k$ be a $(\boldsymbol{\delta},\boldsymbol{d},r)$-regular (n,ℓ,k)-complex where $n\ge n_0$ and $\boldsymbol{\delta}=(\delta_2,\dots,\delta_k)$, $\boldsymbol{d}=(d_2,\dots,d_k)$ and r are chosen as described in Section 5.4.1. We apply Lemma 4.15 to $\boldsymbol{\mathcal{G}}$ with the constant $\tilde\delta_k$, the functions $\tilde{\boldsymbol{\delta}}(\boldsymbol{D})=(\tilde\delta_{k-1}(D_{k-1}),\dots,\tilde\delta_2(D_2,\dots,D_{k-1}))$ and the function $\tilde r(\boldsymbol{D})$ as given in (5.15), (5.17) and (5.18). Lemma 4.15 renders an (n,ℓ,k)-complex $\tilde{\boldsymbol{\mathcal{G}}}=\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$, a real vector of positive coordinates $\tilde{\boldsymbol{d}}=(\tilde d_2,\dots,\tilde d_{k-1})$ componentwise bigger than $\tilde{\boldsymbol{c}}$ and an almost perfect $(\tilde{\boldsymbol{\delta}}(\tilde{\boldsymbol{d}}),\tilde{\boldsymbol{d}},\tilde r,\boldsymbol{b})$-family of partitions $\mathscr{P}=\mathscr{P}(k-1,\bar{\boldsymbol{b}},\boldsymbol{\psi})$ refining $\tilde{\boldsymbol{\mathcal{G}}}$ (cf. Definition 4.12 and Definition 4.14). Note that the choice of r in (5.20) and (v) of Lemma 4.15 ensures that for $2\le j\le k$,

$$r\ge 2^\ell\tilde L_k^k\ge 2^\ell\big|\hat A(k-1,\boldsymbol{b})\big|\ge\big|\hat A(j-1,\boldsymbol{b})\big|\,. \tag{5.22}$$

For $2\le j\le k-1$, we finally fix the constants

$$\tilde\delta_j=\tilde\delta_j(\tilde d_j,\dots,\tilde d_{k-1}) \qquad\text{and}\qquad \tilde r=\tilde r(\tilde{\boldsymbol{d}})\,.$$

From the monotonicity of the functions $\tilde\delta_2$ and $\tilde r$, we infer

$$\tilde\delta_2=\tilde\delta_2(\tilde{\boldsymbol{d}})\ge\tilde\delta_2(\tilde{\boldsymbol{c}})\gg\delta_2 \quad\text{and}\quad \tilde r=\tilde r(\tilde{\boldsymbol{d}})\le\tilde r(\tilde{\boldsymbol{c}})\ll r\,. \tag{5.23}$$

$$
\begin{array}{ccccccccccccc}
 & & d_k & & \dots & & d_3 & & d_2, \varepsilon & & & & & & \\
 & & \vee\!\!\vee & & \dots & & \vee\!\!\vee & & \vee\!\!\vee & & & & & & \\
\gamma & \gg & \delta_k & \gg & \cdots & \gg & \delta_3 & \gg & \varepsilon' & & & & & & \\
 & & & & & & & & \vee\!\!\vee & & & & & & \\
 & & & & & & & & \tilde{\eta} & & & & & & \\
 & & & & & & & & \vee\!\!\vee & & & & & & \\
 & & & & & & & & \nu & & \tilde{d}_{k-1} & & \dots & & \tilde{d}_3 & & \tilde{d}_2 \\
 & & & & & & & & \vee\!\!\vee & & \vee\!\!\vee & & \dots & & \vee\!\!\vee & & \vee\!\!\vee \\
 & & & & & & & & \tilde{\delta}_k & \gg & \tilde{\delta}_{k-1} & \gg & \cdots & \gg & \tilde{\delta}_3 & \gg & \tilde{\delta}_2, \frac{1}{\tilde{r}} \\
 & & & & & & & & & & & & & & & & \vee\!\!\vee \\
 & & & & & & & & & & & & & & & & \tilde{L}_k^2 \delta_2, \frac{1}{r}
\end{array}
$$

Figure 5.2: Flowchart of the constants

For future reference, we summarize, in Figure 5.2, (5.11)–(5.20) and(5.23). For the remainder of the proof of Theorem 5.3 , all constants are fixed as summarized above in Figure 5.2.

Observe that by the choice of the functions $\tilde{\delta}_j$ in (5.17) and part (i) of Definition 4.14, we have for every $2 \leq j < k$ and $j < i \leq \ell$

$$
\prod_{h=2}^{j} (\tilde{d}_h b_h)^{\binom{i}{h}} = \prod_{h=2}^{j} \left(1 \pm \tilde{\delta}_h / \tilde{d}_h\right)^{\binom{i}{h}} = \prod_{h=2}^{j} (1 \pm \nu)^{1/k-2} = 1 \pm \nu \,. \tag{5.24}
$$

Remark 5.4. *Observe that the last two "equality signs" in (5.24) are used in a non-symmetric way. For example, the last equality sign abbreviates the validity of the two inequalities*

$$
(1-\nu) \leq \prod_{h=2}^{j} (1-\nu)^{1/k-2} \qquad \text{and} \qquad \prod_{h=2}^{j} (1+\nu)^{1/k-2} \leq (1+\nu) \,.
$$

We will use this notation occasionally in the calculations throughout this dissertation.

Part (iii) of Lemma 4.15 bounds the difference of the number of $K_\ell^{(k)}$'s in $\mathcal{G}^{(k)}$ and $\tilde{\mathcal{G}}^{(k)}$ by

$$\tilde{\delta}_k \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell \ll \delta_k \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell \,. \tag{5.25}$$

For future reference, we summarize the results of Cleaning Phase I.

Setup 5.5 (After Cleaning Phase I)**.** *Let all constants be chosen as summarized in Figure 5.2 so that also* (5.22) *and* (5.24) *hold. Let* $\boldsymbol{\mathcal{G}}$ *be the* $(\boldsymbol{\delta}, \boldsymbol{d}, r)$*-regular* (n, ℓ, k)*-complex from the input of the Counting Lemma, Theorem 3.6. Let* $\tilde{\boldsymbol{\mathcal{G}}}$ *be the* (n, ℓ, k)*-complex and* $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ *be the almost perfect* $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$*-family of partitions refining* $\tilde{\boldsymbol{\mathcal{G}}}$ *given after Cleaning Phase I, i.e., after an application of Lemma 4.15.*

We now mention a few comments to motivate our next step in the proof. The family of partitions $\mathscr{P}$ given by Lemma 4.15 (cf. Setup 5.5) is an almost perfect family (cf. Definition 4.14); moreover, by (i), $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$. However, while every component of the partition is regular, it is possible that the densities $d\big(\tilde{\mathcal{G}}^{(j)} \big| \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$ may vary across different $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b})$ for which $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq \tilde{\mathcal{G}}^{(j-1)}$.

The goal of the next cleaning phase is to alter $\tilde{\boldsymbol{\mathcal{G}}}$ to form a complex $\boldsymbol{\mathcal{F}}$ where all densities are appropriately uniform. Importantly, we show that the two complexes $\tilde{\boldsymbol{\mathcal{G}}}$ and $\boldsymbol{\mathcal{F}}$ share mostly all their respective cliques. (For technical reasons, we will also need to construct two auxiliary complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$)

5.4.3 Cleaning Phase II

The aim of this section is to construct the complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$ which is promised by Theorem 5.3. For the proof of part (iii) of Theorem 5.3, we define two auxiliary complexes $\boldsymbol{\mathcal{H}}_+ = \{\mathcal{H}_+^{(j)}\}_{j=1}^k$ and $\boldsymbol{\mathcal{H}}_- = \{\mathcal{H}_-^{(j)}\}_{j=1}^k$. Later, in the final phase (see Section 5.4.4), our goal is to apply the Dense Counting Lemma to these auxiliary complexes.

The construction of $\boldsymbol{\mathcal{H}}_+$, $\boldsymbol{\mathcal{H}}_-$ and $\boldsymbol{\mathcal{F}}$ splits into two parts. First (cf. upcoming Lemma 5.6), we construct an auxiliary (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^k$ which will have the required properties for $1 \leq j < k$ (we have $\mathcal{H}_+^{(j)} = \mathcal{H}_-^{(j)} = \mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $1 \leq j < k$).

In the second part, we use upcoming Lemma 5.7 to overcome a 'slight imperfection' of $\mathcal{H}^{(k)}$ and construct $\mathcal{H}_+^{(k)}$, $\mathcal{H}_-^{(k)}$, and $\mathcal{F}^{(k)}$ so that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ (as we will later show in Lemma 5.9) satisfy the assumptions of the Dense Counting Lemma. Moreover, $\mathcal{H}_+^{(k)}$ and $\mathcal{H}_-^{(k)}$ will "sandwich" $\mathcal{F}^{(k)}$ and $\mathcal{H}^{(k)}$ (i.e. $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$).

We need the following definition in order to state Lemma 5.6. For a $(j-1)$-uniform hypergraph $\mathcal{H}^{(j-1)}$, we denote by $\hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right) \subseteq \hat{A}(j-1, \boldsymbol{b})$ the set of polyad addresses $\hat{\boldsymbol{x}}^{(j-1)}$ such that

$$\hat{\mathcal{P}}^{(j-1)}\left(\hat{\boldsymbol{x}}^{(j-1)}\right) \subseteq \mathcal{H}^{(j-1)}\,. \tag{5.26}$$

Lemma 5.6 (Cleaning Phase II, Part 1)**.** *Given Setup 5.5, there exists an (n, ℓ, k)-complex $\boldsymbol{\mathcal{H}} = \left\{\mathcal{H}^{(j)}\right\}_{j=1}^k$ such that:*

(*a*) $\mathcal{H}^{(1)} = \tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$ *(and consequently $\hat{A}\left(\mathcal{H}^{(1)}, 1, \boldsymbol{b}\right) = \hat{A}\left(1, \boldsymbol{b}\right)$) and $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$.*

(*b*) *For every $2 \leq j < k$, the following holds:*

(*b1*) *For any $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, there exist a set of indices $I\left(\hat{\boldsymbol{x}}^{(j-1)}\right) \subseteq [b_j]$ of size $\left|I\left(\hat{\boldsymbol{x}}^{(j-1)}\right)\right| = d_j b_j$ such that*

$$\mathcal{H}^{(j)} \cap \mathcal{K}_j^{(j-1)}\left(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\right) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\left(\left(\hat{\boldsymbol{x}}^{(j-1)}, \alpha\right)\right)\,.$$

(*b2*) *For every* $j \le i \le \ell$,

$$\left|\mathcal{K}_i^{(j)}\left(\mathcal{H}^{(j)}\right) \triangle \mathcal{K}_i^{(j)}\left(\tilde{\mathcal{G}}^{(j)}\right)\right| \le \delta_j^{1/3}\left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i \,.$$

(*c*) *Finally, the* (n,ℓ,k)*-cylinder* $\mathcal{H}^{(k)}$ *satisfies the following two properties:*

(*c1*) *For every* $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\big(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\big)$, $\mathcal{H}^{(k)}$ *is* $\big(\tilde{\delta}_k, \bar{d}_k(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r}\big)$*-regular w.r.t.* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ *where* $\bar{d}_k\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = d_k \pm \sqrt{\delta_k}$.

(*c2*)

$$\left|\mathcal{K}_\ell^{(k)}\left(\mathcal{H}^{(k)}\right) \triangle \mathcal{K}_\ell^{(k)}\left(\tilde{\mathcal{G}}^{(k)}\right)\right| \le \delta_k^{1/3}\left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell \,.$$

We prove Lemma 5.6 in Section 7.2.

Consider the subcomplex $\boldsymbol{\mathcal{H}}^{(\boldsymbol{k-1})} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1}$. The complex $\boldsymbol{\mathcal{H}}^{(\boldsymbol{k-1})}$ is 'absolutely perfect' by having the following two properties for every $2 \le j < k$:

perfectly equitable (*PE*) For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ and every $\beta \in I(\hat{\boldsymbol{x}}^{(j-1)})$, the $(n/b_1, j, j)$-cylinder $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\big)$ is $(\tilde{\delta}_j, \tilde{d}_j, \tilde{r})$-regular with respect to its underlying polyad $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

uniformly dense (*UD*) For every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$,

$$d\big(\mathcal{H}^{(j)}\big|\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = (d_j b_j)(\tilde{d}_j \pm \tilde{\delta}_j)\,. \tag{5.27}$$

The property (*PE*) is an immediate consequence of the fact that $\mathscr{P}$ is an almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions. Property (*UD*) easily follows from (*b1*) combined with (*PE*).

We now rewrite the right-hand side of (5.27) in a more convenient form (cf. (5.28)). Using (*i*) from Definition 4.14, we infer

$$d\big(\mathcal{H}^{(j)}\big|\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = d_j(1 \pm \tilde{\delta}_j/\tilde{d}_j) \pm b_j\tilde{\delta}_j = d_j \pm (\tilde{\delta}_j/\tilde{d}_j + b_j\tilde{\delta}_j)\,.$$

As a consequence of Definition 4.14 (i) and $\tilde{d}_j > \tilde{\delta}_j$, we have $b_j < 2/\tilde{d}_j$. Due to the choice of the constants (cf. Figure 5.2) $\tilde{\delta}_j \ll \tilde{d}_j$. We therefore infer

$$d\big(\mathcal{H}^{(j)}\big|\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = d_j \pm (\tilde{\delta}_j/\tilde{d}_j + b_j\tilde{\delta}_j) = d_j \pm 3\tilde{\delta}_j/\tilde{d}_j = d_j \pm \sqrt{\tilde{\delta}_j}\,. \quad (5.28)$$

For each $2 \leq j < k$ consider $\mathcal{H}^{(j)}$ as the union

$$\mathcal{H}^{(j)} = \bigcup \big\{\mathcal{H}^{(j)} \cap \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \colon\ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})\big\}\,.$$

From property (PE) and (5.28) we will infer that $\mathcal{H}^{(j)}$ is $(\tilde{\delta}_j^{1/3}, \tilde{d}_j, 1)$-regular (and, therefore, also $(\varepsilon', \tilde{d}_j, 1)$-regular) w.r.t. $\mathcal{H}^{(j-1)}$ (see proof of Lemma 5.9 in Section 8.2). This means, however, that the complex $\boldsymbol{\mathcal{H}}^{(\boldsymbol{k-1})}$ is 'ready' for an application of the Dense Counting Lemma, Theorem 4.1.

The proof of $(b2)$ is based on the induction assumption (cf. $\mathbf{IHC}_{k-1,\ell}$). The treatment of $\mathcal{H}^{(k)}$ will necessarily have to be different. We shall construct two (n,ℓ,k)-cylinders $\mathcal{H}_+^{(k)}$ and $\mathcal{H}_-^{(k)}$ so that $\mathcal{H}_+^{(k)} \supseteq \mathcal{H}^{(k)} \supseteq \mathcal{H}_-^{(k)}$. Moreover, we construct $\mathcal{F}^{(k)}$, incomparable with respect to $\mathcal{H}^{(k)}$, but with $\mathcal{H}_+^{(k)} \supseteq \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$. To this end, we use the following lemma whose proof we defer to Section 7.3.

Lemma 5.7 (Cleaning Phase II, Part 2)**.** *Given Setup 5.5 and the (n,ℓ,k)-complex $\boldsymbol{\mathcal{H}}$ from Part 1 of Cleaning Phase II, Lemma 5.6, there are (n,ℓ,k)-cylinders $\mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}$ such that:*

- (α) $\boldsymbol{\mathcal{H}}_- = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \mathcal{H}_-^{(k)}$, $\boldsymbol{\mathcal{H}}_+ = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \mathcal{H}_+^{(k)}$, *and* $\boldsymbol{\mathcal{F}} = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1} \cup \mathcal{F}^{(k)}$ *are (n,ℓ,k)-complexes and* $\mathcal{H}_-^{(k)} \subseteq \mathcal{H}^{(k)} \subseteq \mathcal{H}_+^{(k)}$.
- (β) *For every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\left(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\right)$, the following holds:*
  - $(\beta 1)$ $\mathcal{H}_-^{(k)}$ *is* $\big(3\tilde{\delta}_k, d_k - \sqrt{\delta_k}, \tilde{r}\big)$*-regular w.r.t. to* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ *and*
  - $(\beta 2)$ $\mathcal{H}_+^{(k)}$ *is* $\big(3\tilde{\delta}_k, d_k + \sqrt{\delta_k}, \tilde{r}\big)$*-regular w.r.t. to* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$,
  - $(\beta 3)$ $\mathcal{F}^{(k)}$ *is* $\big(21\tilde{\delta}_k, d_k, \tilde{r}\big)$*-regular w.r.t. to* $\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$.

Cleaning Phase II is now concluded. For future reference, we summarize the effects of Cleaning Phase II.

Setup 5.8 (After Cleaning Phase II)**.** *Let all constants be chosen as summarized in Figure 5.2 so that* (5.22) *and* (5.24) *hold.*

- *Let* $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ *be the almost perfect* $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$*-family of partitions given after Cleaning Phase I, i.e., after an application of Lemma 4.15.*

- *Let* $\boldsymbol{\mathcal{H}}$ *be the* (n, ℓ, k)*-complex given from Part 1 of Cleaning Phase II, Lemma 5.6. For every* $2 \leq j < k$ *and* $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$*, let* $I(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq [b_j]$ *be the index set satisfying* (*b1*) *of Lemma 5.6.*

- *Moreover, let* $\boldsymbol{\mathcal{H}}_+$*,* $\boldsymbol{\mathcal{H}}_-$*, and* $\boldsymbol{\mathcal{F}}$ *be the* (n, ℓ, k)*-complexes given by Part 2 of Cleaning Phase II, Lemma 5.7.*

5.4.4 The Final Phase

We now finish the proof Theorem 5.3. The first goal is to show that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ satisfy the assumptions of the Dense Counting Lemma. To this end, we use the upcoming Union Lemma, Lemma 5.9, stated below. After stating the Union Lemma, we finish the proof of Theorem 5.3.

Lemma 5.9 (Union lemma)**.** *Given Setup 5.8 and let* $* \in \{+, -\}$*, the complex* $\boldsymbol{\mathcal{H}}_*$ *is* $(\boldsymbol{\varepsilon'}, \boldsymbol{d}_*, 1)$*-regular where* $\boldsymbol{\varepsilon'} = (\varepsilon', \dots, \varepsilon') \in \mathbb{R}^{k-1}$ *and* $\boldsymbol{d}_* = (d_2^*, \dots, d_k^*)$ *with*

$$d_j^* = \begin{cases} d_j & \text{if } \ 2 \leq j \leq k-1 \\ d_k + \sqrt{\delta_k} & \text{if } \ j = k \ \text{ and } * = + \\ d_k - \sqrt{\delta_k} & \text{if } \ j = k \ \text{ and } * = - \,. \end{cases} \tag{5.29}$$

Similarly, the (n, ℓ, k)*-complex* $\boldsymbol{\mathcal{F}} = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \cup \mathcal{F}^{(k)}$ *is* $(\boldsymbol{\varepsilon'}, \boldsymbol{d}, 1)$*-regular where* $\boldsymbol{\varepsilon'} = (\varepsilon', \dots, \varepsilon') \in \mathbb{R}^{k-1}$ *and* $\boldsymbol{d} = (d_2, \dots, d_k)$*.*

We give the proof of Lemma 5.9 in Chapter 8. We now finish this section with the proof of Theorem 5.3.

Proof of Theorem 5.3. Set $\mathcal{F}^{(j)} = \mathcal{H}^{(j)}$ for $1 \leq j < k$ and let $\mathcal{F}^{(k)}$ be given by Lemma 5.7. Consequently, $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$ is an (n, ℓ, k)-complex and Lemma 5.9 gives (i) of Theorem 5.3. Moreover, due to part (a) of Lemma 5.6, we have $\mathcal{G}^{(1)} = \mathcal{H}^{(1)} = \mathcal{F}^{(1)}$ and $\mathcal{G}^{(2)} = \mathcal{H}^{(2)} = \mathcal{F}^{(2)}$ which yields (ii) of Theorem 5.3. It is left to verify part (iii) of the theorem.

As an intermediate step, we first consider $\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})$. Since $\mathcal{H}_+^{(k)} \supseteq \mathcal{H}^{(k)} \cup \mathcal{F}^{(k)}$ and $\mathcal{H}^{(k)} \cap \mathcal{F}^{(k)} \supseteq \mathcal{H}_-^{(k)}$ (cf. Lemma 5.7), we have

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_+^{(k)}) \setminus \mathcal{K}_\ell^{(k)}(\mathcal{H}_-^{(k)})\right|. \tag{5.30}$$

We infer from Lemma 5.9 and the choice of ε' in (5.14) and n_0 in (5.21) that $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ satisfy the assumptions of the Dense Counting Lemma, Theorem 4.1. Consequently,

$$\begin{aligned}
\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_+^{(k)})\right| &\leq \left(1 + \sqrt{\delta_k}\right)\left(d_k + \sqrt{\delta_k}\right)^{\binom{\ell}{k}} \prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}} \times n^\ell \\
&\leq \left(1 + \sqrt{\delta_k}\right)\left(1 + 2\binom{\ell}{k}\frac{\sqrt{\delta_k}}{d_k}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell \\
&\leq \left(1 + \delta_k^{1/3}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell.
\end{aligned} \tag{5.31}$$

Similarly,

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}_-^{(k)})\right| \geq \left(1 - \delta_k^{1/3}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{5.32}$$

Therefore, from (5.30), (5.31) and (5.32), we infer

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \leq 2\delta_k^{1/3} \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell\,. \tag{5.33}$$

We now prove (iii) of Theorem 5.3. Using the triangle-inequality, we infer

$$\begin{aligned}\left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \le \left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right| + \\ + \left|\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)})\right| + \\ + \left|\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| .\end{aligned} \tag{5.34}$$

Then (5.25), part $(c2)$ of Lemma 5.6, and (5.33) bound the right-hand side of (5.34) and, hence,

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{G}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\mathcal{F}^{(k)})\right| \le \left(\delta_k + 3\delta_k^{1/3}\right) \prod_{h=2}^{k} d_h^{\binom{\ell}{h}} \times n^\ell . \tag{5.35}$$

Part (iii) of Theorem 5.3 now follows from $\gamma \gg \delta_k$ (cf. Figure 5.2). This concludes the proof of Theorem 5.3. $\square$

Chapter 6

Proof of Cleaning Phase I

The proof of Lemma 4.15 is organized as follows. We first fix all constants involved in the proof (as usual). We then inductively construct the almost perfect family of partitions $\mathscr{P}$ and the complex $\tilde{\boldsymbol{\mathcal{G}}}$ promised by Lemma 4.15. Finally, we verify that $\mathscr{P}$ and $\tilde{\boldsymbol{\mathcal{G}}}$ have the desired properties.

6.1 Constants

Let $\boldsymbol{d} = (d_2, \dots, d_k)$ be a vector of positive reals and let $\delta_3, \dots, \delta_k$ satisfy $0 \le \delta_j \le d_j/2$ for $j = 3, \dots, k$. Moreover, let $\tilde{\delta}_k$ be a positive real and let $\tilde{\boldsymbol{\delta}}(\boldsymbol{D})$ and $\tilde{r}(B_1, \boldsymbol{D})$ be the arbitrary positive functions in variables $\boldsymbol{D} = (D_2, \dots, D_{k-1})$ and B_1 given by the lemma. The proof of Lemma 4.15 relies on the Regularity Lemma, and more specifically, on Corollary 4.13. The proof also relies on the Induction Hypothesis on the Counting Lemma ($\mathbf{IHC}_{k-1,\ell}$), Statement 5.1, with $\ell = k$. Therefore, for the proof of Lemma 4.15 presented here, we assume that $\mathbf{IHC}_{k-1,\ell}$ holds (cf. (5.2)).

We set

$$\eta = \frac{1}{4}, \quad \sigma_j = \begin{cases} d_2 & \text{if} \quad j = 2 \\ 1 & \text{if} \quad 3 \le j \le k-1 \end{cases} \quad \text{and} \quad \delta'_k = \mu = \frac{\tilde{\delta}_k}{2^{\ell+k}} \prod_{h=2}^{k} d_h^{\binom{\ell}{h}}. \tag{6.1}$$

We also fix functions in variables $D_j, \dots, D_{k-1}$ for $j = k-1, \dots, 2$ so that

$$\begin{aligned}
&\delta'_j\left(D_j, \dots, D_{k-1}\right) < \min\left\{\frac{D_j}{18}\tilde{\delta}_j\left(D_j, \dots, D_{k-1}\right), \frac{D_j^3}{36}\right\} < \frac{D_j^2}{9} \\
&\delta'_j\left(D_j, \dots, D_{k-1}\right) < \frac{D_j}{9}\delta_j\big(\mathbf{IHC}_{k-1,\ell}(\eta, D_{k-1}, \delta'_{k-1}(D_{k-1}), D_{k-2}, \dots \\
&\qquad\qquad \dots, D_{j+1}, \delta'_{j+1}(D_{j+1}\dots, D_{k-1}), D_j)\big)\,.
\end{aligned} \tag{6.2}$$

(Observe that the right-hand side of the last inequality is a function in variables $D_j, \dots, D_{k-1}$.) Similarly, we set

$$\begin{aligned}
&r'\left(B_1, \boldsymbol{D}\right) \geq \tilde{r}\left(B_1, \boldsymbol{D}\right), \\
&r'\left(B_1, \boldsymbol{D}\right) \geq r\Big(\mathbf{IHC}_{k-1,\ell}\left(\eta, D_{k-1}, \delta'_{k-1}(D_{k-1}), D_{k-2}, \dots\right. \\
&\qquad\qquad \dots, D_{j+1}, \delta'_{j+1}(D_{j+1}\dots, D_{k-1}), D_j)\Big)
\end{aligned} \tag{6.3}$$

where, without loss of generality, we may assume that the functions given in (6.2) and (6.3) are monotone. Corollary 4.13 then yields the integer constants n_k and L_k. Next we define the constants promised by Lemma 4.15 as follows

$$\begin{aligned}
&\tilde{c}_j = \frac{1}{2^{\ell+2}L_k^k} \text{ for } j = 2, \dots, k-1\,, \quad \tilde{\boldsymbol{c}} = \left(\tilde{c}_2, \dots, \tilde{c}_{k-1}\right), \\
&\tilde{L}_k = 2^{\ell+k^2} L_k^{k-1} \prod_{j=2}^{k-1}\left(\frac{1}{\tilde{c}_j}\right)^{\binom{k-1}{j}} \quad \text{and} \quad \delta_2 = \frac{\delta'_2(\tilde{\boldsymbol{c}})}{\tilde{L}_k^2}\,.
\end{aligned} \tag{6.4}$$

Finally, let $m_{k-1,\ell}$ be the integer given by Statement 5.1 applied to the constants η, $\tilde{c}_{k-1}$, $\delta'_{k-1}(\tilde{c}_{k-1}), \dots, \tilde{c}_2$, $\delta'_2(\tilde{\boldsymbol{c}})$ and set $\tilde{n}_k = \max\{n_k, L_k m_{k-1,\ell}\}$.

6.2 Getting started

Let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ be an (n, ℓ, k)-complex with $n \geq \tilde{n}_k$. We apply Corollary 4.13 to $\boldsymbol{\mathcal{G}}$ to obtain a $\big(\mu, \boldsymbol{\delta'}(\tilde{\boldsymbol{d}}), \tilde{\boldsymbol{d}}, r'(\tilde{\boldsymbol{d}})\big)$-equitable $\big(\delta'_k, r'(\tilde{\boldsymbol{d}})\big)$-regular family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(j)}\}_{j=1}^{k-1}$ refining $\boldsymbol{\mathcal{G}}$ (cf. Definition 4.12) where $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \dots, \tilde{d}_{k-1})$ is the density vector of the partition $\mathscr{R}$. Note that it follows from our choice of σ_j in (6.1) that

$$d_2/\tilde{d}_2 \textit{ and, for all } j = 3, \dots, k-1,\ 1/\tilde{d}_j, \textit{ are integers.} \tag{6.5}$$

We now make a few preparations concerning notation. Having $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \dots, \tilde{d}_{k-1})$ as an outcome of Corollary 4.13, we derive the constants δ'_j, $\tilde{\delta}_j$ for $j = 2, \dots, k-1$ and r' and $\tilde{r}$ from the functions given in (6.2) and (6.3) by setting

$$\begin{aligned} \delta'_j = \delta'_j(\tilde{d}_j, \dots, \tilde{d}_{k-1}) &< \tilde{\delta}_j = \tilde{\delta}_j(\tilde{d}_j, \dots, \tilde{d}_{k-1}) \\ \text{and} \qquad r' = r'(a_1, \tilde{\boldsymbol{d}}) &\geq \tilde{r}(a_1, \tilde{\boldsymbol{d}}) = \tilde{r}\,, \end{aligned}$$

(the inequalities above follow immediately from (6.2) and (6.3)). Moreover, we set $\boldsymbol{\delta'} = (\delta'_2, \dots, \delta'_{k-1})$ and $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \dots, \tilde{\delta}_{k-1})$.

For every $j = 2, \dots, k-1$ and every $\hat{\boldsymbol{y}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{a})$, let $a_j^{\mathrm{reg}} = a_j^{\mathrm{reg}}(\hat{\boldsymbol{y}}^{(j-1)})$ be the number of $(\delta'_j, \tilde{d}_j, r')$-regular $(n/a_1, j, j)$-cylinders belonging to $\hat{\mathcal{R}}^{(j-1)}(\hat{\boldsymbol{y}}^{(j-1)})$. We then observe that

$$a_j^{\mathrm{reg}} = a_j^{\mathrm{reg}}(\hat{\boldsymbol{y}}^{(j-1)}) \leq \frac{1}{\tilde{d}_j - \delta'_j} \leq \frac{2}{\tilde{d}_j}\,. \tag{6.6}$$

Finally, we fix the integer vector $\boldsymbol{b} = (b_1, \dots, b_{k-1})$. We set

$$b_1 = a_1, \quad b_2 = \left\lceil \frac{1}{\tilde{d}_2 + 9\delta'_2/\tilde{d}_2} \right\rceil \leq \frac{2}{\tilde{d}_j}\,, \ \text{ and } \ b_j \overset{(6.5)}{=} \frac{1}{\tilde{d}_j} \text{ for } j = 3, \dots, k-1\,. \tag{6.7}$$

We then define $\bar{\boldsymbol{b}} = (b_1, b_2 + 1, b_3, \dots, b_{k-1})$.

Before we begin constructing the promised almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family of partitions $\mathscr{P} = \mathscr{P}\left(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi}\right) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ (cf. Definition 4.14) and the (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$, we proceed with the following simple observation.

Observation regarding 'bad' j-tuples. Since $\mathscr{R}$ is a $(\mu, \boldsymbol{\delta'}, \tilde{\boldsymbol{d}}, r')$-equitable (δ'_k, r')-regular partition, all but at most μn^k crossing (with respect to $\mathcal{G}^{(1)}$) k-tuples belong to $(\boldsymbol{\delta'}, \tilde{\boldsymbol{d}}, r')$-regular $(n/a_1, k, k-1)$-complexes $\hat{\boldsymbol{\mathcal{R}}}(\hat{\boldsymbol{x}}^{(k-1)}) = \big\{\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(k-1)})\big\}_{j=1}^{k-1}$ given by the family of partitions $\mathscr{R}$. We assert that

$$\begin{gathered}\textit{for each } 2 \le j \le k, \textit{ at most } \mu\binom{k}{j} n^j \textit{ crossing } j\textit{-tuples do not belong to} \\ \big((\delta'_2, \dots, \delta'_{j-1}), (\tilde{d}_2, \dots, \tilde{d}_{j-1}), r'\big)\textit{-regular } (n/a_1, j, j-1)\textit{-complexes of } \mathscr{R}.\end{gathered} \tag{6.8}$$

Indeed, a j-tuple belonging to an irregular $(n/a_1, j, j-1)$-complex can be extended to $\binom{\ell-j}{k-j} n^{k-j}$ crossing k-tuples and at most $\binom{k}{j}$ such j-tuples can be extended to the same k-tuple. Each such k-tuple necessarily belongs to an irregular $(n/a_1, k, k-1)$-complex.

Itinerary. We define complex $\tilde{\boldsymbol{\mathcal{G}}}$ and family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi})$ so that $\mathscr{P}$ is an almost perfect family of partitions refining $\tilde{\boldsymbol{\mathcal{G}}}$. Our plan is to alter the family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ into the family of partitions $\mathscr{P} = \mathscr{P}(k-1, \bar{\boldsymbol{b}}, \boldsymbol{\psi}) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$. The families $\mathscr{P}$ and $\mathscr{R}$ will overlap in the regular elements of $\mathscr{R}$. The elements of $\mathscr{R}$ which are not regular are replaced by random cylinders.

We construct $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ inductively for $j = 1, \dots, k-1$. First set $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$. Since $b_1 = a_1$, we have $A(1, \boldsymbol{a}) = A(1, \boldsymbol{b}) = A(1, \bar{\boldsymbol{b}})$ and $\hat{A}(1, \boldsymbol{a}) = \hat{A}(1, \boldsymbol{b}) = \hat{A}(1, \bar{\boldsymbol{b}})$. We set $\psi_1 \equiv \varphi_1$ and define $\mathscr{P}^{(1)} = \mathscr{R}^{(1)}$. In other words, both $\mathscr{R}$ and $\mathscr{P}$ split the sets V_λ for $\lambda \in [\ell]$ into the same pieces $V_\lambda = V_{\lambda,1} \cup \dots \cup V_{\lambda,b_1}$.

For $2 \le j < k$, we shall define $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ in such a way that the following statement $(\mathcal{C}_j)$ holds:

$(\mathcal{C}_j)$ There is a partition $\mathscr{P}^{(j)} = \mathscr{P}^{(j)}_{\mathrm{orig}} \cup \mathscr{P}^{(j)}_{\mathrm{new}}$ of $K^{(j)}_\ell(V_1, \dots, V_\ell)$ where, for $* \in \{\mathrm{orig}, \mathrm{new}\}$, we define

$$\mathcal{P}^{(j)}_* = \bigcup \{\mathcal{P}^{(j)} \colon \mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_*\},$$

and an (n, ℓ, j)-cylinder $\tilde{\mathcal{G}}^{(j)} \subseteq K_\ell^{(j)}(V_1, \ldots, V_\ell)$ such that (I)–(III) below hold:

(I) $$\mathscr{P}^{(j)}_{\mathrm{orig}} = \Big\{ \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}) \colon \ \boldsymbol{y}^{(j)} \in A(j, \boldsymbol{a}) \text{ and } \boldsymbol{\mathcal{R}}^{(j)}(\boldsymbol{y}^{(j)}) = \{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)})\}_{h=1}^{j} \text{is a } \big((\delta_2', \ldots, \delta_j'), (\tilde{d}_2, \ldots, \tilde{d}_j), r'\big)\text{-regular complex} \Big\},$$

(II) $$\tilde{\mathcal{G}}^{(j)} = \begin{cases} \mathcal{G}^{(2)} & \text{if} \quad j = 2 \\ \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\mathrm{orig}} = \mathcal{G}^{(j)} \setminus \mathcal{P}^{(j)}_{\mathrm{new}} & \text{if} \quad 3 \le j < k \end{cases} \quad \text{and}$$

(III) the family of partitions $\mathscr{P}_j = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(j)}\}$ is an almost perfect $\big((9\delta_2'/\tilde{d}_2, \ldots, 9\delta_j'/\tilde{d}_j), (\tilde{d}_2, \ldots, \tilde{d}_j), r', \boldsymbol{b}\big)$-family.

Before we give an inductive proof of statement $(\mathcal{C}_j)$, we list a few of its consequences in Fact 6.1. The properties *(1)*–*(5)* of Fact 6.1 will be derived directly from $(\mathcal{C}_j)$. They are utilized in our proof, in particular, we use Fact 6.1 with $j-1$ to establish $(\mathcal{C}_j)$.

Fact 6.1 (Consequences of $(\mathcal{C}_j)$)**.** *Let $2 \le j \le k-1$ be fixed. If $(\mathcal{C}_{j'})$ holds for $2 \le j' \le j$ and if $\mathscr{P}^{(2)}$ refines $\tilde{\mathcal{G}}^{(2)}$, then the following is true:*

(1) $\tilde{\mathcal{G}}^{(j)} \subseteq \mathcal{G}^{(j)}$,

(2) $\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{(j)}} = \{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^{j}$ *is an (n, ℓ, j)-complex, and in particular, for each $2 \le h \le j$, $\tilde{\mathcal{G}}^{(h)} \subseteq \mathcal{K}_h^{(h-1)}\big(\tilde{\mathcal{G}}^{(h-1)}\big)$,*

(3) *$\mathscr{P}_j$ refines the complex $\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{(j)}}$,*

(4) *for every $j \le i \le \ell$,*

$$\Big| \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\mathcal{G}^{(j)}) \Big| \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i$$

and

(*5*) *for every* $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \boldsymbol{b})$,

$$\left|\mathcal{K}_{j+1}^{(j)}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)\right| = (1 \pm \eta) \prod_{h=2}^{j} \tilde{d}_h^{\binom{j+1}{h}} \times \left(\frac{n}{b_1}\right)^{j+1} > \frac{(n/b_1)^{j+1}}{\ln(n/b_1)}.$$

Proof of Fact 6.1. Part (*1*) follows clearly from (II). We prove (*2*) by induction on j. For $j = 1$ or 2 there is nothing to prove. Let $j \geq 3$. Suppose $(\mathcal{C}_i)$ is true for $2 \leq i \leq j$ and suppose, by induction, (*2*) holds for $j-1$, i.e, $\tilde{\boldsymbol{\mathcal{G}}}^{\boldsymbol{(j-1)}}$ is an $(n, \ell, j-1)$-complex. We show that every j-tuple $J \in \tilde{\mathcal{G}}^{(j)}$ satisfies $J \in \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$.

Let $J \in \tilde{\mathcal{G}}^{(j)}$ be fixed. It then follows from (II) of $(\mathcal{C}_j)$ that

$$J \in \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\text{orig}}\,. \tag{6.9}$$

We first confirm

$$J \in \mathcal{K}_j^{(j-1)}(\mathcal{P}^{(j-1)}_{\text{orig}})\,. \tag{6.10}$$

To that end, since $J \in \mathcal{P}^{(j)}_{\text{orig}}$, it follows from (I) of $(\mathcal{C}_j)$ that there exists $\boldsymbol{y}^{(j)} \in A(j, \boldsymbol{a})$ such that $J \in \mathcal{R}^{(j)}(\boldsymbol{y}^{(j)})$ and the complex $\boldsymbol{\mathcal{R}}^{(j)}(\boldsymbol{y}^{(j)}) = \{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)})\}_{h=1}^{j}$ is $\big((\delta_2', \dots, \delta_j'), (\tilde{d}_2, \dots, \tilde{d}_j), r'\big)$-regular. Consequently, $J \in \mathcal{K}_j^{(j-1)}\big(\mathcal{R}^{(j-1)}(\boldsymbol{y}^{(j)})\big)$ and by (I) of $(\mathcal{C}_{j-1})$ we have that $\mathcal{R}^{(j-1)}(\boldsymbol{y}^{(j)}) \subseteq \mathcal{P}^{(j-1)}_{\text{orig}}$. This yields $J \in \mathcal{K}_j^{(j-1)}(\mathcal{P}^{(j-1)}_{\text{orig}})$ as claimed in (6.10).

Now from (6.9) and (6.10), we infer that $J \in \mathcal{K}_j^{(j-1)}(\mathcal{G}^{(j-1)} \cap \mathcal{P}^{(j-1)}_{\text{orig}})$ (since $\boldsymbol{\mathcal{G}}$ is a complex), and so by (II) of $(\mathcal{C}_{j-1})$ we have $J \in \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. This completes the proof of (*2*).

Next we show part (*3*), again by induction on j. The statement is trivial for $j = 1$. It holds for $j = 2$ by assumption of Fact 6.1. So let $j \geq 3$ and assume that $\mathscr{P}_{j-1}$ refines $\{\tilde{\mathcal{G}}^{(h)}\}_{j=1}^{j-1}$. We have to show that either $\mathcal{P}^{(j)} \subseteq \tilde{\mathcal{G}}^{(j)}$ or $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$ for every $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$. So let $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ be fixed. If $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\text{new}}$, then $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$ by (II) of $(\mathcal{C}_j)$. Therefore, we may assume that $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\text{orig}}$. Now, if $\mathcal{P}^{(j)} \cap \mathcal{G}^{(j)} = \varnothing$, then again by (II) of $(\mathcal{C}_j)$ we infer $\mathcal{P}^{(j)} \cap \tilde{\mathcal{G}}^{(j)} = \varnothing$. On the other hand, if $\mathcal{P}^{(j)} \cap \mathcal{G}^{(j)} \neq \varnothing$, then $\mathcal{P}^{(j)} \subseteq \mathcal{G}^{(j)}$ since

$\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}_{\rm orig}$, (I) of $(\mathcal{C}_j)$, and the fact that the original family of partitions $\mathscr{R}$ refines the complex $\boldsymbol{\mathcal{G}}$. Therefore, $\mathcal{P}^{(j)} \subseteq \mathcal{G}^{(j)} \cap \mathcal{P}^{(j)}_{\rm orig} = \tilde{\mathcal{G}}^{(j)}$ by *(1)* of Fact 6.1. This verifies *(3)* of Fact 6.1.

Next we show *(4)* of Fact 6.1. From (6.8) and (I) and (II) of $(\mathcal{C}_j)$ we infer that

$$\left|\mathcal{G}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}\right| = \left|\mathcal{G}^{(j)} \setminus \tilde{\mathcal{G}}^{(j)}\right| \le \mu \binom{k}{j} n^j .$$

Consequently, by the choice of μ in (6.1)

$$\left|\mathcal{K}^{(j)}_i(\tilde{\mathcal{G}}^{(j)}) \triangle \mathcal{K}^{(j)}_i(\mathcal{G}^{(j)})\right| \le \mu \binom{k}{j} n^j \times \binom{\ell - j}{i - j} n^{i-j} \le \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i ,$$

which yields *(4)*.

Finally, we note that *(5)* follows from (III) and $\mathbf{IHC}_{k-1,\ell}$ (cf. (5.2)) since $j \le k-1$. In particular, *(5)* is a consequence of the choice of δ'_j and r' in (6.2) and (6.3), (III) of $(\mathcal{C}_{k-1})$, and (5.2). $\square$

6.3 Proof of Statement $(\mathcal{C}_j)$

As mentioned earlier, we verify $(\mathcal{C}_j)$ by induction on j.

6.3.1 The Induction Start

In the immediate sequel, we define $\mathscr{P}^{(2)} = \mathscr{P}^{(2)}_{\rm new} \cup \mathscr{P}^{(2)}_{\rm orig}$ of $K^{(2)}_\ell(V_1, \dots, V_\ell)$. In our construction, we use that due to (6.2) and (6.4), our constants satisfy

$$a_1^2 \delta_2 < L_k^2 \delta_2 < \tilde{L}_k^2 \delta_2 < \delta'_2 < \tilde{d}_2 \le d_2 \tag{6.11}$$

and also use that $d_2/\tilde{d}_2$ is an integer (see (6.5)). Before constructing the partition $\mathscr{P}^{(2)}$, we require some notation.

Notation. Recall that the partition $\mathscr{R}^{(2)} = \{\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) \colon \boldsymbol{y}^{(2)} \in A(2, \boldsymbol{a})\}$ refines the partition $\mathcal{G}^{(2)} \cup \overline{\mathcal{G}^{(2)}}$ (here, $\overline{\mathcal{G}^{(2)}} = K^{(2)}_\ell(V_1, \dots, V_\ell) \setminus \mathcal{G}^{(2)}$). Therefore, for each $\hat{\boldsymbol{y}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a})$, there exist disjoint sets of indices $I_2^{\mathrm{reg}} = I_2^{\mathrm{reg}}(\hat{\boldsymbol{y}}^{(1)})$ and $\overline{I}_2^{\mathrm{reg}} = \overline{I}_2^{\mathrm{reg}}(\hat{\boldsymbol{y}}^{(1)})$ so that $\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\}_{\alpha \in I_2^{\mathrm{reg}}}$ and $\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\}_{\alpha \in \overline{I}_2^{\mathrm{reg}}}$ are the collections of all $(\delta_2', \tilde{d}_2, 1)$-regular graphs $\mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) = \mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)$ whose edge sets are subsets of $\mathcal{G}^{(2)}(\boldsymbol{y}^{(1)}) = \mathcal{G}^{(2)}[V_{\lambda,\beta} \cup V_{\lambda',\beta'}]$ and $\overline{\mathcal{G}^{(2)}}(\boldsymbol{y}^{(1)}) = V_{\lambda,\beta} \times V_{\lambda',\beta'} \setminus \mathcal{G}^{(2)}$, respectively.

Plan for constructing $\mathscr{P}^{(2)}$. We now outline our plan for constructing $\mathscr{P}^{(2)} = \{\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)})\colon \boldsymbol{x}^{(2)} \in A(2, \bar{\boldsymbol{b}})\}$. Later we fill in the technical details. With $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a}) = \hat{A}(1, \bar{\boldsymbol{b}})$ fixed, we define a partition $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ of $\mathcal{K}_2^{(1)}\big(\hat{\mathcal{P}}^{(1)}(\hat{\boldsymbol{x}}^{(1)})\big) = V_{\lambda,\beta} \times V_{\lambda',\beta'}$. More precisely, with $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)}$ defining a pair of sets $V_{\lambda,\beta}$, $V_{\lambda',\beta'}$, we consider all regular subgraphs of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$ from the partition $\mathscr{R}^{(2)}$ and leave them in the "original part" $(\mathscr{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)}))$ of $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$. In other words, for $\hat{\boldsymbol{x}}^{(1)} = \hat{\boldsymbol{y}}^{(1)}$ we set

$$\begin{aligned} &\mathscr{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)}) \\ &\qquad = \Big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\Big\}_{\alpha \in I_2^{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})} \cup \Big\{\mathcal{R}^{(2)}\big((\hat{\boldsymbol{y}}^{(1)}, \alpha)\big)\Big\}_{\alpha \in \overline{I}_2^{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})}. \end{aligned} \tag{6.12}$$

This collection of graphs consist of all subgraphs of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$ belonging to $\mathscr{R}^{(2)}$ which are $(\delta_2', \tilde{d}_2, 1)$-regular. In order to simplify the notation, we set

$$\mathcal{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)}) = \bigcup \Big\{\mathcal{P}^{(2)}\colon\ \mathcal{P}^{(2)} \in \mathscr{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)})\Big\}.$$

For the construction of the partition of $V_{\lambda,\beta} \times V_{\lambda',\beta'} \setminus \mathcal{P}^{(2)}_{\mathrm{orig}}$, we will use the Slicing Lemma to introduce new $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular graphs that do not belong to $\mathscr{R}^{(2)}$. We shall call the collection of those graphs $\mathscr{P}^{(2)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(1)})$. We now provide the technical details to the plan described above.

Technical details for constructing $\mathscr{P}^{(2)}$. Let $\hat{\boldsymbol{x}}^{(1)} = \big((\lambda, \lambda'), (\beta, \beta')\big) \in \hat{A}(1, \boldsymbol{a})$ remain fixed. Let

$$\mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)}) \cap \mathcal{G}^{(2)}$$

be the union of all graphs $\mathcal{P}^{(2)} \subseteq \mathcal{G}^{(2)}$ in $\mathscr{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)})$. Similarly, we define

$$\overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)}) \cap \overline{\mathcal{G}^{(2)}}.$$

Note that while $\overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ and $\mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})$ are disjoint, they are not not necessarily complements of each other. Moreover, observe that $\mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})$ is the union of $\alpha_2^{\mathrm{reg}} = |I_2^{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})| \le a_2^{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)}) \le 2/\tilde{d}_2$ (see (6.6)) $(\delta_2', \tilde{d}_2, 1)$-regular graphs. Consequently, $\mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \alpha_2^{\mathrm{reg}}\tilde{d}_2, 1)$-regular (cf. Proposition 8.1). Similarly, $\overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \overline{\alpha}_2^{\mathrm{reg}}\tilde{d}_2, 1)$-regular, where $\overline{\alpha}_2^{\mathrm{reg}} = |\overline{I}_2^{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})|$.

Since $\mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular by the assumption of Lemma 4.15, we infer that $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \mathcal{G}^{(2)}[V_{\lambda,\beta} \cup V_{\lambda',\beta'}]$ is $(a_1^2\delta_2, d_2, 1)$-regular. Therefore, $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ is $(\delta_2', d_2, 1)$-regular by (6.11). Consequently, since $2\delta_2'/\tilde{d}_2 + \delta_2' \le 3\delta_2'/\tilde{d}_2$ we have that $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})$ is $(3\delta_2'/\tilde{d}_2, d_2 - \overline{\alpha}_2^{\mathrm{reg}}\tilde{d}_2, 1)$-regular. We now apply the Slicing Lemma, Lemma 4.16, to $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})$.

To this end, recall $d_2/\tilde{d}_2$ is an integer (see (6.5)) and set $p = \tilde{d}_2(d_2 - \alpha_2^{\mathrm{reg}}\tilde{d}_2)^{-1}$ so that $1/p = d_2/\tilde{d}_2 - \alpha_2^{\mathrm{reg}}$ is an integer. We apply the Slicing Lemma with $\varrho = d_2 - \alpha_2^{\mathrm{reg}}\tilde{d}_2$, $\delta = 3\delta_2'/\tilde{d}_2$, p as above and $r_{\mathrm{SL}} = 1$ to decompose $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \setminus \mathcal{G}^{(2)}_{\mathrm{reg}}(\hat{\boldsymbol{x}}^{(1)})$ into $1/p = d_2/\tilde{d}_2 - \alpha_2^{\mathrm{reg}}$ pairwise edge-disjoint $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular graphs. Denote the family of these bipartite graphs by $\mathscr{P}^{(2)}_{\mathrm{new},\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$.

The partition $\mathscr{P}^{(2)}_{\mathrm{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$ of $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ will be defined similarly. Indeed, the graph $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$ is $(a_1^2\delta_2, 1 - d_2, 1)$-regular since it is the complement of the $(a_1\delta_2, d_2, 1)$-regular graph $\mathcal{G}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$. By (6.11), the graph $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)})$ is then also $(\delta_2'/\tilde{d}_2, 1-d_2, 1)$-regular. Furthermore, $\overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is $(2\delta_2'/\tilde{d}_2, \overline{\alpha}_2^{\mathrm{reg}}\tilde{d}_2, 1)$-regular (since $\overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is the union of $\overline{\alpha}_2^{\mathrm{reg}}$ disjoint $(\delta_2', \tilde{d}_2, 1)$-regular graphs and $\overline{\alpha}_2^{\mathrm{reg}} \le a_2^{\mathrm{reg}} \le 2/\tilde{d}_2$ by (6.6)). Consequently, $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ is $(3\delta_2'/\tilde{d}_2, 1 - d_2 - \overline{\alpha}_2^{\mathrm{reg}}\tilde{d}_2, 1)$-regular.

We apply the Slicing Lemma with $\varrho = 1 - d_2 - \overline{\alpha}_2^{\mathrm{reg}}\tilde{d}_2$, $\delta = 3\delta_2'/\tilde{d}_2$, $p = \tilde{d}_2/\varrho$ and $r_{\mathrm{SL}} = 1$ to decompose $\overline{\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \setminus \overline{\mathcal{G}^{(2)}_{\mathrm{reg}}}(\hat{\boldsymbol{x}}^{(1)})$ into a family $\mathscr{P}^{(2)}_{\mathrm{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$ of bipartite graphs. We conclude that all but at most one of which are $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular. Indeed, note that since (6.5) guaranteed that $d_2/\tilde{d}_2$ is an integer, we are unable to ensure that $1/p = (1 - d_2 - \overline{\alpha}_2^{\mathrm{reg}}\tilde{d}_2)/\tilde{d}_2$ is an integer as well. Consequently, the application of the Slicing Lemma may admit at most one sparse bipartite graph.

For $\hat{\boldsymbol{x}}^{(1)} = \big((\lambda,\lambda'),(\beta,\beta')\big)$, set

$$\mathscr{P}^{(2)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(1)}) = \mathscr{P}^{(2)}_{\mathrm{new},\mathcal{G}^{(2)}}(\hat{\boldsymbol{x}}^{(1)}) \cup \mathscr{P}^{(2)}_{\mathrm{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)})$$

and

$$\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) = \mathscr{P}^{(2)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(1)}) \cup \mathscr{P}^{(2)}_{\mathrm{orig}}(\hat{\boldsymbol{x}}^{(1)})\,.$$

Also set $z(\hat{\boldsymbol{x}}^{(1)}) = |\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})|$. The partition $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ has the following properties:

(A) $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ is a partition of $V_{\lambda,\beta} \times V_{\lambda',\beta'}$.

(B) $z(\hat{\boldsymbol{x}}^{(1)}) \in \{b_2, b_2+1\}$. Indeed, since all graphs but at most 1 from $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ have density within $\tilde{d}_2 \pm 9\delta_2'/\tilde{d}_2$, it therefore follows that

$$\frac{1}{\tilde{d}_2 + 9\delta_2'/\tilde{d}_2} \le z(\hat{\boldsymbol{x}}^{(1)}) \le \frac{1}{\tilde{d}_2 - 9\delta_2'/\tilde{d}_2} + 1. \tag{6.13}$$

It follows from (6.2) that $9\delta_2'/\tilde{d}_2 < (\tilde{d}_2/2)^2$ yielding $(\tilde{d}_2 - 9\delta_2'/\tilde{d}_2)^{-1} - (\tilde{d}_2 + 9\delta_2'/\tilde{d}_2)^{-1} < 1$. Consequently, $z(\hat{\boldsymbol{x}}^{(1)}) \in \{b_2, b_2+1\}$ follows from (6.7).

(C) $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ refines $\mathcal{G}^{(2)} = \tilde{\mathcal{G}}^{(2)}$ in the sense that for every $\alpha \in [z(\hat{\boldsymbol{x}}^{(1)})]$ either $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)},\alpha)\big) \subseteq \mathcal{G}^{(2)}$ or $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)},\alpha)\big) \cap \mathcal{G} = \varnothing$.

(D) All graphs but at most one from $\mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)})$ are $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, 1)$-regular. Moreover, the exceptional graph belongs to the family $\mathscr{P}^{(2)}_{\mathrm{new},\overline{\mathcal{G}^{(2)}}}(\hat{\boldsymbol{x}}^{(1)}) \subseteq \mathscr{P}^{(2)}_{\mathrm{new}}(\hat{\boldsymbol{x}}^{(1)})$ and we may assume with an appropriate addressing the exceptional graph is always $\mathcal{P}^{(2)}\big((\hat{\boldsymbol{x}}^{(1)}, b_2+1)\big)$.

Now, we set

$$\begin{aligned}\tilde{\mathcal{G}}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) &= \bigcup\Big\{\mathcal{P}^{(2)} \in \mathscr{P}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \colon\ \mathcal{P}^{(2)} \subseteq \mathcal{G}^{(2)}\Big\}\\ \tilde{\mathcal{G}}^{(2)} &= \bigcup\Big\{\tilde{\mathcal{G}}^{(2)}(\hat{\boldsymbol{x}}^{(1)}) \colon\ \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1,\bar{\boldsymbol{b}}) = \hat{A}(1,\boldsymbol{a})\Big\}\end{aligned} \tag{6.14}$$

and we set

$$\mathscr{P}^{(2)}_{\text{new}} = \bigcup \left\{ \mathscr{P}^{(2)}_{\text{new}}(\hat{\boldsymbol{x}}^{(1)})\colon\ \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1, \bar{\boldsymbol{b}}) \right\},$$

$$\mathscr{P}^{(2)}_{\text{orig}} = \bigcup \left\{ \mathscr{P}^{(2)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(1)})\colon\ \hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1, \bar{\boldsymbol{b}}) \right\} \quad \text{and} \quad \mathscr{P}^{(2)} = \mathscr{P}^{(2)}_{\text{new}} \cup \mathscr{P}^{(2)}_{\text{orig}}.$$

It is left to verify (I)–(III) of the statement $(\mathcal{C}_2)$. Due to (6.12) and the definition of $I_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$ and $\overline{I}_2^{\text{reg}}(\hat{\boldsymbol{x}}^{(1)})$, for every $\hat{\boldsymbol{x}}^{(1)} \in \hat{A}(1, \bar{\boldsymbol{b}})$, we infer that

$$\mathscr{P}^{(2)}_{\text{orig}} = \left\{ \mathcal{R}^{(2)}(\boldsymbol{y}^{(2)})\colon\ \mathcal{R}^{(2)}(\boldsymbol{y}^{(2)}) \text{ is } (\delta_2', \tilde{d}_2, 1)\text{-regular} \right\},$$

which yields (I) of $(\mathcal{C}_2)$. Owing to (C) from above and (6.14), we have $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ (which is (II)) and

$$\mathscr{P}^{(2)} \text{ refines } \tilde{\mathcal{G}}^{(2)}. \tag{6.15}$$

Finally, from (B) (cf. (6.13)) and $\delta_2' \le \tilde{d}_2\tilde{\delta}_2/18$ (cf. (6.2)), we infer

$$1 - \frac{\tilde{\delta}_2}{\tilde{d}_2} \overset{(6.2)}{\le} \frac{\tilde{d}_2^2}{\tilde{d}_2^2 + 9\delta_2'} \overset{(6.13)}{\le} \tilde{d}_2 b_2 \overset{(6.13)}{\le} \frac{\tilde{d}_2^2}{\tilde{d}_2^2 - 9\delta_2'} \overset{(6.2)}{\le} 1 + \frac{\tilde{\delta}_2}{\tilde{d}_2}. \tag{6.16}$$

Now, (6.16) and (D) yield that the family of partitions $\mathscr{P}_2 = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ is an almost perfect $(9\delta_2'/\tilde{d}_2, \tilde{d}_2, r', \boldsymbol{b})$-family (see Definition 4.14), which gives part (III) of $(\mathcal{C}_2)$.

We again remind the reader that we choose the addressing of the partition classes $\mathscr{P}^{(2)}$ in such a way that for each $\boldsymbol{x}^{(2)} \in A(2, \boldsymbol{b})$, the graph $\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)})$ is $(9\delta_2', /\tilde{d}_2, \tilde{d}_2, r')$-regular. The graph $\mathcal{P}^{(2)}(\boldsymbol{x}^{(2)})$ may not be $(9\delta_2', /\tilde{d}_2, \tilde{d}_2, r')$-regular if and only if $\boldsymbol{x}^{(2)} \in A(2, \bar{\boldsymbol{b}}) \setminus A(2, \boldsymbol{b})$.

This concludes the construction of $\mathscr{P}^{(2)}$ which satisfies $(\mathcal{C}_2)$ and, therefore, we established the induction start of our construction of $\mathscr{P}$ and $\tilde{\boldsymbol{\mathcal{G}}}$. Also note that we additionally verified (6.15).

6.3.2 The Inductive Step

We proceed to the inductive step and construct partition $\mathscr{P}^{(j+1)}$ and $(n, \ell, j+1)$-cylinder $\tilde{\mathcal{G}}^{(j+1)}$ which will satisfy (I)–(III) of $(\mathcal{C}_{j+1})$. Moreover, we assume that $\mathscr{P}^{(h)}$ and $\tilde{\mathcal{G}}^{(h)}$ satisfying $(\mathcal{C}_h)$, $2 \le h \le j$, are given. Moreover, due to (6.15), we assume Fact 6.1 holds as well for $2 \le h \le j$.

Our work in constructing $\mathscr{P}^{(j+1)}$ will be quite similar, albeit easier, than our work for constructing $\mathscr{P}^{(2)}$. This is in part because we do not require that $\tilde{\mathcal{G}}^{(j+1)} = \mathcal{G}^{(j+1)}$ for $j \ge 2$. It will be necessary to construct $\mathscr{P}^{(j+1)}$ before constructing $\tilde{\mathcal{G}}^{(j+1)}$ as the partition ends up defining the hypergraph.

Construction of $\mathscr{P}^{(j+1)}$ and $\tilde{\mathcal{G}}^{(j+1)}$. We define the partition $\mathscr{P}^{(j+1)} = \mathscr{P}^{(j+1)}_{\text{new}} \cup \mathscr{P}^{(j+1)}_{\text{orig}}$ of $K^{(j+1)}_\ell(V_1, \dots, V_\ell)$ separately for each set $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ of $(j+1)$-tuples with $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \bar{\boldsymbol{b}})$.

Fix $\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \bar{\boldsymbol{b}})$. We define the partition $\mathscr{P}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)}) \cup \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ of $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ by distinguishing three cases.

Case 1 ($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \bar{\boldsymbol{b}}) \setminus \hat{A}(j, \boldsymbol{b})$). Observe that $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ touches at least one of the exceptional graphs from the construction of $\mathscr{P}^{(2)}$. For the sake of consistency only (i.e., the partition $\mathscr{P}^{(j+1)}$ should contain a $(n/b_1, j+1, j+1)$-cylinder $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ for every $\boldsymbol{x}^{(j+1)} \in A(j+1, \bar{\boldsymbol{b}})$), we split $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ arbitrarily into b_{j+1} possibly empty classes. Clearly, all the $(n/b_1, j+1, j+1)$-cylinders $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ constructed in this way satisfy $\boldsymbol{x}^{(j+1)} \in \hat{A}(j+1, \bar{\boldsymbol{b}}) \setminus \hat{A}(j+1, \boldsymbol{b})$. The collection of these b_{j+1} disjoint $(n/b_1, j+1, j+1)$-cylinders defines $\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)})$. We set $\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \varnothing$.

Case 2 ($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j, \boldsymbol{b})$ and there is $1 \le s \le j+1$ s.t. $\mathcal{P}^{(j)}(\partial_s \hat{\boldsymbol{x}}^{(j)}) \in \mathscr{P}^{(j)}_{\text{new}}$). We appeal to (*5*) of Fact 6.1 for j. Indeed, observe that $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ is $(\delta, 1, r)$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ for any positive δ and integer r. Consequently, we may apply the Slicing Lemma, Lemma 4.16, with $\varrho = 1$, $p = \tilde{d}_{j+1}$, $\delta = 3\delta'_{j+1}/\tilde{d}_{j+1}$, and $r_{\text{SL}} = r'$ to $\mathcal{F}^{(j+1)} = \mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$. (Observe that $3\delta = 9\delta'/\tilde{d}_{j+1} < \tilde{d}_{j+1} = p\varrho$ by (6.2).) Since $1/p = 1/\tilde{d}_{j+1} = b_{j+1}$ by (6.7), we obtain a collection of $1/\tilde{d}_{j+1}$ pairwise edge-disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders $\mathcal{P}^{(j+1)}\big(\hat{\boldsymbol{x}}^{(j)}, \alpha)\big)$ with $\alpha \in [b_{j+1}]$. Denote by

$$\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)}) = \Big\{\mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j)},\alpha)\big)\colon\ \alpha \in [b_{j+1}]\Big\}$$

the family of $(n/b_1, j{+}1, j{+}1)$-cylinders newly created. Set $\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) = \varnothing$. This concludes our treatment of Case 2.

Case 3 ($\hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\boldsymbol{b})$ and $\mathcal{P}^{(j)}(\partial_s \hat{\boldsymbol{x}}^{(j)}) \in \mathscr{P}^{(j)}_{\text{orig}}$ for every $1 \le s \le j+1$)**.** By the assumption of this case and (I) of $(\mathcal{C}_j)$, we infer that there exists $\hat{\boldsymbol{y}}^{(j)} \in \hat{A}(j,\boldsymbol{a})$ such that $\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{y}}^{(j)}) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$. Recall the definition of $a^{\text{reg}}_{j+1} = a^{\text{reg}}_{j+1}(\hat{\boldsymbol{y}}^{(j)})$ (preceding (6.6)). Without loss of generality, let $\big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\big\}_{\alpha\in[a^{\text{reg}}_{j+1}]}$ be an enumeration of the $(\delta'_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders (regular w.r.t. $\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{y}}^{(j)}) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$). We set

$$\begin{aligned}\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) &= \Big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\Big\}_{\alpha\in[a^{\text{reg}}_{j+1}]} \qquad \text{and}\\ \mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)}) &= \bigcup\Big\{\mathcal{P}^{(j+1)}\colon\ \mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})\Big\} \\ &= \bigcup\Big\{\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)},\alpha)\big)\colon\ \alpha \in [a^{\text{reg}}_{j+1}]\Big\}.\end{aligned} \tag{6.17}$$

Observe that $\mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ is $(a^{\text{reg}}_{j+1}\delta'_{j+1}, a^{\text{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$ (cf. Proposition 8.1) and, as a consequence of (6.6), also

$$(3\delta'_{j+1}/\tilde{d}_{j+1}, a^{\text{reg}}_{j+1}\tilde{d}_{j+1}, r')\text{-regular}\,.$$

Then, $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big) \setminus \mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ is $(3\delta'_{j+1}/\tilde{d}_{j+1}, 1 - a^{\text{reg}}_{j+1}\tilde{d}_{j+1}, r')$-regular. We apply the Slicing Lemma to $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big) \setminus \mathcal{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ with $\varrho = 1 - a^{\text{reg}}_{j+1}\tilde{d}_{j+1}$, $p = \tilde{d}_{j+1}/\varrho$, $\delta = 3\delta'_{j+1}/\tilde{d}_{j+1}$ (yielding $3\delta < p\varrho$ by (6.2)) and $r_{\text{SL}} = r'$. Note that $1/p = \varrho/\tilde{d}_{j+1} = 1/\tilde{d}_{j+1} - a^{\text{reg}}_{j+1}$ is an integer by (6.5). We thus obtain collection $\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)})$ of $1/\tilde{d}_{j+1} - a^{\text{reg}}_{j+1}$ pairwise edge-disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders $\mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j)},\alpha)\big)$. Setting $\mathscr{P}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) = \mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)}) \cup \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})$ yields a partition of $\mathcal{K}^{(j)}_{j+1}\big(\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})\big)$ into $1/\tilde{d}_{j+1} - a^{\text{reg}}_{j+1} + a^{\text{reg}}_{j+1} = 1/\tilde{d}_{j+1} = b_{j+1}$ (by (6.7)) disjoint $(9\delta'_{j+1}/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders. This concludes our treatment of Case 3.

Now, we set

$$\begin{aligned}\tilde{\mathcal{G}}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)}) &= \bigcup\left\{\mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})\colon\ \mathcal{P}^{(j+1)} \subseteq \mathcal{G}^{(j+1)}\right\}\\ \tilde{\mathcal{G}}^{(j+1)} &= \bigcup\left\{\tilde{\mathcal{G}}^{(j+1)}(\hat{\boldsymbol{x}}^{(j)})\colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\bar{\boldsymbol{b}})\right\}\end{aligned} \tag{6.18}$$

and we set

$$\begin{aligned}\mathscr{P}^{(j+1)}_{\text{new}} &= \bigcup\left\{\mathscr{P}^{(j+1)}_{\text{new}}(\hat{\boldsymbol{x}}^{(j)})\colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\bar{\boldsymbol{b}})\right\},\\ \mathscr{P}^{(j+1)}_{\text{orig}} &= \bigcup\left\{\mathscr{P}^{(j+1)}_{\text{orig}}(\hat{\boldsymbol{x}}^{(j)})\colon\ \hat{\boldsymbol{x}}^{(j)} \in \hat{A}(j,\bar{\boldsymbol{b}})\right\},\\ \mathscr{P}^{(j+1)} &= \mathscr{P}^{(j+1)}_{\text{new}} \cup \mathscr{P}^{(j+1)}_{\text{orig}}\,.\end{aligned}$$

It is left to verify (I)–(III) of statement $(\mathcal{C}_{j+1})$.

Confirmation of $(\mathcal{C}_{j+1})$**.** First we verify (I). To this end, we establish the equality of sets in (I) by decomposing the equality into its respective '$\subseteq$' and '$\supseteq$' parts, and begin by considering the former.

We verify the '$\subseteq$' component of equality of sets in (I) of $(\mathcal{C}_{j+1})$. Let $\mathcal{P}^{(j+1)} = \mathcal{P}^{(j+1)}\big((\hat{\boldsymbol{x}}^{(j+1)},\alpha)\big) \in \mathscr{P}^{(j+1)}_{\text{orig}}$. Owing to the construction of $\mathscr{P}^{(j+1)}$ above, $\mathcal{P}^{(j+1)}$ originates from Case 3. By the assumption of Case 3, we know that $\mathcal{P}^{(j)}(\partial_s\hat{\boldsymbol{x}}^{(j+1)}) \in \mathscr{P}^{(j)}_{\text{orig}}$ for every $s \in [j+1]$. Consequently, from (I) of $(\mathcal{C}_j)$ we infer that for each $s \in [j+1]$, there exists $\boldsymbol{y}^{(j)}_s$ such that $\boldsymbol{\mathcal{R}}^{(j)}(\boldsymbol{y}^{(j)}_s) = \big\{\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)}_s)\big\}_{h=1}^{j}$ is a $\big((\delta'_2,\dots,\delta'_j),(\tilde{d}_2,\dots,\tilde{d}_j),r'\big)$-regular $(n/a_1,j,j)$-complex and $\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s) = \mathcal{P}^{(j)}(\partial_s\hat{\boldsymbol{x}}^{(j+1)})$. Clearly,

$$\left\{\bigcup_{s\in[j+1]}\mathcal{R}^{(h)}(\boldsymbol{y}^{(j)}_s)\right\}_{h=1}^{j} \text{ is } \big((\delta'_2,\dots,\delta'_j),(\tilde{d}_2,\dots,\tilde{d}_j),r'\big)\text{-regular} \tag{6.19}$$

and $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)}) = \bigcup_{s\in[j+1]}\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s)$. Moreover, by the construction in Case 3 and $\mathcal{P}^{(j+1)} \in \mathscr{P}^{(j+1)}_{\text{orig}}$, there exists $\mathcal{R}^{(j+1)} \in \mathscr{R}^{(j+1)}$ such that $\mathcal{P}^{(j+1)} = \mathcal{R}^{(j+1)}$ and $\mathcal{R}^{(j+1)}$ is $(\delta'_{j+1},\tilde{d}_{j+1},r')$-regular w.r.t. $\bigcup_{s\in[j+1]}\mathcal{R}^{(j)}(\boldsymbol{y}^{(j)}_s) = \hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j)})$. Then (6.19) yields the '$\subseteq$' component of the equality in (I) of $(\mathcal{C}_{j+1})$.

We now verify the '$\supseteq$' component of the equality in (I). To that end, let $\hat{\boldsymbol{y}}^{(j)} \in \hat{A}(j, \boldsymbol{a})$ and $\alpha \in [a_{j+1}]$ be given so that $\boldsymbol{\mathcal{R}}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)}, \alpha)\big) = \big\{\mathcal{R}^{(h)}\big((\boldsymbol{y}^{(j)}, \alpha)\big)\big\}_{h=1}^{j+1}$ is a $\big((\delta_2', \dots, \delta_{j+1}'), (\tilde{d}_2, \dots, \tilde{d}_{j+1}), r'\big)$-regular complex. Hence, $\mathcal{R}^{(j+1)}(\partial_s \hat{\boldsymbol{y}}^{(j)}) \in \mathscr{P}_{\text{orig}}^{(j)}$ for every $s \in [j+1]$ by the induction assumption (more precisely by (I) of $(\mathcal{C}_j)$). Moreover, the $(n, j+1, j+1)$-cylinder $\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)}, \alpha)\big)$ is $(\delta_{j+1}', \tilde{d}_{j+1}, r')$-regular (i.e., $\alpha \in [a_{j+1}^{\text{reg}}(\hat{\boldsymbol{x}}^{(j)})]$) and, consequently, $\mathcal{R}^{(j+1)}\big((\hat{\boldsymbol{y}}^{(j)}, \alpha)\big) \in \mathscr{P}_{\text{orig}}^{(j+1)}$ (cf. (6.17) in Case 3). This concludes the proof of (I) of $(\mathcal{C}_j)$.

Since $j+1 \geq 3$, part (II) follows directly from (6.18) (recall, that we defined $\tilde{\mathcal{G}}^{(2)}$ slightly differently in (6.14) so that $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$).

In order to verify (III), we appeal to the induction assumption, and in particular, to (III) of $(\mathcal{C}_j)$. Observe that we only need to consider $\mathcal{P}^{(j+1)}(\boldsymbol{x}^{(j+1)})$ for $\boldsymbol{x}^{(j+1)} \in A(j+1, \boldsymbol{b})$. Hence, it suffices to consider the constructions from Case 2 and Case 3. It is clear from the construction that in both of these cases we partitioned $\mathcal{K}_{j+1}^{(j)}\big(\hat{\mathcal{P}}^{(j)}(\boldsymbol{x}^{(j)})\big)$ into b_{j+1} different $(9\delta_{j+1}'/\tilde{d}_{j+1}, \tilde{d}_{j+1}, r')$-regular $(n/b_1, j+1, j+1)$-cylinders. Consequently, (III) of $(\mathcal{C}_{j+1})$ holds and $(\mathcal{C}_{j+1})$ is verified.

This finishes the inductive proof of statement $(\mathcal{C}_i)$ for $2 \leq i \leq k-1$.

6.4 Finale

Having inductively defined partitions $\mathscr{P}^{(j)}$ and hypergraphs $\tilde{\mathcal{G}}^{(j)}$, $2 \leq j \leq k-1$, we proceed to construct the promised hypergraph $\tilde{\mathcal{G}}^{(k)}$ (see (6.20) below). Then we shall show that the conclusions of Lemma 4.15 hold for $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ and $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$.

Let $\hat{A}_{\text{reg}}\big(\mathcal{P}_{\text{orig}}^{(k-1)}, \mathcal{G}^{(k)}, k-1, \boldsymbol{b}\big)$ denote the set of $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$ for which $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{P}_{\text{orig}}^{(k-1)}$ and $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular with respect to

$\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. We set

$$\tilde{\mathcal{G}}^{(k)} = \bigcup \Big\{ \mathcal{G}^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) \colon \\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}_{\mathrm{reg}}\big(\mathcal{P}^{(k-1)}_{\mathrm{orig}}, \mathcal{G}^{(k)}, k-1, \boldsymbol{b}\big) \Big\}. \tag{6.20}$$

It is left to verify that the earlier constructed family of partitions $\mathscr{P} = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ and $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k}$ satisfy the conclusion of Lemma 4.15.

Recall that for $2 \le j \le k-1$, we constructed $\mathscr{P}^{(j)}$ and $\tilde{\mathcal{G}}^{(j)}$ so that $(\mathcal{C}_j)$ and (6.15) holds. Consequently, by Fact 6.1 assertions (*1*)–(*5*) hold for every $j = 2, \dots, k-1$. The verification of Lemma 4.15 will rely on these assertions.

We first show that

$$\tilde{\boldsymbol{\mathcal{G}}} \text{ is an } (n, \ell, k)\text{-complex}. \tag{6.21}$$

By (*2*) of Fact 6.1 for $j = k-1$ we see that $\{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^{k-1}$ is an $(n, \ell, k-1)$-complex. Now, let $K \in \tilde{\mathcal{G}}^{(k)}$. We have to show that $K \in \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})$. From (6.20), we infer that $K \in \mathcal{K}_k^{(k-1)}(\mathcal{G}^{(k-1)} \cap \mathcal{P}^{(k-1)}_{\mathrm{orig}})$ and, consequently, by (II) of $(\mathcal{C}_{k-1})$, we have $K \in \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})$. Therefore, $\tilde{\mathcal{G}}^{(k-1)}$ underlies $\tilde{\mathcal{G}}^{(k)}$ and (6.21) follows.

Now we show that

$$\tilde{\boldsymbol{d}} \text{ is componentwise bigger than } \tilde{\boldsymbol{c}}. \tag{6.22}$$

Suppose $\tilde{d}_j \le \tilde{c}_j$ for some $2 \le j \le k-1$. Recall, that $\tilde{\boldsymbol{d}}$ was given by Corollary 4.13 as the density vector of $\mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$. Moreover, $L_k \ge |A(k-1, \boldsymbol{a})|$ and hence $|\hat{A}(j-1, \boldsymbol{a})| < 2^\ell L_k^k$ for $j = 2, \dots, k$. Therefore, the assumption $\tilde{d}_j \le \tilde{c}_j = 1/(2^{\ell+2} L_k^k)$ (see (6.4)) implies that the number of j-tuples in $(\delta'_j, \tilde{d}_j, r')$-regular polyads of $\mathscr{R}$ is at most $2^\ell L_k^k(\tilde{d}_j + \delta'_j) n^j \le 2^{\ell+1} L_k^k \tilde{c}_j n^j = n^j/2$. On the other hand, by (6.8), all but at most $\mu\binom{k}{j} n^j$ crossing j-tuples belong to $(\delta'_j, \tilde{d}_j, r')$-regular polyads of $\mathscr{R}$. Since $(1/2 + \mu\binom{k}{j}) n^j \le \binom{\ell}{j} n^j$ the assumption $\tilde{d}_j \le \tilde{c}_j$ must be wrong and we infer that $\tilde{d}_j > \tilde{c}_j$ for every $2 \le j \le k-1$, as claimed in (6.22).

Using (III) of $(\mathcal{C}_{k-1})$ combined with (6.2) and (6.3) yields that

$$\mathscr{P} = \mathscr{P}_{k-1} \text{ is an almost perfect } (\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})\text{-family of partitions}. \tag{6.23}$$

Moreover, *(3)* of Fact 6.1 for $j = k-1$ states that

$$\mathscr{P} = \mathscr{P}_{k-1} \text{ refines } \tilde{\boldsymbol{\mathcal{G}}}. \tag{6.24}$$

From (6.21)–(6.24) we infer that it is left to show *(i)*–*(v)* of Lemma 4.15, only. We observe that *(i)* is immediate from the construction of $\tilde{\mathcal{G}}^{(k)}$ in (6.20). Also, due to (6.20), (II) of $(\mathcal{C}_j)$ for $j = 2, \ldots, k-1$ (see also *(1)* of Fact 6.1), and the definition of $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$ we have *(ii)* of Lemma 4.15.

Now we verify *(iii)* of Lemma 4.15. For $3 \le j < k$ it is given by part *(4)* of Fact 6.1. For $j = k$, we recall the definition of $\tilde{\mathcal{G}}^{(k)}$ in (6.20) and consider $\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}$. There are two reasons for a k-tuple $K \in \mathcal{G}^{(k)}$ to be in $\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}$. Either $K \notin \mathcal{K}_k^{(k-1)}(\mathcal{P}^{(k-1)}_{\text{orig}})$ or K belongs to a polyad $\hat{\mathcal{P}}^{(k-1)}$ such that $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.

Consider a k-tuple of the first type, i.e., $K \notin \mathcal{K}_k^{(k-1)}(\mathcal{P}^{(k-1)}_{\text{orig}})$. Owing to (I) of $(\mathcal{C}_{k-1})$ we see that K belongs to a $\big((\delta'_2, \ldots, \delta'_{k-1}), (\tilde{d}_2, \ldots, \tilde{d}_{k-1}), r'\big)$-irregular $(n/a_1, k, k-1)$-complex of the original family of partitions $\mathscr{R}$. Consequently, by (6.8) (with $j = k$) there are at most μn^k k-tuples K of the first type ($K \notin \mathcal{K}_k^{(k-1)}(\mathcal{P}^{(k-1)}_{\text{orig}})$).

Now consider a k-tuple K, which is not of the first type, but of the second type. In particular, $K \in \mathcal{K}_k^{(k-1)}(\mathcal{P}^{(k-1)}_{\text{orig}})$ and $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, the underlying polyad of K in the family of partitions $\mathscr{P}$. From (I) of $(\mathcal{C}_{k-1})$ we infer that $\hat{\mathcal{P}}^{(k-1)}$ corresponds to k different $(n, k-1, k-1)$-cylinders, which are all elements of $\mathscr{R}^{(k-1)}$. Since $\mathscr{R}$ is a (δ'_k, r')-regular partition w.r.t. $\mathcal{G}^{(k)}$ and $\tilde{\delta}_k \ge \delta'_k$ and $\tilde{r} \le r'$ (cf. (6.2) and (6.3)), there are at most $\delta'_k n^k$ k-tuples $K \in \mathcal{G}^{(k)} \cap \mathcal{K}_k^{(k-1)}(\mathcal{P}^{(k-1)}_{\text{orig}})$ so that $\mathcal{G}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-irregular w.r.t. to the underlying polyad $\hat{\mathcal{P}}^{(k-1)}$ of K.

Summarizing the above, we infer that

$$|\mathcal{G}^{(k)} \triangle \tilde{\mathcal{G}}^{(k)}| = |\mathcal{G}^{(k)} \setminus \tilde{\mathcal{G}}^{(k)}| \le (\mu + \delta'_k) n^k \overset{(6.1)}{=} 2\delta'_k n^k.$$

Consequently, by the choice of δ_k' in (6.1), the following holds for every $k \leq i \leq \ell$,

$$\left|\mathcal{K}_i^{(k)}(\tilde{\mathcal{G}}^{(k)}) \triangle \mathcal{K}_i^{(k)}(\mathcal{G}^{(k)})\right| \leq \mu n^k \times \binom{\ell-k}{i-k} n^{i-k} \overset{(6.1)}{\leq} \tilde{\delta}_k \prod_{h=2}^{k} d_h^{\binom{i}{h}} \times n^i$$

which completes the verification of (iii) of Lemma 4.15.

We further note that (iv) of Lemma 4.15 is an immediate consequence of $b_1 = a_1 \leq \operatorname{rank} \mathscr{R} \leq L_k \leq \tilde{L}_k$ (cf. (6.4)), $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ and the assumption of Lemma 4.15 that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, 1)$-regular complex.

Finally, we show (v) of Lemma 4.15 as follows:

$$\begin{aligned}
\operatorname{rank} \mathscr{P} = |A(k-1, \bar{\boldsymbol{b}})| &= \binom{\ell}{k-1} b_1^{k-1} (b_2+1)^{\binom{k-1}{2}} \prod_{j=3}^{k-1} b_j^{\binom{k-1}{j}} \\
&\leq \binom{\ell}{k-1} a_1^{k-1} (2b_2)^{\binom{k-1}{2}} \prod_{j=3}^{k-1} b_j^{\binom{k-1}{j}} \overset{(6.7)}{\leq} 2^{\ell + 2\binom{k-1}{2}} L_k^{k-1} \prod_{j=2}^{k-1} \left(\frac{1}{\tilde{d}_j}\right)^{\binom{k-1}{j}}.
\end{aligned}$$

Then (v) follows from $\tilde{\boldsymbol{c}} \leq \tilde{\boldsymbol{d}}$ and the choice of $\tilde{L}_k$ in (6.4).

This completes the proof of Lemma 4.15.

Chapter 7

Proofs concerning Cleaning Phase II

We prove Lemma 5.6 and Lemma 5.7 in this chapter. We work in the context of Setup 5.5, the environment after Cleaning Phase I (after an application of Lemma 4.15) with the constants from Figure 5.2. The main objective of this chapter is to construct the complexes $\boldsymbol{\mathcal{H}}_+$ and $\boldsymbol{\mathcal{H}}_-$ defined in Lemma 5.6 and Lemma 5.7. We prove these lemmas in Section 7.2 and Section 7.3, respectively. The following section, Section 7.1, contains some preliminary facts, which are immediate consequences of the choice of constants given in Section 5.4.1 (see Figure 5.2).

7.1 Preliminary Facts

We start with the following facts which we apply liberally in the remainder of this chapter. The first two facts are immediate consequences of $\mathbf{IHC}_{k-1,\ell}$ and the choice of constants in Section 5.4.1 (applied to differing setups).

Fact 7.1. *For all integers $2 \le j < k$ and $j < i \le \ell$ and every $\Lambda_i \in \binom{[\ell]}{i}$,*

$$\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| = (1 \pm \eta) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,, \tag{7.1}$$

$$\left|\mathcal{K}_i^{(j)}\big(\tilde{\mathcal{G}}^{(j)}[\Lambda_i]\big)\right| = \big(1 \pm (\eta + \tilde{\delta}_k)\big) \prod_{h=2}^{j} d_h^{\binom{i}{h}} \times n^i\,. \tag{7.2}$$

Consequently, by the choice of η in (5.12) and $\tilde{\delta}_k \le 1/8$ in (5.15),

$$\left|\mathcal{K}_i^{(j)}\big(\tilde{\mathcal{G}}^{(j)}[\Lambda_i]\big)\right| \ge \frac{1 - 1/4 - \tilde{\delta}_k}{1 + 1/4}\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right| \ge \frac{1}{2}\left|\mathcal{K}_i^{(j)}\big(\mathcal{G}^{(j)}[\Lambda_i]\big)\right|. \tag{7.3}$$

Proof. Due to the choice of $\boldsymbol{\delta} = (\delta_2, \dots, \delta_{k-1})$ and r (cf. (5.13), (5.19), and (5.20)) for $2 \le j < k$, the complex $\boldsymbol{\mathcal{G}}^{(\boldsymbol{j})} = \{\mathcal{G}^{(h)}\}_{h=1}^{j}$ satisfies the assumption of $\mathbf{IHC}_{k-1,\ell}$. As such, we conclude that (7.1) holds. Since $\tilde{\boldsymbol{\mathcal{G}}}$ is given by Lemma 4.15, it satisfies (*iii*) of that lemma and (7.2) follows. □

In the following fact, $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j-1})}$ represents an arbitrary regular $(n/b_1, i, j-1)$-complex arising from an application of Lemma 4.15 (i.e., the complex $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j-1})}$ is "built from blocks" of the partition $\mathscr{P}$).

Fact 7.2. *If $1 \le j-1 < k$ and $j \le i \le \ell$ and $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j-1})} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ is a $\big((\tilde{\delta}_2, \dots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \dots, \tilde{d}_{j-1}), \tilde{r}\big)$-regular $(n/b_1, i, j-1)$-complex, then*

$$\left|\mathcal{K}_i^{(j-1)}\big(\check{\mathcal{H}}^{(j-1)}\big)\right| = (1 \pm \tilde{\eta}) \prod_{h=2}^{j-1} \tilde{d}_h^{\binom{i}{h}} \times \left(\frac{n}{b_1}\right)^i. \tag{7.4}$$

In particular, for every $1 \le j-1 < k$ and every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(j-1, \boldsymbol{b})$,

$$\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\right| = (1 \pm \tilde{\eta}) \prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}} \times \left(\frac{n}{b_1}\right)^j. \tag{7.5}$$

Proof. Similarly as in the proof of Fact 7.1, by the choice of $\tilde{\eta}$ and $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \dots, \tilde{\delta}_{k-1})$ and $\tilde{r}$ (cf. (5.15) and (5.17)), we infer that $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j-1})}$ for $2 \le j-1 \le k-1$ satisfies the assumptions of $\mathbf{IHC}_{k-1,\ell}$ and, consequently, (7.4) of Fact 7.2 holds. □

Recall that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular complex where by (ii) and (iii) of Lemma 4.15 (with $i = j$)

$$\begin{aligned} \mathcal{G}^{(1)} = \tilde{\mathcal{G}}^{(1)}\,, \quad \mathcal{G}^{(2)} = \tilde{\mathcal{G}}^{(2)} \qquad &\text{and} \\ \left|\mathcal{G}^{(j)} \setminus \tilde{\mathcal{G}}^{(j)}\right| \leq \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \text{ for } 3 \leq j \leq k\,. \end{aligned} \tag{7.6}$$

Since $\tilde{\delta}_k$ is significantly smaller than δ_j, $3 \leq j \leq k$ (cf. Figure 5.2), we infer the following fact by a standard argument.

Fact 7.3. *The (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}}$ is $(2\boldsymbol{\delta}, \boldsymbol{d}, r)$-regular.*

Proof. By the choice of the constants in Section 5.4.1, we infer that $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ (see Lemma 4.15 (ii)) and hence $\tilde{\mathcal{G}}^{(2)} = \mathcal{G}^{(2)}$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)}$.

We now show that $\tilde{\mathcal{G}}^{(j)}$ is $(2\delta_j, d_j, r)$-regular w.r.t. $\tilde{\mathcal{G}}^{(j-1)}$ for each $j \geq 3$. Let j and $\Lambda_j \in \binom{[\ell]}{j}$ be fixed. Let $\boldsymbol{\mathcal{Q}^{(j-1)}} = \left\{\mathcal{Q}_s^{(j-1)}\right\}_{s \in [r]}$ be a family of subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j] \subseteq \mathcal{G}^{(j-1)}[\Lambda_j]$ such that

$$\left|\bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \geq 2\delta_j \left|\mathcal{K}_j^{(j-1)}\left(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\right)\right|.$$

From (7.1) and (7.3), we then infer

$$\begin{aligned} \left|\bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| &> \delta_j \left|\mathcal{K}_j^{(j-1)}\left(\mathcal{G}^{(j-1)}[\Lambda_j]\right)\right| \\ &\geq \delta_j(1-\eta) \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j. \end{aligned} \tag{7.7}$$

Since $\boldsymbol{\mathcal{Q}^{(j-1)}}$ is a family of subhypergraphs of $\mathcal{G}^{(j-1)}[\Lambda_j]$ and since $\mathcal{G}^{(j)}[\Lambda_j]$ is (δ_j, d_j, r)-regular with respect to $\mathcal{G}^{(j-1)}[\Lambda_j]$, we see

$$\left|\mathcal{G}^{(j)}[\Lambda_j] \cap \bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| = (d_j \pm \delta_j) \left|\bigcup_{s \in [r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|. \tag{7.8}$$

On the other hand, (7.6) and (7.8) imply

$$\begin{aligned}
&\left|\tilde{\mathcal{G}}^{(j)}\left[\Lambda_j\right] \cap \bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \\
&\qquad \overset{(7.6)}{=} \left|\mathcal{G}^{(j)}\left[\Lambda_j\right] \cap \bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \pm \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \\
&\qquad \overset{(7.8)}{=} (d_j \pm \delta_j)\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right| \pm \tilde{\delta}_k \prod_{h=2}^{j} d_h^{\binom{j}{h}} \times n^j \\
&\qquad = (d_j \pm 2\delta_j)\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}\left(\mathcal{Q}_s^{(j-1)}\right)\right|,
\end{aligned}$$

where the last equality uses (7.7) and $\tilde{\delta}_k d_j \le \delta_j^2(1-\eta)$ for $j \ge 3$. □

7.2 Proof of Lemma 5.6

The proof of Lemma 5.6 will take place in stages. Setting $\mathcal{H}^{(1)} = \tilde{\mathcal{G}}^{(1)}$ and $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)}$ satisfies part (*a*) of Lemma 5.6. We prove part (*b*) of Lemma 5.6 in Section 7.2.1 and part (*c*) in Section 7.2.3.

7.2.1 Proof of Property (*b*) of Lemma 5.6

We prove part (*b*) by induction on j.

Induction Start. Recall that we set $\mathcal{H}^{(2)} = \tilde{\mathcal{G}}^{(2)}$. Consequently, the symmetric difference considered in part (*b2*) of Lemma 5.6 is empty. Hence, (*b2*) holds trivially for $j = 2$ and it is left to verify (*b1*). To that end, let $\hat{\boldsymbol{x}}^{(1)} = ((\lambda_1, \lambda_2), (\beta_1, \beta_2)) \in \hat{A}(\mathcal{H}^{(1)}, 1, \boldsymbol{b}) = \hat{A}(1, \boldsymbol{b})$ be fixed. From part (*iv*) of Lemma 4.15, we infer $d\big(\tilde{\mathcal{G}}^{(2)}\big|\hat{\mathcal{P}}^{(1)}\big(\hat{\boldsymbol{x}}^{(1)}\big)\big) = d_2 \pm \tilde{L}_k^2\delta_2$. From (*ii*) of Definition 4.14, we then infer

$$\frac{d_2 - \tilde{L}_k^2\delta_2}{\tilde{d}_2 + \tilde{\delta}_2} \le \left|I\big(\hat{\boldsymbol{x}}^{(1)}\big)\right| \le \frac{d_2 + \tilde{L}_k^2\delta_2}{\tilde{d}_2 - \tilde{\delta}_2}\,.$$

As such, to verify (*b1*), we may show that the left-hand side of the last inequality is bigger than d_2b_2-1 and the right-hand side is less than d_2b_2+1. Consequently, it suffices to verify

$$(\tilde{d}_2+\tilde{\delta}_2)(d_2b_2-1)<d_2-\tilde{L}_k^2\delta_2 \quad \text{and} \quad d_2+\tilde{L}_k^2\delta_2<(d_2b_2+1)(\tilde{d}_2-\tilde{\delta}_2)\,. \tag{7.9}$$

The proofs of both inequalities are similar and we only present the details for the first one here.

We consider the left-hand side of the first inequality in (7.9) and see

$$\begin{aligned}(\tilde{d}_2+\tilde{\delta}_2)(d_2b_2-1)&<\tilde{d}_2d_2b_2-\tilde{d}_2+\tilde{\delta}_2d_2b_2\\&\le d_2(1+\tilde{\delta}_2/\tilde{d}_2)-\tilde{d}_2+\tilde{\delta}_2b_2\\&\le d_2+\tilde{\delta}_2/\tilde{d}_2-\tilde{d}_2+\tilde{\delta}_2b_2\,.\end{aligned} \tag{7.10}$$

where we use (*i*) of Definition 4.14 for the last inequality. Again, from (*i*) of Definition 4.14 and $\tilde{d}_2>\tilde{\delta}_2$, we know $b_2<2/\tilde{d}_2$. Therefore, using $\tilde{\delta}_2\ll\tilde{d}_2$ gives the following bound for the right-hand side of (7.10)

$$d_2+\tilde{\delta}_2/\tilde{d}_2-\tilde{d}_2+\tilde{\delta}_2b_2<d_2-\tilde{d}_2+3\tilde{\delta}_2/\tilde{d}_2<d_2-\tilde{d}_2+\sqrt{\tilde{\delta}_2}\,. \tag{7.11}$$

Summarizing (7.10) and (7.11), the first inequality of (7.9) follows from the choice of constants $\tilde{d}_2\gg\tilde{\delta}_2\gg\tilde{L}_k^2\delta_2$ (see Figure 5.2), by

$$(\tilde{d}_2+\tilde{\delta}_2)(d_2b_2-1)<d_2-\tilde{d}_2+\sqrt{\tilde{\delta}_2}<d_2-\tilde{L}_k^2\delta_2\,.$$

Induction Step. Assume that for $2\le j<k$, part (*b*) of Lemma 5.6 holds for $j-1$ with inductively defined complex $\boldsymbol{\mathcal{H}}^{(j-1)}=\{\mathcal{H}^{(h)}\}_{h=1}^{j-1}$. We construct the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{A}\left(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b}\right)$, and hypergraph $\mathcal{H}^{(j)}$ satisfying (*b1*) and (*b2*). We first define the following set of indices crucial for our constructions.

For a vector $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{A}\left(\mathcal{H}^{(j-1)},j-1,\boldsymbol{b}\right)$, set

$$\mathcal{J}\left(\hat{\boldsymbol{x}}^{(j-1)}\right)=\left\{\beta\in[b_j]\colon\ \mathcal{P}^{(j)}\left((\hat{\boldsymbol{x}}^{(j-1)},\beta)\right)\subseteq\tilde{\mathcal{G}}^{(j)}\right\}\,. \tag{7.12}$$

For $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\left(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\right)$, observe

$$\left|\tilde{\mathcal{G}}^{(j)} \cap \mathcal{K}_j^{(j-1)}\left(\mathcal{P}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\right)\right| = \sum_{\beta \in \mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})} \left|\mathcal{P}^{(j)}\left((\hat{\boldsymbol{x}}^{(j-1)}, \beta)\right)\right| . \tag{7.13}$$

Now we construct the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$ for every $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$.

- *If* $\left|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})\right| \geq d_j b_j$, *then remove* $\left|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})\right| - d_j b_j$ *arbitrary indices from* $\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})$ *to construct* $I(\hat{\boldsymbol{x}}^{(j-1)})$.
- *If* $\left|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})\right| < d_j b_j$, *then add* $d_j b_j - \left|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})\right|$ *arbitrary indices from* $[b_j] \setminus \mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})$ *to* $\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})$ *to construct* $I(\hat{\boldsymbol{x}}^{(j-1)})$.

This defines the sets $I(\hat{\boldsymbol{x}}^{(j-1)})$.

For upcoming considerations, we define the set $\hat{B}^{(j-1)}$ of addresses $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ for which $|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})|$ is 'too big'. More precisely, we define

$$\hat{B}^{(j-1)} = \left\{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \colon \left|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})\right| > \sqrt{\delta_j} d_j b_j\right\}.$$

We prove the following claim in Section 7.2.2.

Claim 7.4. $\left|\hat{B}^{(j-1)}\right| < 2\sqrt{\delta_j} \prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}} \times b_1^j$.

We define hypergraph $\mathcal{H}^{(j)}$ as

$$\begin{aligned}\mathcal{H}^{(j)} = \bigcup \Big\{ & \mathcal{P}^{(j)}\left((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\right) \colon \\ & \qquad \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \text{ and } \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})\Big\} .\end{aligned} \tag{7.14}$$

We now prove Property (b) of Lemma 5.6, and to that end, we establish both parts $(b1)$ and $(b2)$. Note, however, that with $I(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, and $\mathcal{H}^{(j)}$ constructed above, Property $(b1)$ of Lemma 5.6 follows immediately. Thus, it remains to prove Property $(b2)$.

Let $j \leq i \leq \ell$ be fixed and consider the set $\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$. Clearly, for every i-tuple $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$, there exists a j-tuple $J_0 \in \binom{I_0}{j}$

such that $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$. We note that one possibility for $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$ is that $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. Since we have some control over the cardinality of $\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$ (by the induction assumption on (*b1*)), it is natural to split the i-tuples $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$ into two parts, $\mathfrak{K}_i^{(j)}(1)$ and $\mathfrak{K}_i^{(j)}(2)$, depending on whether there is a $J_0 \in \binom{I_0}{j}$ as described above. More precisely, we define

$$\begin{aligned}\mathfrak{K}_i^{(j)}(1) = \Bigg\{ I_0 \in \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\right) &: \\ \exists\, J_0 \in \binom{I_0}{j} \text{ so that } J_0 \in &\left(\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})\right) \Bigg\}\end{aligned}$$

and

$$\begin{aligned}\mathfrak{K}_i^{(j)}(2) &= \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\right) \Big\backslash\, \mathfrak{K}_i^{(j)}(1) \\ &= \Bigg\{ I_0 \in \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\right) : \\ &\qquad \forall\, J_0 \in \binom{I_0}{j} \quad J_0 \notin \left(\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})\right) \Bigg\}.\end{aligned}$$

Observe that we may rewrite $\mathfrak{K}_i^{(j)}(2)$ as

$$\begin{aligned}\mathfrak{K}_i^{(j)}(2) = \Bigg\{ I_0 \in \left(\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})\right) &: \\ \forall\, J_0 \in \binom{I_0}{j} \quad J_0 \in &\left(\mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \cap \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})\right) \Bigg\}.\end{aligned} \tag{7.15}$$

Indeed, for the equality (of sets) in (7.15), the inclusion '$\supseteq$' is obvious. The opposite inclusion '$\subseteq$' follows from the fact that $\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)}) \subseteq \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \cup \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$ and, consequently, for every considered I_0 and $J_0 \in \binom{I_0}{j}$, we have $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \cup \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. The '$\subseteq$' inclusion then follows. Note that from (7.15), we infer

$$\mathfrak{K}_i^{(j)}(2) \subseteq \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_i^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})\,. \tag{7.16}$$

We now consider a subdivision of $\mathfrak{K}_i^{(j)}(2)$. From (7.15), we infer that all $I_0 \in \mathfrak{K}_i^{(j)}(2)$ only 'touch' polyads $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. Let $\mathfrak{K}_i^{(j)}(2,1)$ be the set of all $I_0 \in \mathfrak{K}_i^{(j)}(2)$ which 'touch' a bad polyad $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ (bad in the sense of Claim 7.4) with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}^{(j-1)} \subseteq \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. Formally, set

$$\mathfrak{K}_i^{(j)}(2,1) = \Bigg\{ I_0 \in \mathfrak{K}_i^{(j)}(2) \colon \exists\ J_0 \in \binom{I_0}{j} \text{ and } \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}^{(j-1)} \\ \text{so that } J_0 \in \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \Bigg\}.$$

The remaining $I_0 \in \mathfrak{K}_i^{(j)}(2) \setminus \mathfrak{K}_i^{(j)}(2,1)$ 'touch' only good polyads. However, as observed earlier, for every such I_0, there exists a $J_0 \in \binom{I_0}{j}$ such that $J_0 \in \mathcal{H}^{(j)} \triangle \tilde{\mathcal{G}}^{(j)}$. Recall that the union of the sets $\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})$ with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ represents $\tilde{\mathcal{G}}^{(j)} \cap \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)})$ (cf. (7.12)) and similarly the union of $I(\hat{\boldsymbol{x}}^{(j-1)})$ represents $\mathcal{H}^{(j)}$ (cf. (7.14)). Consequently, $\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ represents the difference of $\tilde{\mathcal{G}}^{(j)}$ and $\mathcal{H}^{(j)}$ on the underlying polyad having address $\hat{\boldsymbol{x}}^{(j-1)}$. Hence, we infer that for every $I_0 \in \mathfrak{K}_i^{(j)}(2) \setminus \mathfrak{K}_i^{(j)}(2,1)$, there exist a $J_0 \in \binom{I_0}{j}$, $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)}$ and $\alpha \in \mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ so that $J_0 \in \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$. We therefore set

$$\mathfrak{K}_i^{(j)}(2,2) = \Bigg\{ I_0 \in \mathfrak{K}_i^{(j)}(2) \colon \exists J_0 \in \binom{I_0}{j}, \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} \\ \text{and } \alpha \in \mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)}) \text{ so that } J_0 \in \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) \Bigg\}.$$

Note that $\mathfrak{K}_i^{(j)}(2,1)$ and $\mathfrak{K}_i^{(j)}(2,2)$ are not necessarily disjoint. However, $\mathfrak{K}_i^{(j)}(2) = \mathfrak{K}_i^{(j)}(2,1) \cup \mathfrak{K}_i^{(j)}(2,2)$ and therefore

$$\Big| \mathcal{K}_i^{(j)}\big(\mathcal{H}^{(j)}\big) \triangle \mathcal{K}_i^{(j)}\big(\tilde{\mathcal{G}}^{(j)}\big) \Big| \le \big|\mathfrak{K}_i^{(j)}(1)\big| + \big|\mathfrak{K}_i^{(j)}(2,1)\big| + \big|\mathfrak{K}_i^{(j)}(2,1)\big|. \tag{7.17}$$

In what follows, we derive an upper bound for each term of the right-hand side of (7.17) which all combined yield part $(b2)$ of Lemma 5.6.

Bounding $|\mathfrak{K}_i^{(j)}(1)|$. The upper bound on $|\mathfrak{K}_i^{(j)}(1)|$ follows from the induction assumption on part (b) of Lemma 5.6. First, observe that

$$\mathfrak{K}_i^{(j)}(1) \subseteq \mathcal{K}_i^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_i^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\,. \tag{7.18}$$

Indeed, if $I_0 \in \mathfrak{K}_i^{(j)}(1)$, then (immediately) $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \triangle \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$. Assume, without loss of generality, that $I_0 \in \mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \setminus \mathcal{K}_i^{(j)}(\tilde{\mathcal{G}}^{(j)})$ (the other case is symmetric). Since, $\mathcal{K}_i^{(j)}(\mathcal{H}^{(j)}) \subseteq \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)})$ we have

$$I_0 \in \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)})\,. \tag{7.19}$$

On the other hand, due to the definition of $\mathfrak{K}_i^{(j)}(1)$, for each such I_0 there exists $J_0 \in \binom{I_0}{j}$ satisfying $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. From (7.19), we also have $J_0 \in \mathcal{K}_j^{(j-1)}(\mathcal{H}^{(j-1)})$ and hence $J_0 \notin \mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$. Consequently, $I_0 \notin \mathcal{K}_i^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$ and thus $I_0 \in \mathcal{K}_i^{(j-1)}(\mathcal{H}^{(j-1)}) \triangle \mathcal{K}_i^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)})$ which yields (7.18).

Now the induction assumption on $(b2)$, with j replaced by $j-1$, gives the following:

$$\begin{aligned}\left|\mathfrak{K}_i^{(j)}(1)\right| \leq \left|\mathcal{K}_i^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_i^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\right| &\leq \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{i}{h}}\right) n^i \\ &\leq \frac{1}{3}\delta_j^{1/3}\left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i\end{aligned} \tag{7.20}$$

where the last inequality follows from the choice of constants summarized in Figure 5.2 ensuring $\delta_{j-1}^{1/3} \ll \delta_j^{1/3} d_j^{\binom{i}{j}}$.

Bounding $|\mathfrak{K}_i^{(j)}(2,1)|$. By $(b1)$, there are $\binom{\ell-j}{i-j}\big(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\big) b_1^{i-j}$ ways to complete any given $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ with $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ to a $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \ldots, \tilde{d}_{j-1}), \tilde{r}\big)$-regular $(n/b_1, i, j-1)$-subcomplex $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j-1})} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ of $\boldsymbol{\mathcal{H}}^{(\boldsymbol{j-1})}$. Then, (7.4) of Fact 7.2 yields that for each such $\check{\mathcal{H}}^{(j-1)}$,

$$\left|\mathcal{K}_i^{(j-1)}\big(\check{\mathcal{H}}^{(j-1)}\big)\right| \leq (1+\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{i}{h}}\right)\left(\frac{n}{b_1}\right)^i \leq 2\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{i}{h}}\right)\left(\frac{n}{b_1}\right)^i.$$

Using (7.16) and Claim 7.4 (for the second inequality below), we therefore see

$$
\begin{aligned}
\left|\mathfrak{K}_i^{(j)}(2,1)\right| &\le \left|\hat{B}^{(j-1)}\right| \times \binom{\ell-j}{i-j}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) b_1^{i-j} \times \\
&\qquad \times\ 2\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{i}{h}}\right)\left(\frac{n}{b_1}\right)^i \\
&\le 4\binom{\ell-j}{i-j}\sqrt{\delta_j}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{i}{h}}\right)\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{i}{h}}\right) n^i \\
&\le 4\binom{\ell-j}{i-j}\sqrt{\delta_j}\left(\prod_{h=2}^{j-1}(d_h b_h \tilde{d}_h)^{\binom{i}{h}}\right) n^i\,,
\end{aligned}
$$

and so, by (5.24) and the choice of $\delta_j \ll d_j$, we have the upper bound

$$
\begin{aligned}
\left|\mathfrak{K}_i^{(j)}(2,1)\right| &\le 4\binom{\ell-j}{i-j}(1+\nu)\sqrt{\delta_j}\left(\prod_{h=2}^{j-1} d_h^{\binom{i}{h}}\right) n^i \\
&\le \frac{1}{3}\delta_j^{1/3}\left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i\,.
\end{aligned}
\tag{7.21}
$$

Bounding $|\mathfrak{K}_i^{(j)}(2,2)|$. First, let $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)}$ and $\alpha \in \mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ be fixed and consider the $(n/b_1, j, j)$-complex implicitly given by $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$. By Property *(b1)*, there are

$$
\binom{\ell-j}{i-j} b_1^{i-j}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) b_j^{\binom{i}{j}-1} = d_j^{1-\binom{i}{j}}\binom{\ell-j}{i-j} b_1^{i-j}\prod_{h=2}^{j}(d_h b_h)^{\binom{i}{h}-\binom{j}{h}}
$$

ways to complete $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ to a $\big((\tilde{\delta}_2, \dots, \tilde{\delta}_j), (\tilde{d}_2, \dots, \tilde{d}_j), \tilde{r}\big)$-regular $(n/b_1, i, j)$-complex $\check{\boldsymbol{\mathcal{H}}}^{(\boldsymbol{j})} = \{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j}$ in such a way that $\{\check{\mathcal{H}}^{(h)}\}_{h=1}^{j-1}$ is a subcomplex of $\boldsymbol{\mathcal{H}}^{(\boldsymbol{j-1})}$.

Then (7.4) of Fact 7.2 yields

$$
\left|\mathcal{K}_i^{(j)}(\check{\mathcal{H}}^{(j)})\right| \le (1+\tilde{\eta})\prod_{h=2}^{j}\tilde{d}_h^{\binom{i}{h}} \times \left(\frac{n}{b_1}\right)^i
$$

for every such $\check{\mathcal{H}}^{(j)}$. Now, summing over all choices $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(j-1)} \subseteq \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$ and $\alpha \in \mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})$ gives

$$\begin{aligned}\left|\mathfrak{K}_i^{(j)}(2,2)\right| \overset{(7.16)}{\leq} \left|\hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\right| \times \\ \times \ \max\left\{\left|\mathcal{J}\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \triangle I\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| : \ \hat{\boldsymbol{x}}^{(j-1)} \notin \hat{B}^{(j-1)}\right\} \times \\ \times \ d_j^{1-\binom{i}{j}} \binom{\ell-j}{i-j} b_1^{i-j} \left(\prod_{h=2}^{j} (d_h b_h)^{\binom{i}{h}-\binom{j}{h}}\right) \times \\ \times \ (1+\tilde{\eta}) \left(\prod_{h=2}^{j} \tilde{d}_h^{\binom{i}{h}}\right) \left(\frac{n}{b_1}\right)^i .\end{aligned} \tag{7.22}$$

By Property $(b1)$,

$$\left|\hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\right| = \binom{\ell}{j} \left(\prod_{h=2}^{j-1} (d_h b_h)^{\binom{j}{h}}\right) b_1^j .$$

Also note that for each $\hat{\boldsymbol{x}}^{(j-1)} \notin \hat{B}^{(j-1)}$, $\left|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)}) \triangle I(\hat{\boldsymbol{x}}^{(j-1)})\right| \leq \sqrt{\delta_j} d_j b_j$. Consequently, the right-hand side of (7.22) is less than

$$d_j^{1-\binom{i}{j}} \binom{\ell}{j} \binom{\ell-j}{i-j} \sqrt{\delta_j}(1+\tilde{\eta}) \left(\prod_{h=2}^{j} (d_h b_h \tilde{d}_h)^{\binom{i}{h}}\right) n^i$$

Now, using (5.24), the choice of $\tilde{\eta}$ and $\delta_j \ll d_j$ yields

$$\left|\mathfrak{K}_i^{(j)}(2,2)\right| \leq \frac{1}{3} \delta_j^{1/3} \left(\prod_{h=2}^{j} d_h^{\binom{i}{h}}\right) n^i . \tag{7.23}$$

Finally, (7.17) combined with (7.20), (7.21), and (7.23) yields part $(b2)$ of Lemma 5.6. In order to complete the proof of part (b) of Lemma 5.6 we still have to verify Claim 7.4.

7.2.2 Proof of Claim 7.4

The proof is rather straightforward in the genre of hypergraph regularity. We first split the set $\hat{B}^{(j-1)}$ into two parts $\hat{B}_+^{(j-1)}$ and $\hat{B}_-^{(j-1)}$ as follows:

$$\begin{aligned}\hat{B}_+^{(j-1)} &= \left\{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\colon\ \left|\mathcal{J}\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| > \left(1+\sqrt{\delta_j}\right) d_j b_j\right\}\\ \hat{B}_-^{(j-1)} &= \left\{\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\colon\ \left|\mathcal{J}\big(\hat{\boldsymbol{x}}^{(j-1)}\big)\right| < \left(1-\sqrt{\delta_j}\right) d_j b_j\right\}.\end{aligned} \tag{7.24}$$

We prove the following claim which in view of Fact 7.3 is a slightly stronger statement than Claim 7.4.

Claim 7.4′. *If for some* $* \in \{+,-\}$,

$$\left|\hat{B}_*^{(j-1)}\right| \geq \sqrt{\delta_j}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{j}{h}}\right) b_1^j\,,$$

then $\tilde{\mathcal{G}}^{(j)}$ *is not* $(2\delta_j, d_j, r)$*-regular w.r.t.* $\tilde{\mathcal{G}}^{(j-1)}$*.*

Proof. We prove the case $* = -$ only with the other case very similar. We assume there exists an ordered set $\Lambda_j \in \binom{[\ell]}{j}_<$ such that

$$\left|\hat{B}_-^{(j-1)}[\Lambda_j]\right| \geq \frac{\sqrt{\delta_j}}{\binom{\ell}{j}}\left(\prod_{h=2}^{j-1}(d_h b_h)^{\binom{j}{h}}\right) b_1^j \tag{7.25}$$

where $\hat{B}_-^{(j-1)}[\Lambda_j]$ is the set of $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \ldots, \hat{\boldsymbol{x}}_{j-1}) \in \hat{B}_-^{(j-1)}$ such that $\hat{\boldsymbol{x}}_0 = \Lambda_j$.

We show that (7.25) implies that $\tilde{\mathcal{G}}^{(j)}$ is irregular. Note that the polyad addresses $\hat{\boldsymbol{x}}^{(j-1)}$ in $\hat{B}_-^{(j-1)}[\Lambda_j]$ considered in (7.25) correspond to subhypergraphs of $\mathcal{H}^{(j-1)}$ and not necessarily to subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}$. The set $\hat{\Gamma}_-^{(j-1)}[\Lambda_j]$ which we define below is the subset of those polyad addresses of $\hat{B}_-^{(j-1)}[\Lambda_j]$ which correspond to subhypergraphs of $\tilde{\mathcal{G}}^{(j-1)}$ as well. Only those addresses are useful to verify Claim 7.4′. We therefore set

$$\hat{\Gamma}_-^{(j-1)}[\Lambda_j] = \hat{B}_-^{(j-1)}[\Lambda_j] \cap \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big) \tag{7.26}$$

and

$$\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)} = \left\{\hat{\mathcal{P}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big) \colon\ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_{-}^{(j-1)}[\Lambda_j]\right\} = \left\{\hat{\mathcal{Q}}_1^{(j-1)}, \dots, \hat{\mathcal{Q}}_t^{(j-1)}\right\}$$

where $t = \big|\hat{\Gamma}_{-}^{(j-1)}[\Lambda_j]\big|$. In what follows, we show

$$\left|\bigcup_{s\in[t]}\left\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right\}\right| > 2\delta_j \left|\mathcal{K}_j^{(j-1)}\left(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]\right)\right| \tag{7.27}$$

and

$$d\left(\tilde{\mathcal{G}}^{(j)}\middle|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right) < d_j - 2\delta_j\,. \tag{7.28}$$

From (5.22), we see that $r \geq \big|\hat{A}(j-1,\boldsymbol{b})\big| \geq t$. Therefore, establishing (7.27) and (7.28) proves Claim 7.4′.

We first verify (7.27). Observe that due to the definition of $\hat{\Gamma}_{-}^{(j-1)}[\Lambda_j]$ in (7.26),

$$\begin{aligned}\hat{B}_{-}^{(j-1)}[\Lambda_j] \setminus \hat{\Gamma}_{-}^{(j-1)}[\Lambda_j] &\subseteq \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big)\Big\backslash \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\\ &\subseteq \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) \triangle \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\end{aligned} \tag{7.29}$$

and since $\mathscr{P}^{(j-1)}$ respects $\mathcal{H}^{(j-1)}$ (cf. part (*b1*)) and $\mathscr{P}^{(j-1)}$ respects $\tilde{\mathcal{G}}^{(j-1)}$ (cf. Setup 5.5),

$$\begin{aligned}&\bigcup\left\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\big(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}\big) \triangle \hat{A}\big(\tilde{\mathcal{G}}^{(j-1)}, j-1, \boldsymbol{b}\big)\right\}\\ &\qquad = \mathcal{K}_j^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_j^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\,.\end{aligned} \tag{7.30}$$

Combining (7.29) and (7.30) with the induction hypothesis on (*b2*) for $j-1$ yields

$$\begin{aligned}&\left|\bigcup\left\{\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \colon\ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}_{-}^{(j-1)}[\Lambda_j] \setminus \hat{\Gamma}_{-}^{(j-1)}[\Lambda_j]\right\}\right|\\ &\qquad \leq \left|\mathcal{K}_j^{(j-1)}\big(\mathcal{H}^{(j-1)}\big) \triangle \mathcal{K}_j^{(j-1)}\big(\tilde{\mathcal{G}}^{(j-1)}\big)\right| < \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\,.\end{aligned}$$

Consequently, we have

$$\left|\bigcup_{s\in[t]}\left\{\mathcal{K}_j^{(j-1)}(\hat{\mathcal{Q}}_s^{(j-1)})\right\}\right| = \left|\bigcup\left\{\mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}))\colon \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right\}\right|$$

$$\geq \sum\left\{\left|\mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}))\right|\colon \hat{\boldsymbol{x}}^{(j-1)}\in\hat{B}_-^{(j-1)}[\Lambda_j]\right\} - \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\,.$$

Applying (7.5) of Fact 7.2 to each term in the sum above yields the further lower bound

$$\left|\hat{B}_-^{(j-1)}[\Lambda_j]\right|(1-\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j - \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j\,.$$

Finally, from our assumption (7.25) and inequality (5.24), we infer

$$\begin{aligned}
&\left|\bigcup_{s\in[t]}\left\{\mathcal{K}_j^{(j-1)}(\hat{\mathcal{Q}}_s^{(j-1)})\right\}\right| \\
&\qquad\geq \binom{\ell}{j}^{-1}(1-\tilde{\eta})\sqrt{\delta_j}\left(\prod_{h=2}^{j-1}\left(d_h b_h \tilde{d}_h\right)^{\binom{j}{h}}\right) n^j - \delta_{j-1}^{1/3}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j \\
&\qquad\geq \left(\binom{\ell}{j}^{-1}(1-\tilde{\eta})(1-\nu)\sqrt{\delta_j} - \delta_{j-1}^{1/3}\right)\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j \\
&\qquad\geq \delta_j^{3/4}\left(\prod_{h=2}^{j-1} d_h^{\binom{j}{h}}\right) n^j
\end{aligned}\tag{7.31}$$

where the last inequality follows from the choice of $\tilde{\eta}$, ν, and $\delta_j \gg \delta_{j-1}$. Now, (7.27) follows from (7.31) combined with (7.2) of Fact 7.1 for $j-1$ and $i = j$.

It is left to verify (7.28). First, observe that from the definition of $\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}$ and (7.13), we have

$$\begin{aligned}
&\left|\tilde{\mathcal{G}}^{(j)}\cap\bigcup_{s\in[t]}\mathcal{K}_j^{(j-1)}(\hat{\mathcal{Q}}_s^{(j-1)})\right| \\
&\qquad= \sum\left\{\left|\tilde{\mathcal{G}}^{(j)}\cap\mathcal{K}_j^{(j-1)}(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}))\right|\colon\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right\} &(7.32)\\
&\qquad= \sum\sum\left\{\left|\mathcal{P}^{(j)}((\hat{\boldsymbol{x}}^{(j-1)},\beta))\right|\colon\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j],\ \beta\in\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})\right\}.
\end{aligned}$$

Recall that by Definition 4.14, part (ii), every $\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)$ is $(\tilde{\delta}_j,\tilde{d}_j,\tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$, $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]$ and $\beta\in\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})$. Consequently, from (7.5) of Fact 7.2, we note

$$\left|\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)},\beta)\big)\right| \le \left(\tilde{d}_j+\tilde{\delta}_j\right)(1+\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j$$

for every $\hat{\boldsymbol{x}}^{(j-1)}$ and β considered in (7.32). Consequently, we may bound (7.32) using that for every $\hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\subseteq\hat{B}_-^{(j-1)}$, $|\mathcal{J}(\hat{\boldsymbol{x}}^{(j-1)})|<(1-\sqrt{\delta_j})d_jb_j$ (cf. (7.24)) as

$$\left|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right|\times\left(1-\sqrt{\delta_j}\right)d_jb_j\times\left(\tilde{d}_j+\tilde{\delta}_j\right)(1+\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j . \tag{7.33}$$

On the other hand, we infer again from (7.5) of Fact 7.2 that

$$\begin{aligned}\left|\bigcup_{s\in[t]}\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| &= \sum\left\{\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)\right|:\ \hat{\boldsymbol{x}}^{(j-1)}\in\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right\}\\ &\ge\left|\hat{\Gamma}_-^{(j-1)}[\Lambda_j]\right|\times(1-\tilde{\eta})\left(\prod_{h=2}^{j-1}\tilde{d}_h^{\binom{j}{h}}\right)\left(\frac{n}{b_1}\right)^j . \end{aligned}\tag{7.34}$$

Comparing (7.33) and (7.34) yields

$$d\left(\tilde{\mathcal{G}}^{(j)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right)<d_j\frac{(1-\sqrt{\delta_j})\left(b_j\tilde{d}_j+b_j\tilde{\delta}_j\right)(1+\tilde{\eta})}{1-\tilde{\eta}} .$$

From (5.24) and $\tilde{\eta}\ll\delta_j$ (observe $j>2$ here), we infer

$$d\left(\tilde{\mathcal{G}}^{(j)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(j-1)}\right)<d_j\left(1-\delta_j^{3/4}\right)\left(1+\nu+b_j\tilde{\delta}_j\right) . \tag{7.35}$$

Finally, we observe that by Definition 4.14 (i) and $\tilde{d}_j>\tilde{\delta}_j$ we have $b_j<2/\tilde{d}_j$. Therefore, (7.28) follows from (7.35) and the choice of constants $\delta_j\gg\nu\gg\tilde{d}_j\gg\tilde{\delta}_j$. This completes the proof of Claim 7.4′. □

7.2.3 Proof of Property (c) of Lemma 5.6

In this section, we define the promised hypergraph $\mathcal{H}^{(k)}$ and confirm the Properties $(c1)$ and $(c2)$. We first observe that the hypergraph $\tilde{\mathcal{G}}^{(k)}$ 'almost' satisfies the properties of the promised $\mathcal{H}^{(k)}$. In particular, due to Lemma 4.15 (i), the hypergraph $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular w.r.t. every polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ for $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{b})$. However, the relative density $d\big(\tilde{\mathcal{G}}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ of $\tilde{\mathcal{G}}^{(k)}$ w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ may be 'wrong' (that is, differing substantially from d_k) for some $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$. We intend to replace $\tilde{\mathcal{G}}^{(k)}$ on those polyads. To that end, define

$$\hat{B}^{(k-1)} = \Big\{\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}\big(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\big) : \Big|d\big(\tilde{\mathcal{G}}^{(k)} \big| \hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big) - d_k\Big| > \delta_k^{1/2}\Big\}.$$

Similarly as in Section 7.2.1 (cf. Claim 7.4), we claim $\Big|\hat{B}^{(k-1)}\Big|$ is small.

Claim 7.5. $\Big|\hat{B}^{(k-1)}\Big| < 2\sqrt{\delta_k} \prod_{h=2}^{k-1} (d_h b_h)^{\binom{k}{h}} \times b_1^k$.

Again, we defer the rather technical but standard proof of Claim 7.5 to a different section, Section 7.2.4.

We prepare to define $\mathcal{H}^{(k)}$. To that end, we first define auxiliary hypergraphs $\mathcal{S}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ for $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$. While our work below is straightforward, we do need to distinguish two cases depending on whether $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$ or $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b}) \setminus \hat{B}^{(k-1)}$.

Case 1 ($\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \setminus \hat{B}^{(k-1)}$). We set

$$\mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = \tilde{\mathcal{G}}^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\,. \tag{7.36}$$

Case 2 ($\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$). Observe that

$$\big|\mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\big| > \frac{(n/b_1)^k}{\ln(n/b_1)}$$

by (7.5) of Fact 7.2. Hence, we may apply the Slicing Lemma, Lemma 4.16, with $m = n/b_1$, $p = d_k$, $\varrho = 1$, $\delta = \tilde{\delta}_k/3$ and $r_{\mathrm{SL}} = \tilde{r}$, and conclude that for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$ there exists a hypergraph

$$\mathcal{S}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) \subseteq \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) \tag{7.37}$$

which is $(\tilde{\delta}_k, d_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

We now define the promised hypergraph $\mathcal{H}^{(k)}$ as

$$\mathcal{H}^{(k)} = \bigcup \left\{ \mathcal{S}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \colon\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \right\}. \tag{7.38}$$

With $\mathcal{H}^{(k)}$ defined above, we claim that property $(c1)$ of Lemma 5.6 is immediately satisfied. Indeed, assertion $(c1)$ is clearly satisfied whenever $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$. On the other hand, by part (i) of Lemma 4.15, for every $\hat{\boldsymbol{x}}^{(k-1)}$ in $\hat{A}\left(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}\right)$, $\tilde{\mathcal{G}}^{(k)}$ is $(\tilde{\delta}_k, \tilde{r})$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. Moreover, by the definition of $\hat{B}^{(k-1)}$ above and $\mathcal{H}^{(k)}$ in (7.38), for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \setminus \hat{B}^{(k-1)}$,

$$\begin{aligned} d\left(\mathcal{H}^{(k)}\middle|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\right) &= d\left(\mathcal{S}^{(k)}\middle|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\right) \\ &= d\left(\tilde{\mathcal{G}}^{(k)}\middle|\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\right) = d_k \pm \sqrt{\delta_k}\,. \end{aligned}$$

Thus, property $(c1)$ is satisfied with $\mathcal{H}^{(k)}$ as defined above. The remainder of this section is therefore devoted to the proof of property $(c2)$ for $\mathcal{H}^{(k)}$.

The proof of property $(c2)$ is similar (but somewhat simpler) than the proof of $(b2)$. Here, we partition the ℓ-tuples $L_0 \in \mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})$ into

$$\begin{aligned} \mathfrak{K}_\ell^{(k)}(1) = \Bigg\{ & L_0 \in \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right) \colon \\ & \exists\, K_0 \in \binom{L_0}{k} \text{ so that } K_0 \in \left(\mathcal{K}_k^{(k-1)}(\mathcal{H}^{(k-1)}) \triangle \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})\right) \Bigg\} \\ \mathfrak{K}_\ell^{(k)}(2) = & \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right) \Big\backslash\, \mathfrak{K}_\ell^{(k)}(1) = \\ = \Bigg\{ & L_0 \in \left(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right) \colon \\ & \forall\, K_0 \in \binom{L_0}{k} \quad K_0 \in \left(\mathcal{K}_k^{(k-1)}(\mathcal{H}^{(k-1)}) \cap \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})\right) \Bigg\}. \end{aligned}$$

The last equality follows from an argument similar to the one given after (7.15).

Let L_0 be in $\mathfrak{K}_\ell^{(k)}(2)$. Note that $L_0 \in \mathcal{K}_\ell^{(k-1)}(\mathcal{H}^{(k-1)}) \cap \mathcal{K}_\ell^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)})$ (i.e., every $K_0 \in \binom{L_0}{k}$ 'touches' polyads $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$ only) and $L_0 \in \mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})$. Recall $\mathcal{H}^{(k)}$ and $\tilde{\mathcal{G}}^{(k)}$ only differ on 'bad' polyads (see the construction of $\mathcal{H}^{(k)}$ in (7.37)–(7.38)). Hence, there exists $K_0 \in \binom{L_0}{k}$ that 'touches' some 'bad' polyad $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$. Summarizing the above, we observe

$$\begin{aligned}\mathfrak{K}_\ell^{(k)}(2) = \Big\{ L_0 \in \Big(\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\Big) \colon \exists\, K_0 \in \binom{L_0}{k} \quad \text{and} \\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)} \text{ so that } K_0 \in \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\Big\}.\end{aligned} \tag{7.39}$$

By definition, $\mathfrak{K}_\ell^{(k)}(1) \cup \mathfrak{K}_\ell^{(k)}(2)$ is a partition of $\mathcal{K}_\ell^{(k)}(\mathcal{H}^{(k)}) \triangle \mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})$ and so we have

$$\left|\mathcal{K}_\ell^{(k)}\big(\mathcal{H}^{(k)}\big) \triangle \mathcal{K}_\ell^{(k)}\big(\tilde{\mathcal{G}}^{(k)}\big)\right| = \left|\mathfrak{K}_\ell^{(k)}(1)\right| + \left|\mathfrak{K}_\ell^{(k)}(2)\right|. \tag{7.40}$$

We now bound $\left|\mathfrak{K}_\ell^{(k)}(1)\right|$ and $\left|\mathfrak{K}_\ell^{(k)}(2)\right|$ to obtain part (c) of Lemma 5.6.

Bounding $|\mathfrak{K}_\ell^{(k)}(1)|$. The upper bound again easily follows from $(b2)$ of Lemma 5.6 for $j = k-1$ and $i = \ell$. Indeed, observe

$$\mathfrak{K}_\ell^{(k)}(1) \subseteq \mathcal{K}_\ell^{(k-1)}\big(\mathcal{H}^{(k-1)}\big) \triangle \mathcal{K}_\ell^{(k-1)}\big(\tilde{\mathcal{G}}^{(k-1)}\big)$$

holds by the same argument presented after (7.18). We therefore see

$$\left|\mathfrak{K}_\ell^{(k)}(1)\right| \le \delta_{k-1}^{1/3} \left(\prod_{h=2}^{k-1} d_h^{\binom{\ell}{h}}\right) n^\ell \le \frac{1}{2}\delta_k^{1/3} \left(\prod_{h=2}^{k} d_h^{\binom{\ell}{h}}\right) n^\ell, \tag{7.41}$$

where the last inequality follows from $\delta_{k-1}^{1/3} \ll \delta_k^{1/3} d_k$ as given in Figure 5.2.

Bounding $|\mathfrak{K}_\ell^{(k)}(2)|$. As a consequence of assertion $(b1)$ for $2 \le j < k$ and $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}^{(k-1)}$, we infer there are $\prod_{h=2}^{k-1}(d_h b_h)^{\binom{\ell}{h}-\binom{k}{h}} \times b_1^{\ell-k}$ ways to complete

$\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)$ to a $\big((\tilde{\delta}_2,\ldots,\tilde{\delta}_{k-1}),(\tilde{d}_2,\ldots,\tilde{d}_{k-1}),\tilde{r}\big)$-regular $(n/b_1,\ell,k-1)$-subcomplex $\boldsymbol{\check{\mathcal{H}}^{(k-1)}}=\{\check{\mathcal{H}}^{(h)}\}_{h=1}^{k-1}$ of $\mathcal{H}^{\boldsymbol{(k-1)}}$. Note that (7.4) of Fact 7.2 applied with $i=\ell$ and $j=k$ yields

$$\Big|\mathcal{K}_\ell^{(k-1)}(\check{\mathcal{H}}^{(k-1)})\Big|\le(1+\tilde{\eta})\left(\prod_{h=2}^{k-1}\tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell\le 2\left(\prod_{h=2}^{k-1}\tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell$$

for each such $\check{\mathcal{H}}^{(k-1)}$. Since this holds for every $\hat{\boldsymbol{x}}^{(k-1)}\in\hat{B}^{(k-1)}$, the last inequality combined with (7.39) and Claim 7.5 yields

$$\begin{aligned}\Big|\mathfrak{K}_\ell^{(k)}(2)\Big|&\le\Big|\hat{B}^{(k-1)}\Big|\times\left(\prod_{h=2}^{k-1}(d_hb_h)^{\binom{\ell}{h}-\binom{k}{h}}\right)b_1^{\ell-k}\times 2\left(\prod_{h=2}^{k-1}\tilde{d}_h^{\binom{\ell}{h}}\right)\left(\frac{n}{b_1}\right)^\ell\\&\le 4\sqrt{\delta_k}\left(\prod_{h=2}^{k-1}(d_hb_h\tilde{d}_h)^{\binom{\ell}{h}}\right)n^\ell\overset{(5.24)}{\le}4\sqrt{\delta_k}(1+\nu)\left(\prod_{h=2}^{k-1}d_h^{\binom{\ell}{h}}\right)n^\ell\,,\end{aligned}\tag{7.42}$$

and so by the choice of $\delta_k\ll d_k$ we have

$$\Big|\mathfrak{K}_\ell^{(k)}(2)\Big|\le\frac{1}{2}\delta_k^{1/3}\left(\prod_{h=2}^{k}d_h^{\binom{\ell}{h}}\right)n^\ell\,.\tag{7.43}$$

Combining (7.40) with (7.41) and (7.43) yields part (*c2*) of Lemma 5.6. Note that to establish (7.42) we used Claim 7.5 to bound $|\hat{B}^{(k-1)}|$. We prove this last component of Lemma 5.6 in the section below.

7.2.4 Proof of Claim 7.5

This proof follows the lines of the proof of Claim 7.4 presented in Section 7.2.2. We again first split $\hat{B}^{(k-1)}$ as follows (cf. (7.24)):

$$\hat{B}_+^{(k-1)}=\Big\{\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\big(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}\big):d\big(\tilde{\mathcal{G}}^{(k)}\big|\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big)>d_k+\delta_k^{1/2}\Big\},$$
$$\hat{B}_-^{(k-1)}=\Big\{\hat{\boldsymbol{x}}^{(k-1)}\in\hat{A}\big(\mathcal{H}^{(k-1)},k-1,\boldsymbol{b}\big):d\big(\tilde{\mathcal{G}}^{(k)}\big|\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\big)<d_k-\delta_k^{1/2}\Big\}.$$

We verify the following claim which combined with Fact 7.3 immediately implies Claim 7.5.

Claim 7.5′. *If for some* $* \in \{+,-\}$,

$$\left|\hat{B}_{*}^{(k-1)}\right| \geq \sqrt{\delta_k}\left(\prod_{h=2}^{k-1}(d_h b_h)^{\binom{k}{h}}\right) b_1^k\,,$$

then $\tilde{\mathcal{G}}^{(k)}$ *is not* $(2\delta_k, d_k, r)$*-regular w.r.t.* $\tilde{\mathcal{G}}^{(k-1)}$.

Proof of Claim 7.5′. While we showed explicitely the case $* = -$ in the proof of the similar claim, Claim 7.4′, for a change here we present the details for $* = +$ only. Without loss of generality, we assume that there exists $\Lambda_k \in \binom{[\ell]}{k}$ such that

$$\left|\hat{B}_{+}^{(k-1)}[\Lambda_k]\right| \geq \frac{\sqrt{\delta_k}}{\binom{\ell}{k}}\left(\prod_{h=2}^{k-1}(d_h b_h)^{\binom{k}{h}}\right) b_1^k \tag{7.44}$$

where $\hat{B}_{+}^{(k-1)}[\Lambda_k]$ is the set of $\hat{\boldsymbol{x}}^{(k-1)} = (\hat{\boldsymbol{x}}_0, \ldots, \hat{\boldsymbol{x}}_{k-1}) \in \hat{B}_{+}^{(k-1)}$ such that $\hat{\boldsymbol{x}}_0 = \Lambda_k$. Similarly as in the proof of Claim 7.4′, we show that (7.44) implies that $\tilde{\mathcal{G}}^{(k)}$ is irregular w.r.t. $\tilde{\mathcal{G}}^{(k-1)}$. However, again the polyad addresses $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{B}_{+}^{(k-1)}[\Lambda_k]$ considered in (7.44) correspond to subhypergraphs of $\mathcal{H}^{(k-1)}$ and not necessarily to subhypergraphs of $\tilde{\mathcal{G}}^{(k-1)}$. The set $\hat{\Gamma}_{+}^{(k-1)}[\Lambda_k]$, defined below, is the subset of those polyad addresses of $\hat{B}_{+}^{(k-1)}[\Lambda_k]$ which correspond to subhypergraphs of $\tilde{\mathcal{G}}^{(k-1)}$ as well. Only these addresses are useful to verify Claim 7.5′.

We set

$$\hat{\Gamma}_{+}^{(k-1)}[\Lambda_k] = \hat{B}_{+}^{(k-1)}[\Lambda_k] \cap \hat{A}\big(\tilde{\mathcal{G}}^{(k-1)}, k-1, \boldsymbol{b}\big)$$

and

$$\hat{\boldsymbol{\mathcal{Q}}}^{(k-1)} = \left\{\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) \colon \hat{\boldsymbol{x}}^{(k-1)} \in \hat{\Gamma}_{+}^{(k-1)}[\Lambda_k]\right\} = \left\{\hat{\mathcal{Q}}_1^{(k-1)}, \ldots, \hat{\mathcal{Q}}_t^{(k-1)}\right\}$$

for $t = \left|\hat{\Gamma}_{+}^{(k-1)}[\Lambda_k]\right|$. In what follows, we show

$$\left|\bigcup_{s\in[t]}\left\{\mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{Q}}_s^{(k-1)}\big)\right\}\right| > 2\delta_k\left|\mathcal{K}_k^{(k-1)}\left(\tilde{\mathcal{G}}^{(k-1)}[\Lambda_k]\right)\right| \tag{7.45}$$

and

$$d\left(\tilde{\mathcal{G}}^{(k)}\middle|\hat{\boldsymbol{\mathcal{Q}}}^{(k-1)}\right) > d_k + 2\delta_k. \tag{7.46}$$

From (5.22), we see $r \geq \left|\hat{A}(k-1, \boldsymbol{b})\right| \geq t$. Therefore, establishing (7.45) and (7.46) proves Claim 7.5′.

We first verify (7.45). Observe that from (*b2*) (for $j = k-1$),

$$\begin{aligned}
&\mathcal{K}_k^{(k-1)}(\mathcal{H}^{(k-1)}) \triangle \mathcal{K}_k^{(k-1)}(\tilde{\mathcal{G}}^{(k-1)}) \\
&\qquad \supseteq \bigcup \left\{\mathcal{K}_k^{(k-1)}(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}))\colon\ \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}_+^{(k-1)}[\Lambda_k] \setminus \hat{\Gamma}_+^{(k-1)}[\Lambda_k]\right\}
\end{aligned}$$

and so

$$\begin{aligned}
&\left|\bigcup_{s\in[t]} \left\{\mathcal{K}_k^{(k-1)}(\hat{\mathcal{Q}}_s^{(k-1)})\right\}\right| = \left|\bigcup \left\{\mathcal{K}_k^{(k-1)}(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}))\colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{\Gamma}_+^{(k-1)}[\Lambda_k]\right\}\right| \\
&\geq \sum \left\{\left|\mathcal{K}_k^{(k-1)}(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)}))\right|\colon \hat{\boldsymbol{x}}^{(j-1)} \in \hat{B}_+^{(k-1)}[\Lambda_k]\right\} - \delta_{k-1}^{1/3}\left(\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}\right) n^k.
\end{aligned}$$

Applying (7.5) of Fact 7.2 to every term in the sum above yields the further lower bound

$$\left|\hat{B}_+^{(k-1)}[\Lambda_k]\right| (1-\tilde{\eta}) \left(\prod_{h=2}^{k-1} \tilde{d}_h^{\binom{k}{h}}\right) \left(\frac{n}{b_1}\right)^k - \delta_{k-1}^{1/3}\left(\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}\right) n^k .$$

Hence, from our assumption (7.44), we infer

$$\begin{aligned}
&\left|\bigcup_{s\in[t]} \left\{\mathcal{K}_k^{(k-1)}(\hat{\mathcal{Q}}_s^{(k-1)})\right\}\right| \\
&\qquad \geq \binom{\ell}{k}^{-1} (1-\tilde{\eta})\sqrt{\delta_k} \left(\prod_{h=2}^{k-1} \left(d_h b_h \tilde{d}_h\right)^{\binom{k}{h}}\right) n^k - \delta_{k-1}^{1/3}\left(\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}\right) n^k \\
&\qquad \overset{(5.24)}{\geq} \left(\binom{\ell}{k}^{-1} (1-\tilde{\eta})(1-\nu)\sqrt{\delta_k} - \delta_{k-1}^{1/3}\right) \left(\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}\right) n^k .
\end{aligned} \tag{7.47}$$

Finally, the choice of $\tilde{\eta}$, ν, and $\delta_k \gg \delta_{k-1}$ applied to (7.47) and inequality (7.2) of Fact 7.1 for $j = k-1$ and $i = k$ imply (7.45).

It is left to verify (7.46). We set, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{\Gamma}_+^{(k-1)}[\Lambda_k]$,

$$\tilde{\mathcal{G}}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) = \tilde{\mathcal{G}}^{(k)} \cap \mathcal{K}_k^{(k-1)}\Big(\hat{\mathcal{P}}^{(k-1)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\Big).$$

We now apply (7.5) of Fact 7.2 with $j = k$ to each $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{\Gamma}_+^{(k-1)}[\Lambda_k]$ and the assumption on $\hat{B}_+^{(k-1)} \supseteq \hat{\Gamma}_+^{(k-1)}[\Lambda_k]$ to infer

$$\begin{aligned}\left|\tilde{\mathcal{G}}^{(k)} \cap \bigcup_{s\in[t]} \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{Q}}_s^{(k-1)}\big)\right| &= \sum\left\{\left|\tilde{\mathcal{G}}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big)\right|:\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{\Gamma}_+^{(k-1)}[\Lambda_k]\right\}\\ &> \left|\hat{\Gamma}_+^{(k-1)}[\Lambda_k]\right| \times \left(d_k + \sqrt{\delta_k}\right) \times (1-\tilde{\eta}) \left(\prod_{h=2}^{k-1} \tilde{d}_h^{\binom{k}{h}}\right)\left(\frac{n}{b_1}\right)^k. \end{aligned} \tag{7.48}$$

On the other hand, again from (7.5) we infer

$$\begin{aligned}\left|\bigcup_{s\in[t]} \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{Q}}_s^{(k-1)}\big)\right| &= \sum\left\{\left|\mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\right|:\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{\Gamma}_+^{(k-1)}[\Lambda_k]\right\}\\ &\le \left|\hat{\Gamma}_+^{(k-1)}[\Lambda_k]\right| \times (1+\tilde{\eta}) \left(\prod_{h=2}^{k-1} \tilde{d}_h^{\binom{k}{h}}\right)\left(\frac{n}{b_1}\right)^k \end{aligned} \tag{7.49}$$

Comparing (7.48) and (7.49) yields

$$d\left(\tilde{\mathcal{G}}^{(k)}\big|\hat{\boldsymbol{\mathcal{Q}}}^{(k-1)}\right) > \left(d_k + \sqrt{\delta_k}\right)\frac{1-\tilde{\eta}}{1+\tilde{\eta}}$$

which yields (7.46), since $d_k \gg \delta_k \gg \tilde{\eta}$. This finishes the proof of Claim 7.4′.

□

7.3 Proof of Lemma 5.7

Lemma 5.7 follows from a simple and straightforward application of the Slicing Lemma, Lemma 4.16. Recall Setup 5.5 and that $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(h)}\}_{h=1}^k$ is the (n, ℓ, k)-complex given by Lemma 5.6.

Proof of Lemma 5.7. Recall that by (*c*) of Lemma 5.6, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, the (n, ℓ, k)-cylinder

$$\begin{aligned}\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)& \\ \text{is } (\tilde{\delta}_k, \bar{d}(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r})\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})&\end{aligned} \tag{7.50}$$

where $\bar{d}(\hat{\boldsymbol{x}}^{(k-1)}) = d_k \pm \sqrt{\delta_k}$.

Construction of $\boldsymbol{\mathcal{H}_-}$. For $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, apply the Slicing Lemma, Lemma 4.16, with $\varrho = \bar{d}(\hat{\boldsymbol{x}}^{(k-1)})$, $p = (d_k - \sqrt{\delta_k})/\varrho$, $\delta = \tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to $\mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ to obtain a $(3\tilde{\delta}_k, d_k - \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\mathcal{S}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$.

Note that the assumptions of the Slicing Lemma are satisfied. This is due to the fact that the family of partitions $\mathscr{P}$ is an almost perfect $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, \tilde{r}, \boldsymbol{b})$-family and, consequently, $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $\big((\tilde{\delta}_2, \dots, \tilde{\delta}_{k-1}), (\tilde{d}_2, \dots, \tilde{d}_{k-1}), \tilde{r}\big)$-regular. Hence, by (7.5) of Fact 7.2 (with $j = k$),

$$\left|\mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\right| > \frac{(n/b_1)^k}{\ln(n/b_1)}$$

for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$.

We then set

$$\mathcal{H}_-^{(k)} = \bigcup \left\{\mathcal{S}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \colon\ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})\right\}.$$

Obviously, $\mathcal{H}_-^{(k)}$ has the desired properties (α) and ($\beta 1$) by construction.

Construction of $\boldsymbol{\mathcal{H}_+}$. The construction of $\mathcal{H}_+^{(k)}$ is similar and follows by an application of the Slicing Lemma to the complement of $\mathcal{H}^{(k)}$. More precisely, for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, set $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(k-1)})\big) \setminus \mathcal{H}^{(k)}$. Note that, due to (7.50), $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(\tilde{\delta}_k, 1 - \bar{d}(\hat{\boldsymbol{x}}^{(k-1)}), \tilde{r})$-regular. Consequently, we can apply the Slicing Lemma to $\overline{\mathcal{H}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\varrho = 1 - \bar{d}(\hat{\boldsymbol{x}}^{(k-1)})$, $p = (1 - d_k - \sqrt{\delta_k})/\varrho$, $\delta = \tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to obtain a $(3\tilde{\delta}_k, 1 -$

$d_k - \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\overline{\mathcal{S}}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. We then set $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(k-1)})\big) \setminus \overline{\mathcal{S}}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Clearly, $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(3\tilde{\delta}_k, d_k + \sqrt{\delta_k}, \tilde{r})$-regular and $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \supseteq \mathcal{H}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Finally, we define $\mathcal{H}_+^{(k)}$ to be the union of all $\mathcal{S}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ constructed that way.

Construction of $\mathcal{F}$. The construction of $\mathcal{F}^{(k)}$ is more involved owing to the requirement $\mathcal{H}_-^{(k)} \subseteq \mathcal{F}^{(k)} \subseteq \mathcal{H}_+^{(k)}$.

Let $\mathcal{H}_-^{(k)}$ and $\mathcal{H}_+^{(k)}$ be given as constructed above and for $* \in \{+, -\}$ and $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b})$, let $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}_*^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$. Due to (*β1*) and (*β2*), $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(3\tilde{\delta}_k, d_k^*, \tilde{r})$-regular where $d_k^- = d_k - \sqrt{\delta_k}$ and $d_k^+ = d_k + \sqrt{\delta_k}$. Moreover, $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \supseteq \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ and, consequently, $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \setminus \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is $(6\tilde{\delta}_k, 2\sqrt{\delta_k}, \tilde{r})$-regular. We now apply the Slicing lemma to $\mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \setminus \mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ with $\varrho = 2\sqrt{\delta_k}$, $p = \sqrt{\delta_k}/\varrho = 1/2$, $\delta = 6\tilde{\delta}_k$ and $r_{\mathrm{SL}} = \tilde{r}$ to obtain a $(18\tilde{\delta}_k, \sqrt{\delta_k}, \tilde{r})$-regular hypergraph $\mathcal{S}_{\mathcal{F}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Now define $\mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ to be the disjoint union $\mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \cup \mathcal{S}_{\mathcal{F}}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Clearly, $\mathcal{H}_-^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{F}^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) \subseteq \mathcal{H}_+^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$. Moreover, it is straightforward to verify that $\mathcal{F}^{(k)}$ is $(21\tilde{\delta}_k, d_k, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ and, consequently,

$$\mathcal{F}^{(k)} = \bigcup \left\{ \mathcal{F}^{(k)}\big(\hat{\boldsymbol{x}}^{(k-1)}\big) \colon \ \hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(\mathcal{H}^{(k-1)}, k-1, \boldsymbol{b}) \right\},$$

has the desired properties. $\square$

Chapter 8

Proof of the Union Lemma

8.1 Union of regular hypergraphs

Below we present some useful facts regarding regularity properties of the union of regular (m,j,j)-cylinders. We distinguish two cases depending whether the (m,j,j)-cylinder in question has the same underlying polyad or not.

The first proposition says that we may take the union of disjoint regular (m,j,j)-cylinders of the same density which share the same underlying $(m,j,j-1)$-cylinder without spoiling the regularity too much.

Proposition 8.1. *Let $j \geq 2$, t and m be fixed positive integers, let δ and d be positive reals and let $\mathcal{P}_1^{(j)}, \dots, \mathcal{P}_t^{(j)}$ be a family of pairwise edge disjoint (m,j,j)-cylinders with the same underlying $(m,j,j-1)$-cylinder $\hat{\mathcal{P}}^{(j-1)}$. If for every $s \in [t]$, the hypergraph $\mathcal{P}_s^{(j)}$ is $(\delta, d, 1)$-regular with respect to $\hat{\mathcal{P}}^{(j-1)}$, then $\mathcal{P}^{(j)} = \bigcup_{s\in[t]} \mathcal{P}_s^{(j)}$ is $(t\delta, td, 1)$-regular with respect to $\hat{\mathcal{P}}^{(j-1)}$.*

The proof of Proposition 8.1 is straightfoward and short and we therefore omit it. The next proposition gives us control when we unite hypergraphs having different underlying polyads. Before we make this precise, we define the setup for our proposition.

Setup 8.2. *Let $j \geq 3$, t and m be fixed positive integers and let δ and d be positive reals. Let $\{\hat{\mathcal{P}}_s^{(j-1)}\}_{s\in[t]}$ be a family of $(m, j, j-1)$-cylinders such that*

$$\begin{aligned} \mathcal{K}_j^{(j-1)}\Big(\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}\Big) &= \bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big) \quad \text{and} \\ \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big) \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_{s'}^{(j-1)}\big) &= \varnothing \quad \text{for } 1 \leq s < s' \leq t\,. \end{aligned} \tag{8.1}$$

From (8.1), $\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big)$ is a partition of the j-cliques of $\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$. Let $\{\mathcal{P}_s^{(j)}\}_{s\in[t]}$ be a family of (m, j, j)-cylinders such that $\hat{\mathcal{P}}_s^{(j-1)}$ underlies $\mathcal{P}_s^{(j)}$ for any $s \in [t]$. Set $\hat{\mathcal{P}}^{(j-1)} = \bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$ and $\mathcal{P}^{(j)} = \bigcup_{s\in[t]} \mathcal{P}_s^{(j)}$.

Proposition 8.3. *Let $\{\mathcal{P}_s^{(j)}\}_{s\in[t]}$ and $\{\hat{\mathcal{P}}_s^{(j-1)}\}_{s\in[t]}$ satisfy Setup 8.2. If $\mathcal{P}_s^{(j)}$ is $(\delta, d, 1)$-regular w.r.t. $\hat{\mathcal{P}}_s^{(j-1)}$ for every $s \in [t]$, then $\mathcal{P}^{(j)}$ is $(2\sqrt{\delta}, d, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}$.*

Proof. Let $\hat{\mathcal{Q}}^{(j-1)} \subseteq \hat{\mathcal{P}}^{(j-1)}$ be such that

$$\big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\big| \geq \sqrt{\delta}\big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\big|\,. \tag{8.2}$$

For every $s \in [t]$, set $\hat{\mathcal{Q}}_s^{(j-1)} = \hat{\mathcal{Q}}^{(j-1)} \cap \hat{\mathcal{P}}_s^{(j-1)}$. Since $\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big)$ is a partition of the j-cliques of $\bigcup_{s\in[t]} \hat{\mathcal{P}}_s^{(j-1)}$, $\bigcup_{s\in[t]} \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)$ is a partition of the j-cliques of $\hat{\mathcal{Q}}^{(j-1)} = \bigcup_{s\in[t]} \hat{\mathcal{Q}}_s^{(j-1)}$. As such,

$$\sum_{s\in[t]} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| = \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big|\,. \tag{8.3}$$

Define

$$T = \Big\{ s \in [t] \colon \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| \geq \delta \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}_s^{(j-1)}\big)\Big| \Big\}\,.$$

Observe that

$$\sum_{s\notin T} \Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\Big| < \delta\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\Big| \overset{(8.2)}{\leq} \sqrt{\delta}\Big|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\Big|\,. \tag{8.4}$$

Consequently 8.2, (8.3) and (8.4) give

$$\begin{aligned}\sum_{s\in T}\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| &\geq \left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\right| - \delta\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}\big)\right| \\ &\geq \big(1-\sqrt{\delta}\big)\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\right|.\end{aligned} \tag{8.5}$$

If $s \in T$, then the $(\delta, d, 1)$-regularity of $\mathcal{P}_s^{(j)}$ w.r.t. $\hat{\mathcal{P}}_s^{(j-1)}$ implies

$$\left|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| = (d \pm \delta)\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right|.$$

Consequently,

$$\begin{aligned}\left|\mathcal{P}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\right| &\overset{(8.3)}{=} \sum_{s\in[t]}\left|\mathcal{P}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| \\ &= \sum_{s\in T}\left|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| + \sum_{s\notin T}\left|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| \\ &= (d\pm\delta)\sum_{s\in T}\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| + \sum_{s\notin T}\left|\mathcal{P}_s^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right|.\end{aligned}$$

We then see

$$\begin{aligned}&(d-\delta)\sum_{s\in T}\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right| \\ &\qquad\leq \left|\mathcal{P}^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\right| \leq \\ &\qquad\quad (d+\delta)\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}^{(j-1)}\big)\right| + \sum_{s\notin T}\left|\mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{Q}}_s^{(j-1)}\big)\right|.\end{aligned}$$

In view of (8.4) and (8.5), we infer

$$(d-\delta)(1-\sqrt{\delta}) \leq d\big(\mathcal{P}^{(j)}\big|\hat{\mathcal{Q}}^{(j-1)}\big) \leq d+\delta+\sqrt{\delta}$$

from which Proposition 8.3 follows. □

8.2 Proof of Lemma 5.9

Before proving Lemma 5.9, we recall some notation. For $* \in \{+,-\}$, let $\boldsymbol{\mathcal{H}}_* = \{\mathcal{H}^{(j)}\}_{j=1}^{k-1} \cup \mathcal{H}_*^{(k)} = \{\mathcal{H}_*^{(j)}\}_{j=1}^{k}$ be given by Lemma 5.7. It follows that for each $j = 2, \ldots, k-1$, the set $\hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$ of polyad addresses with $\hat{\mathcal{P}}(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq \mathcal{H}^{(j-1)}$ satisfies that for each $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$, there is an index set $I(\hat{\boldsymbol{x}}^{(j-1)}) \subseteq [b_j]$ of size $d_j b_j$ such that

$$\mathcal{H}_*^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big).$$

Moreover, $d\big(\mathcal{H}_*^{(k)} \big| \hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big) = d_k^*$, where d_k^* is defined in (5.29). Recall that for $\Lambda_j = (\lambda_1, \ldots, \lambda_j) \in \binom{[\ell]}{j}_<$, we denote by $\mathcal{H}_*^{(j)}[\Lambda_j]$ the subhypergraph of $\mathcal{H}_*^{(j)}$ induced on $V_{\lambda_1} \cup \cdots \cup V_{\lambda_j}$.

Due to Lemma 5.6, we know that for $j = 2, \ldots, k-1$, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}(\mathcal{H}^{(j-1)}, j-1, \boldsymbol{b})$, the set $I(\hat{\boldsymbol{x}}^{(j-1)})$ satisfies $|I(\hat{\boldsymbol{x}}^{(j-1)})| = d_j b_j$; moreover, for every $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})$, the $(n/b_1, j, j)$-cylinder $\mathcal{P}^{(j)}\big(\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ is $(\tilde{\delta}, \tilde{d}, \tilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Inductively on j, we aim to show that $\mathcal{H}_*^{(j)}[\Lambda_j]$, which is the union of all $\mathcal{P}^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}, \alpha)$, with $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{j-1})$, $\hat{\boldsymbol{x}}_0 = \Lambda_j$ and $\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})$, is regular w.r.t. $\mathcal{H}_*^{(j-1)}[\Lambda_j]$.

Proof of Lemma 5.9. We only prove the statement about $\boldsymbol{\mathcal{H}}_*$ here. The proof for $\boldsymbol{\mathcal{F}}$ is identical. Consider the following statement:

(S_j) For every $\Lambda_j \in \binom{[\ell]}{j}_<$ then $\boldsymbol{\mathcal{H}}_{\boldsymbol{*}}^{\boldsymbol{(j)}}[\Lambda_j]$ is a $\big((\varepsilon', \ldots, \varepsilon'), (d_2^*, \ldots, d_j^*), 1\big)$-regular (n, j, j)-complex.

Lemma 5.9 then follows from (S_k).

We prove statement (S_j) by induction on j. Suppose $j = 2$ and let $\Lambda_2 \in \binom{[\ell]}{2}_<$ be given. By (a) of Lemma 5.6, we have $\tilde{\mathcal{G}}^{(2)} = \mathcal{H}^{(2)} = \mathcal{H}_*^{(2)}$. Consequently, $\mathcal{H}_*^{(2)}[\Lambda_2]$ is $(\delta_2, d_2, 1)$-regular w.r.t. $\mathcal{H}_*^{(1)}[\Lambda_2]$ and (S_2) follows from $\delta_2 \ll \varepsilon'$ (cf. Figure 5.2).

We now proceed to the induction step. Assume $3 \le j \le k$ and (S_{j-1}) holds. Let $\Lambda_j = (\lambda_1, \dots, \lambda_j) \in \binom{[\ell]}{j}_<$ be arbitrary but fixed. The proof of (S_j) consists of three steps and we begin with the easiest.

Step 1. Let $\hat{X}(\Lambda_j) \subseteq \hat{A}(\mathcal{H}_*^{(j-1)}, j-1, \boldsymbol{b})$ be the set of all polyad addresses $\hat{\boldsymbol{x}}^{(j-1)} = (\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \dots, \hat{\boldsymbol{x}}_{j-1})$ with $\hat{\boldsymbol{x}}_0 = \Lambda_j$. In Step 2, we consider

$$\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big) \tag{8.6}$$

for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$. To that end, fix a $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$. This (and only this) step splits into two cases, depending on whether $j = k$ or not.

Case 1 $(3 \le j < k)$**.** We apply Proposition 8.1 to

$$\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) \qquad \text{and} \qquad \Big\{\mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)\colon\ \alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})\Big\}$$

with $\delta = \delta_j$, $d = \tilde{d}_j$ (since $j < k$), and $t = \big|I(\hat{\boldsymbol{x}}^{(j-1)})\big| = d_j b_j = d_j^* b_j$. Consequently, for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$,

$$\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \bigcup_{\alpha \in I(\hat{\boldsymbol{x}}^{(j-1)})} \mathcal{P}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big) = \mathcal{H}_*^{(j)} \cap \mathcal{K}_j^{(j-1)}\big(\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big)$$

is $\big((d_j^* b_j)\tilde{\delta}_j, (d_j^* b_j)\tilde{d}_j, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$. Since $b_j \tilde{\delta}_j \le 2\tilde{\delta}_j/\tilde{d}_j \ll \nu$ and from (5.24), each $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $\big(2\nu, d_j^*, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$.

Case 2 $(j = k)$**.** Here, we infer directly from conclusion (β) of Lemma 5.7 that $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)}) = \mathcal{H}_*^{(k)} \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)$ is $(3\tilde{\delta}_k, d_k^*, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$. Consequently, $\mathcal{H}_*^{(k)}(\hat{\boldsymbol{x}}^{(k-1)})$ is also $\big(2\nu, d_k^*, 1\big)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$.

Summarizing the two cases, we conclude that for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$, the $(n/b_1, j, j)$-cylinder $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$, as given in (8.6), is

$$\big(2\nu, d_j^*, 1\big)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\,. \tag{8.7}$$

Step 2. In this step, we apply the induction assumption (S_{j-1}). For every $\iota \in [j]$, set $\Lambda_j(\iota) = (\lambda_1, \ldots, \lambda_{\iota-1}, \lambda_{\iota+1}, \ldots, \lambda_j)$. We apply (S_{j-1}) to the $(n, j, j-1)$-complex $\boldsymbol{\mathcal{H}}_*^{(\boldsymbol{j-1})}[\Lambda_j(\iota)]$ for every $\iota \in [j]$. As a result, we infer that

$$\boldsymbol{\mathcal{H}}_*^{(\boldsymbol{j-1})}[\Lambda_j] = \bigcup_{\iota\in[j]} \boldsymbol{\mathcal{H}}_*^{(\boldsymbol{j-1})}[\Lambda_j(\iota)] = \left\{ \bigcup_{\iota\in[j]} \mathcal{H}^{(h)}[\Lambda_j(\iota)] \right\}_{h=1}^{j-1}$$

is an $\big((\varepsilon', \ldots, \varepsilon'), (d_2^*, \ldots, d_{j-1}^*), 1\big)$-regular $(n, j, j-1)$-complex.

Step 3. Finally, as the last step, we show that the disjoint union

$$\bigcup_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j)}[\Lambda_j]$$

is $\big(\varepsilon', d_j^*, 1\big)$-regular w.r.t. $\mathcal{H}_*^{(j-1)}[\Lambda_j]$ (cf. (8.6)). Recall that $\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$ is $(2\nu, d_j^*, 1)$-regular w.r.t. $\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})$ for each $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{X}(\Lambda_j)$, as shown in Step 1 (cf. (8.7)). It is easy to see that $\big\{\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\big\}_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)}$ satisfies (8.1). Consequently, the assumptions of Proposition 8.3 are satisfied with $\delta = 2\nu$, $d = d_j^*$, and $t = \big|\hat{X}(\Lambda_j)\big|$ for the families

$$\left\{\hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \quad \text{and} \quad \left\{\mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})\right\}_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)}.$$

Therefore, it follows from Proposition 8.3 that

$$\mathcal{H}_*^{(j)}[\Lambda_j] = \bigcup_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \mathcal{H}_*^{(j)}(\hat{\boldsymbol{x}}^{(j-1)})$$

is $(2\sqrt{2\nu}, d_j^*, 1)$-regular w.r.t. $\bigcup_{\hat{\boldsymbol{x}}^{(j-1)}\in\hat{X}(\Lambda_j)} \hat{\mathcal{P}}^{(j-1)}(\hat{\boldsymbol{x}}^{(j-1)}) = \mathcal{H}_*^{(j-1)}[\Lambda_j]$. Then (S_j) follows from $2\sqrt{2\nu} < \varepsilon'$ (cf. Figure 5.2). $\square$

Chapter 9

Proofs of the generalizations of the Counting Lemma

In this chapter we prove the corollaries, Corollary 3.9 and Corollary 3.12, discussed in Section 3.4 of Theorem 3.6.

9.1 Proof of Corollary 3.9

Let ℓ and k be given. We first fix all constants involved in the proof of the corollary.

Recall the quantification of constants in Corollary 3.9 is

$$\forall \gamma, d_k \ \exists \delta_k \ \forall d_{k-1} \ \exists \delta_{k-1} \ \ldots \ \forall d_2 \ \exists \delta_2, r \, .$$

The corollary is trivially true for $\gamma \geq 1$. Hence, let $1 > \gamma > 0$ be given. Set an auxiliary constant $\bar{\gamma}$ so that

$$(1 - \gamma/2) = (1 - \bar{\gamma})^{2^{\ell}} \, . \tag{9.1}$$

Next, for every given d_j, we define auxiliary constants $\tilde{d}_j$ and $\tilde{\delta}_j$ and then the required δ_j inductively for $j = k, \dots, 2$ as follows:

$$\tilde{d}_j = \bar{\gamma} d_j \tag{9.2}$$

$$\begin{aligned}\tilde{\delta}_j = \min \Big\{ &\delta_j\big(\mathbf{CL}_{k,\ell}(\gamma/2, \tilde{d}_k, \tilde{\delta}_k, \dots, \tilde{d}_j)\big), \\ &\qquad \delta_j\big(\mathbf{IHC}_{k-1,k}(1/2, \tilde{d}_k, \tilde{\delta}_k, \dots, \tilde{d}_j\big)\Big\}\end{aligned} \tag{9.3}$$

$$\delta_j = \min\left\{ \frac{\tilde{\delta}_j}{9} \bar{\gamma}^{2^k}, \frac{\tilde{d}_j}{3}, \delta_j\big(\mathbf{CL}_{k,\ell}(1/2, d_k, \delta_k, \dots, d_j)\big) \right\}. \tag{9.4}$$

Finally, we set r to

$$\begin{aligned} r = \max\Big\{ &r\big(\mathbf{CL}_{k,\ell}(\gamma/3, \tilde{d}_k, \tilde{\delta}_k, \dots, \tilde{\delta}_2)\big), \\ &\qquad r\big(\mathbf{IHC}_{k-1,k}(1/2, \tilde{d}_k, \tilde{\delta}_k, \dots, \tilde{\delta}_2)\big), \\ &\qquad\qquad r\big(\mathbf{IHC}_{k-1,k}(1/2, d_k, \delta_k, \dots, \delta_2)\big)\Big\}. \end{aligned} \tag{9.5}$$

Let $\mathcal{F}^{(k)}$ be a fixed k-uniform hypergraph on ℓ vertices $\{1, \dots, \ell\}$ and let $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^k$ be a $(\boldsymbol{\delta}, \geq\boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular (n, ℓ, k)-complex with $\boldsymbol{\delta} = (\delta_2, \dots, \delta_k)$ and $\boldsymbol{d} = (d_2, \dots, d_k)$. We show $\boldsymbol{\mathcal{G}}$ contains the desired number of copies of $\mathcal{F}^{(k)}$ by using the original Counting Lemma, Theorem 3.6.

Our first step is to construct an "everywhere regular" (n, ℓ, k)-complex $\tilde{\boldsymbol{\mathcal{G}}} = \{\tilde{\mathcal{G}}^{(j)}\}_{j=1}^k$ from $\boldsymbol{\mathcal{G}}$. To do so, set $\tilde{\mathcal{G}}^{(1)} = \mathcal{G}^{(1)} = V_1 \cup \dots \cup V_\ell$ and for every $j = 2, \dots, k$ and $\Lambda_j = \{\lambda_1, \dots, \lambda_j\} \in \binom{[\ell]}{j}$, set

$$\tilde{\mathcal{G}}^{(j)}[\Lambda_j] = \begin{cases} \mathcal{G}^{(j)}[\Lambda_j] & \text{if} \quad \Lambda_j \in \Delta_j(\mathcal{F}^{(k)}) \\ K_j^{(j)}(V_{\lambda_1}, \dots, V_{\lambda_j}) & \text{if} \quad \Lambda_j \notin \Delta_j(\mathcal{F}^{(k)}). \end{cases} \tag{9.6}$$

Observe that the densities $d_{\Lambda_j} = d(\tilde{\mathcal{G}}^{(j)}[\Lambda_j] | \tilde{\mathcal{G}}^{(j-1)}[\Lambda_j])$, $\Lambda_j \in \binom{[\ell]}{j}$, of the complex $\tilde{\boldsymbol{\mathcal{G}}}$ satisfy

$$d_{\Lambda_j} = \begin{cases} d_j & \text{if} \quad j < k \quad \text{and} \quad \Lambda_j \in \Delta_j(\mathcal{F}^{(k)}) \\ d\big(\mathcal{G}^{(k)}[\Lambda_k] \big| \mathcal{G}^{(k-1)}[\Lambda_k]\big) & \text{if} \quad j = k \quad \text{and} \quad \Lambda_k \in \mathcal{F}^{(k)} = \Delta_k(\mathcal{F}^{(k)}) \\ 1 & \text{otherwise}. \end{cases}$$

Clearly, all (n, j, j)-cylinders $\tilde{\mathcal{G}}^{(j)}[\Lambda_j]$ are $(\delta_j, d_{\Lambda_j}, r)$-regular w.r.t $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]$, $2 \leq j \leq k$. As well, observe that owing to the construction of $\tilde{\boldsymbol{\mathcal{G}}}$ and Definition 3.8,

$$\begin{gathered}\text{the number of partite isomorphic copies of } \mathcal{F}^{(k)} \text{ in } \mathcal{G}^{(k)} \\ \text{is equal } |\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})|\,.\end{gathered} \tag{9.7}$$

Unfortunately, the complex $\tilde{\boldsymbol{\mathcal{G}}}$ is not ready for an application of the Counting Lemma, Theorem 3.6, since the densities d_{Λ_j}, $\Lambda_j \in \binom{[\ell]}{j}$, vary depending on whether $\Lambda_j \in \Delta_j(\mathcal{F}^{(k)})$, $2 \leq j \leq k$. We circumvent this minor technicality by (inductively) decomposing $\tilde{\boldsymbol{\mathcal{G}}}$ into a family $\mathscr{S}_\ell^{(k)}$ of $K_\ell^{(k)}$-disjoint $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, r)$-regular (n, ℓ, k)-subcomplexes, $\tilde{\boldsymbol{\delta}} = (\tilde{\delta}_2, \ldots, \tilde{\delta}_k)$, $\tilde{\boldsymbol{d}} = (\tilde{d}_2, \ldots, \tilde{d}_k)$, using the Slicing Lemma, Lemma 4.16. We then apply the Counting Lemma to each such subcomplex and then add the cliques up over the family $\mathscr{S}_\ell^{(k)}$.

We inductively apply the Slicing Lemma to $\tilde{\boldsymbol{\mathcal{G}}}$ to first produce a family $\mathscr{S}_\ell^{(2)}$ of $(\tilde{\delta}_2, \tilde{d}_2, 1)$-regular $(n, \ell, 2)$-subcomplexes. Indeed, from the construction of $\tilde{\boldsymbol{\mathcal{G}}}$, it follows that $\tilde{\mathcal{G}}^{(2)}[\Lambda_2]$ is $(\delta_2, d_{\Lambda_2}, 1)$-regular w.r.t. $\tilde{\mathcal{G}}^{(1)}[\Lambda_2]$ for every $\Lambda_2 \in \binom{[\ell]}{2}$. As such, we may apply the Slicing Lemma to slice these bipartite graphs into the desired family.

Fix $\Lambda_2 \in \binom{[\ell]}{2}$. We apply the Slicing Lemma to $\tilde{\mathcal{G}}^{(2)}[\Lambda_2]$ with $\delta = \delta_2$, $\varrho = d_{\Lambda_2}$, $r_{\mathrm{SL}} = 1$, and $p = \tilde{d}_2/\varrho$ to obtain a family of $\lfloor d_{\Lambda_2}/\tilde{d}_2 \rfloor$ pairwise edge-disjoint regular bipartite subgraphs of $\tilde{\mathcal{G}}^{(2)}[\Lambda_2]$.

Repeating this procedure over all $\Lambda_2 \in \binom{[\ell]}{2}$ yields at least

$$\prod_{\Lambda_2 \in \binom{[\ell]}{2}} \left\lfloor \frac{d_{\Lambda_2}}{\tilde{d}_2} \right\rfloor = \left\lfloor \frac{d_2}{\tilde{d}_2} \right\rfloor^{|\Delta_2(\mathcal{F}^{(k)})|} \times \left\lfloor \frac{1}{\tilde{d}_2} \right\rfloor^{\binom{\ell}{2} - |\Delta_2(\mathcal{F}^{(k)})|}$$

$(\tilde{\delta}_2, \tilde{d}_2, 1)$-regular $(n, \ell, 2)$-subcomplexes of $\tilde{\boldsymbol{\mathcal{G}}}$. Let $\mathscr{S}_\ell^{(2)}$ be the family of all $(\tilde{\delta}_2, \tilde{d}_2, 1)$-regular $(n, \ell, 2)$-subcomplexes constructed above and let $\hat{\mathscr{S}}^{(2)}$ be the family of all $(\tilde{\delta}_2, \tilde{d}_2, 1)$-regular $(n, 3, 2)$-subcomplexes implicitly constructed.

We now proceed with the induction step of the construction. For $3 \leq j \leq k$, assume the family $\mathscr{S}_\ell^{(j-1)}$ of $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \ldots, \tilde{d}_{j-1}), r\big)$-regular $(n, \ell, j-1)$-subcomplexes was constructed with size at least

$$\prod_{h=2}^{j-1} \prod_{\Lambda_h \in \binom{[\ell]}{h}} \left\lfloor \frac{d_{\Lambda_h}}{\tilde{d}_h} \right\rfloor = \prod_{h=2}^{j-1} \left\lfloor \frac{d_h}{\tilde{d}_h} \right\rfloor^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{h=2}^{j-1} \left\lfloor \frac{1}{\tilde{d}_h} \right\rfloor^{\binom{\ell}{h} - |\Delta_h(\mathcal{F}^{(k)})|} . \tag{9.8}$$

Let $\hat{\mathscr{S}}^{(j-1)}$ be the collection of all $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \ldots, \tilde{d}_{j-1}), r\big)$-regular $(n, j, j-1)$-subcomplexes implicitly constructed. We now construct a families $\mathscr{S}_\ell^{(j)}$ and $\hat{\mathscr{S}}^{(j)}$.

Fix $(n, j, j-1)$-complex $\hat{\boldsymbol{\mathcal{S}}}^{\boldsymbol{(j-1)}} = \{\hat{\mathcal{S}}^{(h)}\}_{h=1}^{j-1} \in \hat{\mathscr{S}}^{(j-1)}$ and suppose $\Lambda_j \in \binom{[\ell]}{j}$ is such that $\hat{\mathcal{S}}^{(1)} = \bigcup_{\lambda \in \Lambda_j} V_\lambda$. We slice $\tilde{\mathcal{G}}^{(j)}[\Lambda_j] \cap \mathcal{K}_j^{(j-1)}(\hat{\mathcal{S}}^{(j-1)})$ into regular pieces of density $\tilde{d}_j$. In order to do so, however, we must first verify the following claim to see that the Slicing Lemma applies.

Claim 9.1. *$\tilde{\mathcal{G}}^{(j)}[\Lambda_j] \cap \mathcal{K}_j^{(j-1)}(\hat{\mathcal{S}}^{(j-1)})$ is $(\tilde{\delta}_j/3, d_{\Lambda_j}, r)$-regular w.r.t. $\hat{\mathcal{S}}^{(j-1)}$ for every $\hat{\boldsymbol{\mathcal{S}}}^{\boldsymbol{(j-1)}} \in \hat{\mathscr{S}}^{(j-1)}$.*

Proof. Note that this claim is trivial if $\Lambda_j \notin \Delta_j(\mathcal{F}^{(k)})$ and so it suffices to consider $\Lambda_j \in \Delta_j(\mathcal{F}^{(k)})$. In this case, the claim is still easy to prove because, as we show below, $\hat{\mathcal{S}}^{(j-1)} \subseteq \tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]$ contains a large portion of the cliques of $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]$, and so, $\hat{\mathcal{S}}^{(j-1)}$ inherits (most of) the regularity $\tilde{\mathcal{G}}^{(j)}[\Lambda_j]$ has with respect to $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]$.

In fact, since $\hat{\boldsymbol{\mathcal{S}}}^{\boldsymbol{(j-1)}}$ is $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_{j-1}), (\tilde{d}_2, \ldots, \tilde{d}_{j-1}), r\big)$-regular and by the choice of the constants in (9.3) and (9.5), we may apply $\mathbf{IHC}_{k-1,k}$ to infer

$$\left|\mathcal{K}_j^{(j-1)}(\hat{\mathcal{S}}^{(j-1)})\right| \geq \frac{1}{2} \prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}} \times n^j .$$

On the other hand, since $\Lambda_j \in \Delta_j(\mathcal{F}^{(k)})$ and $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex, we infer from (9.4), (9.5), and (9.6) combined with $\mathbf{IHC}_{k-1,k}$ that

$$\left|\mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j])\right| \leq \frac{3}{2} \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j .$$

Consequently, if $\boldsymbol{\mathcal{Q}} = \{\mathcal{Q}_1^{(j-1)}, \ldots, \mathcal{Q}_r^{(j-1)}\}$ is a family of r distinct $(n, j, j-1)$-subcylinders of $\hat{\mathcal{S}}^{(j-1)} \subseteq \tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]$ such that $\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}(\mathcal{Q}_s^{(j-1)})\right| \geq (\tilde{\delta}_j/3)\left|\mathcal{K}_j^{(j-1)}(\hat{\mathcal{S}}^{(j-1)})\right|$, then we see from from (9.2) and (9.4), that, in fact,

$$\begin{aligned}
\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}(\mathcal{Q}_s^{(j-1)})\right| &\geq \frac{\tilde{\delta}_j}{3}\left|\mathcal{K}_j^{(j-1)}(\hat{\mathcal{S}}^{(j-1)})\right| \geq \frac{\tilde{\delta}_j}{6}\prod_{h=2}^{j-1} \tilde{d}_h^{\binom{j}{h}} \times n^j \\
&= \frac{\tilde{\delta}_j}{6}\prod_{h=2}^{j-1}\left(\frac{\tilde{d}_h}{d_h}\right)^{\binom{j}{h}} \times \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j \\
&\geq \frac{3}{2}\delta_j \prod_{h=2}^{j-1} d_h^{\binom{j}{h}} \times n^j \geq \delta_j\left|\mathcal{K}_j^{(j-1)}(\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j])\right|.
\end{aligned}$$

Consequently, from the $(\delta_j, d_{\Lambda_j}, r)$-regularity of $\tilde{\mathcal{G}}^{(j)}[\Lambda_j]$ w.r.t. $\tilde{\mathcal{G}}^{(j-1)}[\Lambda_j]$, we infer

$$\left|\tilde{\mathcal{G}}^{(j)}[\Lambda_j] \cap \bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}(\mathcal{Q}_s^{(j-1)})\right| = (d_{\Lambda_j} \pm \delta_j)\left|\bigcup_{s\in[r]} \mathcal{K}_j^{(j-1)}(\mathcal{Q}_s^{(j-1)})\right|$$

and so we proved the claim. □

Claim 9.1 allows us to apply the Lemma 4.16 to $\tilde{\mathcal{G}}^{(j)}[\Lambda_j] \cap \mathcal{K}_j^{(j-1)}(\hat{\mathcal{S}}^{(j-1)})$ on $\hat{\mathcal{S}}^{(j-1)}$ with $\delta = \tilde{\delta}_j/3$, $\varrho = d_{\Lambda_j}$, $r_{\mathrm{SL}} = r$, and $p = \tilde{d}_j/\varrho$. As such, we create $\lfloor d_{\Lambda_j}/\tilde{d}_j \rfloor$ pairwise edge-disjoint (n, j, j)-cylinders, each of which is $(\tilde{\delta}_j, \tilde{d}_j, r)$-regular with respect to $\hat{\mathcal{S}}^{(j-1)}$.

Repeating this process for all $(n, j, j-1)$-complexes $\hat{\boldsymbol{\mathcal{S}}}^{\boldsymbol{(j-1)}} = \{\hat{\mathcal{S}}^{(h)}\}_{h=1}^{j-1} \in \hat{\mathscr{S}}^{(j-1)}$, we see from (9.8) that we construct at least

$$\prod_{h=2}^{j} \prod_{\Lambda_h \in \binom{[\ell]}{h}} \left\lfloor \frac{d_{\Lambda_j}}{\tilde{d}_j} \right\rfloor \tag{9.9}$$

$\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_j), (\tilde{d}_2, \ldots, \tilde{d}_j), r\big)$-regular (n, ℓ, j)-subcomplexes of $\tilde{\boldsymbol{\mathcal{G}}}$. Let $\mathscr{S}_\ell^{(j)}$ denote the family of all such and, moreover, let $\hat{\mathscr{S}}^{(j)}$ denote the family of all $\big((\tilde{\delta}_2, \ldots, \tilde{\delta}_j), (\tilde{d}_2, \ldots, \tilde{d}_j), r\big)$-regular $(n, j+1, j)$-subcomplexes implicitly constructed. This finishes the induction step of the decomposition of $\tilde{\boldsymbol{\mathcal{G}}}$.

Observe from (9.9) that in the case of $j < k$,

$$\left|\mathscr{S}_\ell^{(j)}\right| \geq \prod_{h=2}^{j} \prod_{\Lambda_h \in \binom{[\ell]}{h}} \left\lfloor \frac{d_{\Lambda_j}}{\tilde{d}_j} \right\rfloor \geq \prod_{h=2}^{j} \left\lfloor \frac{d_h}{\tilde{d}_h} \right\rfloor^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{h=2}^{j} \left\lfloor \frac{1}{\tilde{d}_h} \right\rfloor^{\binom{\ell}{h} - |\Delta_h(\mathcal{F}^{(k)})|}$$

and, in the case of $j = k$,

$$\begin{aligned} \left|\mathscr{S}_\ell^{(k)}\right| &\geq \prod_{h=2}^{k} \prod_{\Lambda_h \in \binom{[\ell]}{h}} \left\lfloor \frac{d_{\Lambda_j}}{\tilde{d}_j} \right\rfloor \\ &\geq \left(\prod_{h=2}^{k-1} \left\lfloor \frac{d_h}{\tilde{d}_h} \right\rfloor^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} \left\lfloor \frac{d_{\Lambda_k}}{\tilde{d}_k} \right\rfloor \right) \times \prod_{h=2}^{k} \left\lfloor \frac{1}{\tilde{d}_h} \right\rfloor^{\binom{\ell}{h} - |\Delta_h(\mathcal{F}^{(k)})|}, \end{aligned} \tag{9.10}$$

by the assumption that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta}, \geq \boldsymbol{d}, r, \mathcal{F}^{(k)})$-regular complex (cf. Definition 3.7).

Note that

$$\left|\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right| \geq \sum_{\boldsymbol{\mathcal{S}} = \{\mathcal{S}^{(h)}\}_{h=1}^{k} \in \mathscr{S}_\ell^{(k)}} \left|\mathcal{K}_\ell^{(k)}(\mathcal{S}^{(k)})\right| \tag{9.11}$$

where we may apply the Counting Lemma to each such $(\tilde{\boldsymbol{\delta}}, \tilde{\boldsymbol{d}}, r)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{S}} \in \mathscr{S}_\ell^{(k)}$. Fix $\boldsymbol{\mathcal{S}} \in \mathscr{S}_\ell^{(k)}$. By the choice of the constants in (9.3) and (9.5) we can apply $\mathbf{CL}_{k,\ell}$ to $\boldsymbol{\mathcal{S}}$ to conclude

$$\left|\mathcal{K}_\ell^{(k)}(\mathcal{S}^{(k)})\right| \geq \left(1 - \frac{\gamma}{2}\right) \prod_{h=2}^{k} \tilde{d}_h^{\binom{\ell}{h}} \times n^\ell. \tag{9.12}$$

Applying (9.12) and (9.10) to (9.11) then yields that $\left|\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})\right|$ is at least

$$\begin{aligned} &\left(1 - \frac{\gamma}{2}\right) \left(\prod_{h=2}^{k} \tilde{d}_h^{\binom{\ell}{h}} \times n^\ell \right) \times \\ &\qquad \times \left(\prod_{h=2}^{k-1} \left\lfloor \frac{d_h}{\tilde{d}_h} \right\rfloor^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} \left\lfloor \frac{d_{\Lambda_k}}{\tilde{d}_k} \right\rfloor \right) \times \prod_{h=2}^{k} \left\lfloor \frac{1}{\tilde{d}_h} \right\rfloor^{\binom{\ell}{h} - |\Delta_h(\mathcal{F}^{(k)})|}. \end{aligned}$$

Also, note that (9.2) yields that

$$\begin{aligned}\lfloor d_h/\tilde{d}_h\rfloor &\geq (1-\bar{\gamma})d_h/\tilde{d}_h \quad \text{for} \quad h=2,\dots,k-1\,,\\ \lfloor d_{\Lambda_k}/\tilde{d}_h\rfloor &\geq (1-\bar{\gamma})d_{\Lambda_k}/\tilde{d}_h \quad \text{for} \quad \Lambda_k\in\mathcal{F}^{(k)}\,,\ \text{and}\\ \lfloor 1/\tilde{d}_h\rfloor &\geq (1-\bar{\gamma})/\tilde{d}_h \quad \text{for} \quad h=2,\dots,k\,.\end{aligned}$$

Consequently, we see $|\mathcal{K}_\ell^{(k)}(\tilde{\mathcal{G}}^{(k)})|$ is at least

$$(1-\bar{\gamma})^{2^\ell}\prod_{h=2}^{k}\left(\frac{1}{\tilde{d}_h}\right)^{\binom{\ell}{h}}\prod_{h=2}^{k-1}d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\prod_{\Lambda_k\in\mathcal{F}^{(k)}}d_{\Lambda_k}\times\left(1-\frac{\gamma}{2}\right)\prod_{h=2}^{k}\tilde{d}_h^{\binom{\ell}{h}}\times n^\ell$$
$$\geq(1-\gamma)\prod_{h=2}^{k-1}d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\times\prod_{\Lambda_k\in\mathcal{F}^{(k)}}d_{\Lambda_k}\times n^\ell\,,$$

which yields the corollary by (9.7). □

9.2 Proof of Corollary 3.12

We now use Corollary 3.9 to infer Corollary 3.12.

Let ℓ, k and $\gamma>0$ be given. We inductively fix the constants δ_j according to Corollary 3.9 applied with $\gamma/2$. More precisely, for $j=k,\dots,2$ and given d_j we set

$$\delta_j=\delta_j\big(\text{Cor.3.9}(\gamma/2,d_k,\delta_k,\dots,d_j)\big)\,.$$

In the same way we fix r and n_0. Let n be sufficiently large (bigger than n_0 and satisfying one condition mentioned below).

Let $\mathcal{F}^{(k)}$ with $V(\mathcal{F}^{(k)})=[\ell]$ be a k-uniform hypergraph and suppose $\tilde{\mathcal{F}}$ (with $V(\tilde{\mathcal{F}}^{(k)})=[\tilde{\ell}]$) is a homomorphic image of $\mathcal{F}^{(k)}$ under ϑ. In other words, ϑ is an edge-preserving map satisfying

$$\vartheta\colon\ [\ell]\twoheadrightarrow[\tilde{\ell}]\,.$$

Moreover, let $\tilde{\boldsymbol{\mathcal{G}}}=\{\tilde{\mathcal{G}}^{(h)}\}_{h=1}^k$ with $\tilde{\mathcal{G}}^{(1)}=\tilde{V}_1\cup\dots\cup\tilde{V}_{\tilde{\ell}}$ be a $\big(\boldsymbol{\delta},\geq\boldsymbol{d},r,\tilde{\mathcal{F}}^{(k)}\big)$-regular $(n,\tilde{\ell},k)$-complex, where $\boldsymbol{\delta}=(\delta_2,\dots,\delta_k)$ and $\boldsymbol{d}=(d_2,\dots,d_k)$.

Through ϑ we may view $\mathcal{F}^{(k)}$ as an $\tilde{\ell}$-partite hypergraph on the vertex set

$$V(\mathcal{F}^{(k)}) = [\ell] \cong U = U_1 \cup \cdots \cup U_{\tilde{\ell}} \quad \text{where} \quad U_\beta = \beta \times \vartheta^{-1}(\beta) \quad \text{for} \quad \beta \in [\tilde{\ell}]\,.$$

We construct a $\big(\boldsymbol{\delta}, \geq \boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ which is a "homomorphic preimage" of $\tilde{\boldsymbol{\mathcal{G}}}$. For that, we set $\boldsymbol{\mathcal{G}} = \tilde{\boldsymbol{\mathcal{G}}} \times_\vartheta \mathcal{F}^{(k)}$, where the product "$\times_\vartheta$" is defined as follows.

(a) $\mathcal{G}^{(1)} = \bigcup_{\beta \in [\tilde{\ell}]} W_\beta$ where $W_\beta = \bigcup_{\alpha \in \vartheta^{-1}(\beta)} V_\alpha$ and $V_\alpha = \tilde{V}_\beta \times \alpha$, i.e., for every vertex $(\beta, \alpha) \in U$ of $\mathcal{F}^{(k)}$ we introduce in $\mathcal{G}^{(1)}$ a vertex class $\tilde{V}_\beta \times \alpha = V_\alpha$ of n vertices which is a copy of $\tilde{V}_\beta$ or, equivalently, for every $\alpha \in [\ell]$ we introduce a vertex class V_α which corresponds to $\tilde{V}_\beta$, where $\beta = \vartheta(\alpha)$. Consequently,

$$\bigcup_{\alpha \in [\ell]} V_\alpha = \mathcal{G}^{(1)} = \bigcup_{\beta \in [\tilde{\ell}]} \bigcup_{\alpha \in \vartheta^{-1}(\beta)} V_\alpha \quad \text{where} \quad V_\alpha = \tilde{V}_\beta \times \alpha\,.$$

(b) For $j = 2, \ldots, k-1$ we set

$$\mathcal{G}^{(j)} = \Big\{ \big\{ (\tilde{v}_{\beta_1}, \alpha_1), \ldots, (\tilde{v}_{\beta_j}, \alpha_j) \big\} \colon \{\tilde{v}_{\beta_1}, \ldots, \tilde{v}_{\beta_j}\} \in \tilde{\mathcal{G}}^{(j)} \quad \text{and} \quad (\alpha_1, \ldots, \alpha_j) \in \vartheta^{-1}(\beta_1) \times \cdots \times \vartheta^{-1}(\beta_j) \Big\}\,.$$

Below we observe some facts concerning $\boldsymbol{\mathcal{G}} = \tilde{\boldsymbol{\mathcal{G}}} \times_\vartheta \mathcal{F}^{(k)}$. All are straightforward consequences of the definition in (a) and (b) above.

(i) $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(h)}\}_{h=1}^k$ is an (n, ℓ, k)-complex with $\mathcal{G}^{(1)} = V_1 \cup \cdots \cup V_\ell$.

(ii) $\boldsymbol{\mathcal{G}}$ is $\big(\boldsymbol{\delta}, \geq \boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular.

(iii) If $\mathcal{F}_0^{(k)}$ is a partite isomorphic copy (see Definition 3.8) of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ with $V(\mathcal{F}^{(k)}) = \{(\tilde{v}_{\beta_1}, \alpha_1), \ldots, (\tilde{v}_{\beta_\ell}, \alpha_\ell)\}$ (where due to (a) $\vartheta(\alpha_i) = \beta_i$ for $i = 1, \ldots, \ell$), then $\{\tilde{v}_{\beta_1}, \ldots, \tilde{v}_{\beta_\ell}\}$ spans a copy of some homomorphic image of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$. Moreover, if $|\{\tilde{v}_{\beta_1}, \ldots, \tilde{v}_{\beta_\ell}\}| = \ell$, then

$\{\tilde{v}_{\beta_1}, \dots, \tilde{v}_{\beta_\ell}\}$ spans a ϑ-partite isomorphic copy (see Definition 3.11) of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$.

(*iv*) The number of partite isomorphic copies $\mathcal{F}_0^{(k)}$ of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ with $V(\mathcal{F}^{(k)}) = \{(\tilde{v}_{\beta_1}, \alpha_1), \dots, (\tilde{v}_{\beta_\ell}, \alpha_\ell)\}$ for which $|\{\tilde{v}_{\beta_1}, \dots, \tilde{v}_{\beta_\ell}\}| = \ell' < \ell$ is less than

$$\prod_{\beta \in [\tilde{\ell}]} |\vartheta^{-1}(\beta)|! \times n^{\ell'} = o(n^{\ell}) .$$

(*v*) For every ϑ-partite isomorphic copy $\mathcal{F}_1^{(k)}$ of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$ with $V(\mathcal{F}_1^{(k)}) = \bigcup_{\beta \in [\tilde{\ell}]} V(\mathcal{F}_1^{(k)}) \cap \tilde{V}_\beta$ and $V(\mathcal{F}_1^{(k)}) \cap \tilde{V}_\beta = \bigcup_{\alpha \in \vartheta^{-1}(\beta)} \tilde{v}_{\beta,\alpha}$ there exist precisely

$$\prod_{\beta \in [\tilde{\ell}]} |\vartheta^{-1}(\beta)|!$$

corresponding distinct partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$, namely,

$$\left\{ \bigcup_{\beta \in [\tilde{\ell}]} \bigcup_{\alpha \in \vartheta^{-1}(\beta)} \big(\tilde{v}_{\beta,\alpha}, \sigma_\beta(\alpha)\big) \colon \sigma_\beta \text{ is a permutation on } \vartheta^{-1}(\beta) \text{ for } \beta \in [\tilde{\ell}] \right\}.$$

Due to (*i*) and (*ii*), $\boldsymbol{\mathcal{G}} = \tilde{\boldsymbol{\mathcal{G}}} \times_\vartheta \tilde{\mathcal{F}}^{(k)}$ (defined in (*a*) and (*b*)) is a $\big(\boldsymbol{\delta}, \geq \boldsymbol{d}, r, \mathcal{F}^{(k)}\big)$-regular (n, ℓ, k)-complex. Consequently, by the earlier choice of constants we may apply Corollary 3.9 and infer that the number of partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ is at least

$$\Big(1 - \frac{\gamma}{2}\Big) \prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} d_{\Lambda_k} \times n^{\ell} .$$

Then (*iii*) and (*iv*) yield that (for n sufficiently large) at least

$$\Big(1 - \frac{\gamma}{2}\Big) \prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} d_{\Lambda_k} \times n^{\ell} - o(n^{\ell})$$
$$\geq (1 - \gamma) \prod_{h=2}^{k-1} d_h^{|\Delta_h(\mathcal{F}^{(k)})|} \times \prod_{\Lambda_k \in \mathcal{F}^{(k)}} d_{\Lambda_k} \times n^{\ell}$$

partite isomorphic copies of $\mathcal{F}^{(k)}$ in $\mathcal{G}^{(k)}$ correspond to a ϑ-partite isomorphic copy of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$. Finally, (v) then implies the existence of at least

$$(1-\gamma)\prod_{\beta\in[\tilde{\ell}]}\frac{1}{|\vartheta^{-1}(\beta)|!}\prod_{h=2}^{k-1}d_h^{|\Delta_h(\mathcal{F}^{(k)})|}\times\prod_{\Lambda_k\in\mathcal{F}^{(k)}}d_{\Lambda_k}\times n^{\ell}$$

ϑ-partite isomorphic copy of $\mathcal{F}^{(k)}$ in $\tilde{\mathcal{G}}^{(k)}$ and we conclude the proof of Corollary 3.12. $\square$

Chapter 10

Concluding Remarks concerning the Counting Lemma

The methods used here, combined with the result of [48], yield a Hypergraph Regularity Lemma that would be simpler to state and likely more convenient to use. For 3-uniform hypergraphs, such a result immediately follows from Theorem 5.3 and the RS-Lemma with $k = 3$ (equivalently, the FR-Lemma). We focus first on the case $k = 3$ but later continue for general k.

10.1 The 3-uniform case

We prove the following theorem.

Theorem 10.1. *For every positive real ν and a positive real-valued function $\varepsilon(D_2)$, there exist integers L_3 and n_3 such that for any 3-uniform hypergraph $\mathcal{H}^{(3)}$ on $n \geq n_3$ vertices there exists a 3-uniform hypergraph $\mathcal{F}^{(3)}$ and a $\big(\nu, \varepsilon(d_2), d_2, 1\big)$-equitable family of partitions $\mathscr{R} = \mathscr{R}\,(2, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$ such that*

(*i*) *for every $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$*

$$\left|\big(\mathcal{H}^{(3)} \triangle \mathcal{F}^{(3)}\big) \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)\right| \leq \nu d_2^3 \left(\frac{n}{a_1}\right)^3, \tag{10.1}$$

(*ii*) *all but at most* νn^3 *edges of* $K_n^{(3)}$ *belong to some polyad* $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$ *where* $\mathcal{F}^{(3)}$ *is* $(\varepsilon(d_2), 1)$*-regular w.r.t.* $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$*, and*

(*iii*) $\operatorname{rank}\mathscr{R} \leq L_3$.

We give the proof of Theorem 10.1 momentarily. We proceed first with some general discussion.

There is an important single difference between Theorem 10.1 and the FR-Lemma (or, equivalently, RS-Lemma with $k = 3$); Theorem 10.1 provides an environment sufficient for a direct application of the Dense Counting Lemma, Theorem 4.1. Indeed, unlike the the output of the FR-Lemma where one has constants δ_3, d_2, $\delta_2(d_2)$ and r, so that $\delta_3 \gg d_2 \gg \delta_2(d_2), 1/r$, Theorem 10.1 admits sufficiently small $\varepsilon(d_2)$ with $\varepsilon(d_2) \ll d_2$ and no formulation of r such that both the 3-uniform hypergraph $\mathcal{F}^{(3)}$ and the graphs from the underlying partition $\mathscr{R}^{(2)}$ are $\varepsilon(d_2)$-regular.

The only cost of the cleaner environment Theorem 10.1 renders is that our original input hypergraph $\mathcal{H}^{(3)}$ is slightly (albeit negligibly) altered to the output hypergraph $\mathcal{F}^{(3)}$. We proceed to the proof of Theorem 10.1.

Proof of Theorem 10.1. Theorem 10.1 follows from consecutive applications of Theorem 4.11 and Theorem 5.3. Since Theorem 4.11 is formulated for ℓ-partite hypergraphs, we assume that $\mathcal{H}^{(3)}$ is ℓ-partite for some fixed ℓ. The formulation of Theorem 10.1 for arbitrary $\mathcal{H}^{(3)}$ can be derived in precisely the same way by replacing Theorem 4.11 with the main result from [48], or, equivalently, the FR-Lemma.

We mention another allowance we make in our proof. While not stated explicitly earlier, the conclusion of Theorem 5.3 also holds for an $(n, 3, 3)$-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{3}$ satisfying $d\big(\mathcal{G}^{(3)}\big|\mathcal{G}^{(2)}\big) \geq d_3$ (cf. Definition 3.2 with $r = 1$), rather than $d\big(\mathcal{G}^{(3)}\big|\mathcal{G}^{(2)}\big) = d_3$. We use this conclusion in our proof below.

We begin by fixing the constants involved in the proof of Theorem 10.1. As our proof depends on Theorem 4.11 and Theorem 5.3, our constants are determined by these two theorems. As such, we recall that the quantifications of Theorem 4.11 (for $k = 3$) and Theorem 5.3 (for $k = \ell = 3$) are, respectively,

$$\forall l, \mu, \delta_3', \delta_2'(D_2), r'(A_1, D_2)\ \exists L_3', n_3'$$

and

$$\forall \gamma, d_3\ \exists \delta_3\ \forall d_2, \varepsilon\ \exists \delta_2, r, n_0.$$

Given $\nu > 0$ and a positive real-valued function $\varepsilon(D_2)$, set

$$2\mu = \gamma/2 = 2d_3^{\mathrm{low}} = \nu. \tag{10.2}$$

We appeal to Theorem 5.3 to define δ_3 and functions $\delta_2(D_2)$, $r(D_2)$ and $n_0(D_2)$ in the variable D_2

$$\begin{aligned}
\delta_3 &= \delta_3\big(\text{Thm.5.3}(\gamma, d_3^{\mathrm{low}})\big) \\
\delta_2(D_2) &= \delta_2\big(\text{Thm.5.3}(\gamma, d_3^{\mathrm{low}}, \delta_3, D_2, \varepsilon(D_2))\big) \\
r(D_2) &= r\big(\text{Thm.5.3}(\gamma, d_3^{\mathrm{low}}, \delta_3, D_2, \varepsilon(D_2))\big) \\
n_0(D_2) &= n_0\big(\text{Thm.5.3}(\gamma, d_3^{\mathrm{low}}, \delta_3, D_2, \varepsilon(D_2))\big)
\end{aligned}$$

where, w.l.o.g., we assume that the functions defined above are monotone in D_2. We then set

$$\begin{gathered}
\delta_3' = \min\{\delta_3, \nu/2\}\,, \quad \delta_2'(D_2) = \min\{\delta_2(D_2), \varepsilon(D_2), D_2^3/10\}\,, \\
\text{and} \quad r'(a_1, D_2) = r(D_2)\,.
\end{gathered} \tag{10.3}$$

Having defined μ, δ_3', $\delta_2'(D_2)$, and $r'(A_1, D_2)$, Theorem 4.11 determines two integers L_3' and n_3'. We fix L_3 and n_3 (promised by Theorem 10.1) as

$$L_3 = L_3' \quad \text{and} \quad n_3 = \max\{n_3', a_1 n_0(1/(2L_3))\}\,. \tag{10.4}$$

This concludes our determination of the promised constants. We now begin our proof of Theorem 10.1.

Now, let $\mathcal{H}^{(3)}$ be an $(n, \ell, 3)$-cylinder with $n \geq n_3$ and vertex partition $V_1 \cup \dots \cup V_\ell$. We apply Theorem 4.11 to obtain a $\big(\mu, \delta_2'(d_2), d_2, r'(d_2)\big)$-equitable partition $\mathscr{R} = \mathscr{R}(2, \boldsymbol{a}, \boldsymbol{\varphi}) = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$ with

$$\operatorname{rank} \mathscr{R} \leq L_3' = L_3 \,. \tag{10.5}$$

Accordingly, all but $(\mu + \delta_3')n^3 \leq \nu n^3$ edges of $K_\ell^{(3)}(V_1, \dots, V_\ell)$ belong to $(n/a_1, 3, 2)$-complexes $\hat{\boldsymbol{\mathcal{R}}} = \big\{\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(2)})\big\}_{j=1}^{2}$ which are

(α) $\big(\delta_2'(d_2), d_2, r'(d_2)\big)$-regular where

(β) $\mathcal{H}^{(3)}$ is $(\delta_3', r'(d_2))$-regular w.r.t. $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$.

Observe that it must be the case that

$$d_2 > 1/(2L_3) \tag{10.6}$$

since otherwise there is a contradiction to the $(\mu, \delta_2'(d_2), d_2, r'(d_2))$-equitability of $\mathscr{R}$. For a somewhat simpler notation in what follows, we set $d_3(\hat{\boldsymbol{x}}^{(2)}) = d\big(\mathcal{H}^{(3)}\big|\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)$.

To define the promised hypergraph $\mathcal{F}^{(3)}$, we define, for every $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$, an $(n/a_1, 3, 3)$-cylinder $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) \subseteq \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)$. We distinguish three cases.

(A) If $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$ so that either (α) or (β) fails, then we set $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) = \mathcal{H}^{(3)} \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)$. Note, in this case, we don't alter $\mathcal{H}^{(3)}$ on such polyads $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$.

(B) If $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$ such that (α) and (β) hold but $d_3(\hat{\boldsymbol{x}}^{(2)}) < d_3^{\text{low}}$, then we set $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) = \varnothing$.

(C) If $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$ such that (α) and (β) hold and $d_3(\hat{\boldsymbol{x}}^{(2)}) \geq d_3^{\text{low}}$, then we apply Theorem 5.3 to define $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)})$. To this end, consider the $(n/a_1, 3, 3)$-complex $\boldsymbol{\mathcal{G}}(\hat{\boldsymbol{x}}^{(2)}) = \{\mathcal{G}^{(j)}(\hat{\boldsymbol{x}}^{(2)})\}_{h=1}^{3}$ with $\mathcal{G}^{(j)}(\hat{\boldsymbol{x}}^{(2)}) =$

$\hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(2)})$ for $j \in \{1,2\}$ and $\mathcal{G}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) = \mathcal{H}^{(3)} \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)$. Note, that $n/a_1 \geq n_0(d_2)$ due to $n \geq n_3$, the monotonicity of $n_0(D_2)$, (10.6), and (10.4). Also, due to the assumption on this case and the choice of constants, $\boldsymbol{\mathcal{G}}(\hat{\boldsymbol{x}}^{(3)})$ is a $\big((\delta_2(d_2), \delta_3), (d_2, d_3(\hat{\boldsymbol{x}}^{(2)})), r(d_2)\big)$-regular complex satisfying the assumptions of Theorem 5.3. Therefore, there is an $\big((\varepsilon(d_2), \varepsilon(d_2)), (d_2, d_3(\hat{\boldsymbol{x}}^{(2)})), 1\big)$-regular $(n/a_1, 3, 3)$-complex $\boldsymbol{\mathcal{F}}(\hat{\boldsymbol{x}}^{(2)}) = \{\mathcal{F}^{(j)}(\hat{\boldsymbol{x}}^{(2)})\}_{h=1}^3$ such that

$$\mathcal{F}^{(j)}(\hat{\boldsymbol{x}}^{(2)}) = \mathcal{G}^{(j)}(\hat{\boldsymbol{x}}^{(2)}) = \hat{\mathcal{R}}^{(j)}(\hat{\boldsymbol{x}}^{(2)}) \tag{10.7}$$

for $j \in \{1,2\}$ (by (ii) of Theorem 5.3) and

$$\begin{aligned}\Big|\mathcal{G}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) \triangle \mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)})\Big| &= \Big|\Big(\mathcal{H}^{(3)} \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)\Big) \triangle \mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)})\Big| \\ &\leq \frac{\gamma}{2} d_3(\hat{\boldsymbol{x}}^{(2)}) d_2^3 \left(\frac{n}{a_1}\right)^3 \leq \nu d_2^3 \left(\frac{n}{a_1}\right)^3 .\end{aligned} \tag{10.8}$$

Set

$$\mathcal{F}^{(3)} = \bigcup \Big\{\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) \colon \hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})\Big\} .$$

We verify that $\mathcal{F}^{(3)}$ admits the required properties. Recall that the family of partitions $\mathscr{R}$ is $\big(\mu, \delta_2'(d_2), d_2, r'(d_2)\big)$-equitable. It therefore follows from the choice of μ in (10.2), $\delta_2'(D_2)$ in (10.3), and $r' \geq 1$ that the family of partitions $\mathscr{R}$ is $(\nu, \varepsilon(d_2), d_2, 1)$-equitable. It is left to show (i)–(iii) of Theorem 10.1 for $\mathcal{F}^{(3)}$ and $\mathscr{R}$.

We first confirm property (i). We have to verify (10.1) for every $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$. If $\hat{\boldsymbol{x}}^{(2)}$ satisfies the assumption in (A), then the symmetric difference considered in (10.1) is empty.

If $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) = \mathcal{F}^{(3)} \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)$ was constructed in (B) (this means $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)}) = \varnothing$), then

$$\begin{aligned}\Big|\big(\mathcal{H}^{(3)} \triangle \mathcal{F}^{(3)}\big) \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)\Big| &= \Big|\Big(\mathcal{H}^{(3)} \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)\Big) \triangle \mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)})\Big| \\ &= \Big|\mathcal{H}^{(3)} \cap \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)\Big| \leq d_3^{\mathrm{low}} \Big|\mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big)\Big|\end{aligned}$$

due to the assumption of (B). We then use the Counting Lemma for $k = 2$ and $\ell = 3$, or more precisely Fact A from [16] combined with $\delta_2'(d_2) \leq d_2^3/10$ from (10.3), to bound the right-hand side of the last inequality by

$$d_3^{\text{low}} \times 2d_2^3 \left(\frac{n}{a_1}\right)^3 \overset{(10.2)}{\leq} \nu d_2^3 \left(\frac{n}{a_1}\right)^3 ,$$

which yields (10.1) for $\hat{\boldsymbol{x}}^{(2)}$ satisfying the assumption on (B).

Finally, if $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$ satisfies the assumption of (C), then (10.8) yields (10.1). This concludes the proof of property (i) of Theorem 10.1.

Next we verify part (ii) of Theorem 10.1. For $\hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a})$ considered in (B) and (C) the $(n/a_1, 3, 3)$-cylinder $\mathcal{F}^{(3)}(\hat{\boldsymbol{x}}^{(2)})$ is $(\varepsilon(d_2), 1)$-regular w.r.t. $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$ by construction. Consequently, for any triple from $K_\ell^{(3)}(V_1, \ldots, V_\ell)$ belonging to some polyad $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$, the hypergraph $\mathcal{F}^{(3)}$ is $(\varepsilon(d_2), 1)$-regular w.r.t. $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$ if $\hat{\boldsymbol{x}}^{(2)}$ satisfies the assumption of (B) or (C). Therefore, in order to show (ii) of Theorem 10.1 it suffices to estimate the number of triples belonging to polyads $\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})$ considered in (A). Since $\mathscr{R}$ was $\big(\mu, \delta_2'(d_2), d_2, r'(d_2)\big)$-equitable and $(\delta_3', r'(d_2))$-regular w.r.t. $\mathcal{H}^{(3)}$, it follows (as observed above) that all but at most νn^3 triples from $\mathcal{K}_\ell^{(3)}(V_1, \ldots, V_\ell)$ belong to complexes of the partition $\mathscr{R}$ not satisfying (α) or (β). Consequently,

$$\left| \bigcup \left\{ \mathcal{K}_3^{(2)}\big(\hat{\mathcal{R}}^{(2)}(\hat{\boldsymbol{x}}^{(2)})\big) \colon \hat{\boldsymbol{x}}^{(2)} \in \hat{A}(2, \boldsymbol{a}) \text{ satisfies assumption of } (A) \right\} \right| \leq \nu n^3 ,$$

which yields part (ii) of Theorem 10.1.

We conclude our proof of Theorem 10.1 by confirming property (iii). Note, however, that (iii) holds trivially due to (10.5). $\square$

10.2 Extending Theorem 10.1 to k-uniform hypergraphs

We now present a generalization of Theorem 10.1 to k-uniform hypergraphs. In this generalization, we give a slight improvement to Theorem 10.1. We begin with the following definition.

Definition 10.2 (**$(\varepsilon, \boldsymbol{d}, \boldsymbol{a})$-perfect family of partitions**). *Let $\varepsilon > 0$ and $\boldsymbol{d} = \big(d_2, \ldots, d_{k-1}\big)$ be a vector of positive reals and $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ be a vector of positive integers. We say that a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ (as defined in Definition 4.5) is a perfect $(\varepsilon, \boldsymbol{d}, \boldsymbol{a})$-family of partitions if the following holds for every $2 \leq j \leq k-1$:*

(*i*) $d_j a_j = 1$ *and*

(*ii*) *for every $\hat{\boldsymbol{x}}^{(j-1)} \in \hat{A}\,(j-1, \boldsymbol{a})$ and for every $\alpha \in [a_j]$, the $(n/a_1, j, j)$-cylinder $\mathcal{R}^{(j)}\big((\hat{\boldsymbol{x}}^{(j-1)}, \alpha)\big)$ is $\big(\varepsilon, d_j, 1\big)$-regular w.r.t. $\hat{\mathcal{R}}^{(j-1)}\big(\hat{\boldsymbol{x}}^{(j-1)}\big)$.*

Recall that an equitable partition may contain a few irregular parts. In the definition above, observe that an $(\varepsilon, \boldsymbol{d}, \boldsymbol{a})$-perfect family of partitions admits no irregular parts. We now state the generalization of Theorem 10.1 with this one slight improvement.

Theorem 10.3. *Let $k \geq 2$ be an integer. For every positive real ν and any positive real-valued function $\varepsilon(D_2, \ldots, D_{k-1})$, there exist integers L_k and n_k such that for any k-uniform hypergraph $\mathcal{H}^{(k)}$ on $n \geq n_k$ vertices there exists a k-uniform n-vertex hypergraph $\mathcal{F}^{(k)}$ and an $\big(\varepsilon(\boldsymbol{d}), \boldsymbol{d}, \boldsymbol{a}\big)$-perfect family of partitions $\mathscr{R} = \mathscr{R}\,(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ such that*

(*i*) *for every $\hat{\boldsymbol{x}}^{(k-1)} \in \hat{A}(k-1, \boldsymbol{a})$,*

$$\Big|\big(\mathcal{H}^{(k)} \triangle \mathcal{F}^{(k)}\big) \cap \mathcal{K}_k^{(k-1)}\big(\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})\big)\Big| \leq \nu \left(\prod_{h=2}^{k-1} d_h^{\binom{k}{h}}\right)\left(\frac{n}{a_1}\right)^k$$

(*ii*) *all but at most* νn^k *edges of* $K_n^{(k)}$ *belong to some polyads* $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$ *where* $\mathcal{F}^{(k)}$ *is* $(\varepsilon(\boldsymbol{d}), 1)$*-regular w.r.t.* $\hat{\mathcal{R}}^{(k-1)}(\hat{\boldsymbol{x}}^{(k-1)})$*, and*

(*iii*) $\operatorname{rank} \mathscr{R} \leq L_k$.

Observe, as was the case with Theorem 10.1, that Theorem 10.3 provides an environment sufficient for a direct application of the Dense Counting Lemma. This convenience for k-uniform hypergraphs, $k \geq 3$, is at least as appealing as it is for $k = 3$. Indeed, rather than having a vector of $k-1$ regularity measures $\boldsymbol{\delta}(\boldsymbol{d}) = \big(\delta_2(d_2, \ldots, d_{k-1}), \ldots, \delta_{k-1}(d_{k-1})\big)$, we have a single $\varepsilon(d_2, \ldots, d_{k-1}) \ll \min\{d_2, \ldots, d_{k-1}\}$, no formulation of the parameter r and direct access to the Dense Counting Lemma. As was the case with Theorem 10.1, the cleaner formulation of Theorem 10.3 is at the cost of slightly (albeit negligibly) altering the original input hypergraph $\mathcal{H}^{(k)}$ to the output hypergraph $\mathcal{F}^{(k)}$.

The proof of Theorem 10.3 is more technical than that of Theorem 10.1. We prove Theorem 10.3 in [44], where we proceed along different lines than that of Theorem 10.1.

Part II

Applications of the Regularity Method

Chapter 11

An Extremal Problem

In this chapter we apply the Hypergraph Regularity Lemma of [48] (cf. Theorem 4.11) and the main result of Part I, Theorem 3.6 to prove Theorem 1.5. The proof presented here follows the work of Rödl and Skokan in [46] and we include it here for completeness only.

Proof of Theorem 1.5. Let ℓ, k and $\varepsilon > 0$ be given by Theorem 1.5. Keeping in mind that we intend to apply the Hypergraph Regularity Lemma in conjunction with th Counting Lemma in form of Corollary 3.9′, we introduce the auxiliary constants:

$$\gamma = \frac{1}{4}\,, \qquad\qquad d_k' = \frac{\varepsilon}{4}\,. \tag{11.1}$$

We also define a constant δ_k' and functions $\delta_j'(D_j,\dots,D_{k-1})$ for $j = 2,\dots,k-1$ in variables $D_2,\dots,D_{k-1}$ in terms of the functions given by Corollary 3.9′ (applied with ℓ, k, γ, and d_k') as follows

$$\begin{aligned} \delta_k' &= \min\left\{\delta_k\big(\mathrm{Cor.}3.9'(d_k')\big), \frac{1}{8}\right\}\,, \\ \delta_j'(D_j,\dots,D_{k-1}) &= \min\left\{\delta_j\big(\mathrm{Cor.}3.9'(D_j,\dots,D_{k-1},d_k')\big), \frac{D_j}{2}\right\}\,. \end{aligned} \tag{11.2}$$

Moreover, we consider functions $r'(A_1,D_2,\dots,D_{k-1})$ and $n_0(D_2,\dots,D_{k-1})$ given by Corollary 3.9′

$$r'(A_1,D_2,\dots,D_{k-1}) = r\big(\mathrm{Cor.}3.9'(D_2,\dots,D_{k-1},d_k')\big)\,, \tag{11.3}$$

$$n_0(D_2,\dots,D_{k-1}) = n_0\big(\mathrm{Cor.}3.9'(D_2,\dots,D_{k-1},d_k')\big)\,. \tag{11.4}$$

(We may assume w.l.o.g. that the function $n_0(D_2,\dots,D_{k-1})$ is monotone in each variable.) We also fix constants

$$\mu = \frac{1}{8}\,, \qquad\qquad \ell_{\text{reg}} = \max\left\{\frac{1}{\varepsilon}, \binom{k}{\lfloor k/2\rfloor} + k\right\}. \tag{11.5}$$

We recall the quantification of Theorem 4.11, which for fixed constants ℓ_{reg} (ℓ in Theorem 4.11), k, δ'_k, μ and functions $\delta'_2(D_2,\dots,D_{k-1}),\dots,\delta'_{k-1}(D_{k-1})$, and $r'(A_1,D_2,\dots,D_{k-1})$ defined in (11.2)–(11.5) yields constants n_k and L_k. Finally, we define the promised δ and n_0 as

$$\delta = \frac{1}{2\ell!} \times \left(\frac{1}{2^{\ell_{\text{reg}}+1}L_k^k}\right)^{2^\ell} \times (d'_k)^{\binom{\ell}{k}} \times \left(\frac{1}{\ell_{\text{reg}}L_k}\right)^{\ell}, \tag{11.6}$$

$$n_0 = \max\Big\{\ell_{\text{reg}}n_k,\ \ell_{\text{reg}}L_k n_0\big(\underbrace{(2^{\ell_{\text{reg}}+1}L_k)^{-1},\dots,(2^{\ell_{\text{reg}}+1}L_k)^{-1}}_{(k-2)\text{-times}}\big)\Big\}. \tag{11.7}$$

Having all constants determined we are ready to give a proof of Theorem 1.5. Let $\mathcal{F}^{(k)}$ be some k-uniform hypergraph on ℓ-vertices and $\mathcal{H}^{(k)}$ be a k-uniform hypergraph on $n \ge n_0$ vertices which contains at most δn^ℓ copies of $\mathcal{F}^{(k)}$. We want to apply the Regularity Lemma to $\mathcal{H}^{(k)}$. In Part I it was of some notational advantage to state the Regularity Lemma for (n,ℓ_{reg},k)-cylinders. At this place we have to pay a little tribute to this earlier convenience and we first artificially partition the vertex set of $\mathcal{H}^{(k)}$ into ℓ_{reg} classes $V_1,\dots,V_{\ell_{\text{reg}}}$ of size n/ℓ_{reg} (we may ignore floors and ceilings again, since they have no effect on the arguments). Deleting all non-crossing k-tuples from $\mathcal{H}^{(k)}$ w.r.t. to the artificial vertex partition into ℓ_{reg} classes, we obtain an $(n/\ell_{\text{reg}},\ell_{\text{reg}},k)$-cylinder $\mathcal{H}_1^{(k)} \subseteq \mathcal{H}^{(k)}$, where $n/\ell_{\text{reg}} \ge n_k$ due to (11.7). Moreover,

$$\left|\mathcal{H}^{(k)} \setminus \mathcal{H}_1^{(k)}\right| \le \ell_{\text{reg}} \times \binom{n/\ell_{\text{reg}}}{2} \times \binom{n}{k-2} \le \frac{n^k}{2\ell_{\text{reg}}} \overset{(11.5)}{\le} \frac{\varepsilon}{2}n^k\,. \tag{11.8}$$

We apply the Regularity Lemma, Theorem 4.11, to $\mathcal{H}_1^{(k)}$ with ℓ_{reg}, k, δ'_k,μ, $\delta'_2(D_2,\dots,D_{k-1}),\dots,\delta'_{k-1}(D_{k-1})$, and $r'(A_1,D_2,\dots,D_{k-1})$ defined in (11.2)–(11.5). Theorem 4.11 yields a vector of positive reals $\boldsymbol{d'} = (d'_2,\dots,d'_{k-1})$ and a family of partitions $\mathscr{R} = \mathscr{R}(k-1, \boldsymbol{a}, \boldsymbol{\varphi})$ such that for

$$\begin{aligned} \delta'_j &= \delta'_j(d'_j, \dots, d'_{k-1}) \quad \text{for} \quad j = 2, \dots, k-1\,, \\ \boldsymbol{\delta'} &= (\delta'_2, \dots, \delta'_{k-1})\,, \quad \text{and} \\ r' &= r'(a_1, d'_2, \dots, d'_{k-1}) \end{aligned}$$

the following holds:

(i) $\mathscr{R}$ is $(\mu, \boldsymbol{\delta'}, \boldsymbol{d'}, r')$-equitable,

(ii) $\mathscr{R}$ is (δ'_k, r')-regular w.r.t. $\mathcal{H}_1^{(k)}$, and

(iii) $\operatorname{rank} \mathscr{R} \leq L_k$.

Next we define a subhypergraph $\mathcal{H}_2^{(k)}$ of $\mathcal{H}_1^{(k)}$ by deleting those edges of $\mathcal{H}_1^{(k)}$ which which either belong to irregular or sparse polyads of $\mathscr{R}$. (Note that all edges of $\mathcal{H}_1^{(k)}$ are crossing w.r.t. $\mathscr{R}^{(1)}$ since $\mathcal{H}_1^{(k)}$ is ℓ_{reg}-partite and $\mathscr{R}^{(1)}$ refines that given vertex partition of $\mathcal{H}_1^{(k)}$.) More precisely, let K be a k-tuple in $\mathcal{H}_1^{(k)}$ and $\hat{\boldsymbol{\mathcal{R}}} = \{\hat{\mathcal{R}}^{(j)}\}_{j=1}^{k-1}$ be the underlying polyad of K, i.e., $K \in \mathcal{K}_k^{(k-1)}(\hat{\mathcal{R}}^{(k-1)})$. We then delete K if at least one of the following applies:

(a) $\hat{\boldsymbol{\mathcal{R}}}$ is not a $(\boldsymbol{\delta'}, \boldsymbol{d'}, r')$-regular complex,

(b) $\hat{\mathcal{R}}^{(k-1)}$ is not (δ'_k, r')-regular w.r.t. $\mathcal{H}_1^{(k)}$, or

(c) $d\big(\mathcal{H}_1^{(k)} \big| \hat{\mathcal{R}}^{(k-1)}\big) \leq d'_k$.

We bound the number of deleted edges. Due to (i) and Definition 4.9 at most $\mu \times (n/\ell_{\text{reg}})^k$ edges K of the $(n/\ell_{\text{reg}}, \ell_{\text{reg}}, k)$-cylinder $\mathcal{H}_1^{(k)}$ are deleted because of (a). Moreover, (ii) and Definition 4.10 give that at most $\delta'_k \times (n/\ell_{\text{reg}})^k$ edges fail to be in $\mathcal{H}_2^{(k)}$ due to (b). Finally, at most $d'_k n^k$ edges can be deleted because of (c). Summarizing the above considerations then gives

$$\big|\mathcal{H}^{(k)} \setminus \mathcal{H}_2^{(k)}\big| \leq \big|\mathcal{H}^{(k)} \setminus \mathcal{H}_1^{(k)}\big| + \left(\frac{\mu + \delta'_k}{\ell_{\text{reg}}^k} + d'_k\right) n^k \leq \varepsilon n^k\,, \tag{11.9}$$

where we used (11.8), (11.5), (11.2) and (11.1) for the last inequality.

Consequently, it suffices to show that $\mathcal{H}_2^{(k)}$ is $\mathcal{F}^{(k)}$-free in order to verify the theorem. To the contrary, suppose there is a copy $\mathcal{F}_0^{(k)}$ of $\mathcal{F}^{(k)}$ in $\mathcal{H}_2^{(k)}$. Let $V(\mathcal{F}_0^{(k)}) = \{v_1, \ldots, v_\ell\} \subseteq V(\mathcal{H}_2^{(k)})$ and suppose $v_\alpha \in V_{h_\alpha}$ for $\alpha = 1, \ldots, \ell$. Unfortunately, for different vertices $v_\alpha \neq v_{\alpha'}$ the set V_{h_α} may still equal to $V_{h_{\alpha'}}$ and Corollary 3.9′ is not equipped to directly address this problem.

To this end we construct an auxilliary (n, ℓ, k)-complex $\boldsymbol{\mathcal{G}} = \{\mathcal{G}^{(j)}\}_{j=1}^{k}$, which satisfies the assumptions of Corollary 3.9′. More precisely, for each $\alpha = 1, \ldots, \ell$ let W_α be a copy of the set V_{h_α} such that for all $\alpha \neq \alpha'$ we have $W_\alpha \neq W_{\alpha'}$. Let $\vartheta_\alpha \colon W_\alpha \to V_{h_\alpha}$ be a bijection and for every edge $K \in \mathcal{F}_0^{(k)} \subseteq \mathcal{H}_2^{(k)}$ let $\hat{\boldsymbol{\mathcal{R}}}^{(k-1)}(K) \cup \{\mathcal{H}_2^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{R}}^{(k-1)}(K))\}$ be the unique $(n/(\ell_{\mathrm{reg}} a_1), k, k)$-complex determined by the partition $\mathscr{R}$ and $\mathcal{H}_2^{(k)}$, which contains K. We denote this $(n/(\ell_{\mathrm{reg}} a_1), k, k)$-complex by $\boldsymbol{\mathcal{H}}_2(K, \mathscr{R})$. Consider a copy $\boldsymbol{\mathcal{G}}_K$ of $\boldsymbol{\mathcal{H}}_2(K, \mathscr{R})$ on the vertex set $\bigcup_{\alpha \in K} W_\alpha$ with

$$\vartheta_K = \bigcup_{\alpha \in K} \vartheta_\alpha \colon \bigcup_{\alpha \in K} W_\alpha \to \bigcup_{\alpha \in K} V_{h_\alpha}$$

being an isomorphism between $\boldsymbol{\mathcal{G}}_K$ and $\boldsymbol{\mathcal{H}}_2(K, \mathscr{R})$, i.e., an edge preserving bijection for every layer of both complexes. We then set

$$\boldsymbol{\mathcal{G}} = \bigcup_{K \in \mathcal{F}_0^{(k)}} \boldsymbol{\mathcal{G}}_K .$$

It follows from the defintion of $\boldsymbol{\mathcal{G}}$, that $\boldsymbol{\mathcal{G}}$ is a $(\boldsymbol{\delta'}, \geq \boldsymbol{d'}, r', \mathcal{F}^{(k)})$-regular $(n/(\ell_{\mathrm{reg}} a_1), \ell, k)$-complex and owing to the choices of the functions in (11.2)–(11.4) we see that this $\big(\boldsymbol{\delta'}, \geq \boldsymbol{d'}, r', \mathcal{F}^{(k)}\big)$-regular $(n/(\ell_{\mathrm{reg}} a_1), \ell, k)$-subcomplex satisfies the assumptions of Corollary 3.9′. (Note that it follows from upcoming Claim 11.1 below, the monotonicity of the function $n_0(D_2, \ldots, D_{k-1})$ in (11.4), and the final choice of n_0 in (11.7) that $n/(\ell_{\mathrm{reg}} a_1)$ is sufficiently large.) Moreover, all but at most $\ell (n/(\ell_{\mathrm{reg}} a_1))^{\ell-1}$ crossing copies of $\mathcal{F}^{(k)}$ in the $\boldsymbol{\mathcal{G}}$, correspond to labeled copies of $\mathcal{F}^{(k)}$ in $\bigcup_{K \in \mathcal{F}^{(k)}} \boldsymbol{\mathcal{H}}_2(K, \mathscr{R})$ and hence to labeled copies in $\mathcal{H}_2$. (Possible exceptions are those copies which contain two distinct vertices $w \in W_\alpha$ and $w' \in W_{\alpha'}$ for which $V_{h_\alpha} = V_{h_{\alpha'}}$ and $\vartheta_\alpha(w) = \vartheta_{\alpha'}(w')$.) Consequently, $\mathcal{H}_2^{(k)} \subseteq \mathcal{H}^{(k)}$ contains at least

$$\frac{3}{4\ell!}\prod_{h=2}^{k}(d'_h)^{\binom{\ell}{h}} \times \left(\frac{n}{\ell_{\mathrm{reg}}a_1}\right)^{\ell} - \ell\left(\frac{n}{\ell_{\mathrm{reg}}a_1}\right)^{\ell-1} \geq \frac{1}{2\ell!}\prod_{h=2}^{k}(d'_h)^{\binom{\ell}{h}} \times \left(\frac{n}{\ell_{\mathrm{reg}}a_1}\right)^{\ell}$$

unlabeled copies of $\mathcal{F}^{(k)}$, where the last inequality holds, in view of Claim 11.1, for sufficiently large n. If we show that

$$\frac{1}{2\ell!}\prod_{h=2}^{k-1}(d'_h)^{\binom{\ell}{h}} \times (d'_k)^{\binom{\ell}{k}} \times \left(\frac{1}{\ell_{\mathrm{reg}}a_1}\right)^{\ell} > \delta\,, \tag{11.10}$$

we derive a contradiction to the assumption that $\mathcal{H}^{(k)}$ contains at most δn^ℓ copies of $\mathcal{F}^{(k)}$.

Consequently, in order to prove Theorem 1.5 it is left to verify (11.10). For that we first observe that $a_1 < |A(k-1,\boldsymbol{a})| = \operatorname{rank}\mathscr{R} \leq L_k$ by (4.3) and Definition 4.5. Then in view of the choice of δ in (11.6) inequality (11.10) follows from the following claim. □

Claim 11.1. *$d'_j > \frac{1}{2^{\ell_{\mathrm{reg}}+1}L_k^k}$ for every $j = 2,\dots,k-1$.*

Proof. Let $2 \leq j \leq k-1$ and suppose $d'_j \leq 1/(2^{\ell_{\mathrm{reg}}+1}L_k^k)$. Recall, that $\operatorname{rank}\mathscr{R} \leq L_k$ and hence $|\hat{A}(j-1,\boldsymbol{a})| \leq \binom{\ell_{\mathrm{reg}}}{j-1}L_k^k \leq 2^{\ell_{\mathrm{reg}}}L_k^k$. Also the number of j-tuples in (δ'_j, d'_j, r')-regular polyads is at most

$$\begin{aligned}(d'_j+\delta'_j) &\times \left(\frac{n}{\ell_{\mathrm{reg}}a_1}\right)^{j} \times |\hat{A}(j-1,\boldsymbol{a})| \\ &\leq \frac{3d'_j}{2} \times \left(\frac{n}{\ell_{\mathrm{reg}}a_1}\right)^{j} \times 2^{\ell_{\mathrm{reg}}}L_k^k \leq \frac{3}{4}\left(\frac{n}{\ell_{\mathrm{reg}}}\right)^{j}.\end{aligned} \tag{11.11}$$

On the other hand, we show that at most

$$\mu\left(\frac{n}{\ell_{\mathrm{reg}}}\right)^{j}\binom{k}{j}\Big/\binom{\ell_{\mathrm{reg}}-j}{k-j} \overset{(11.5)}{\leq} \mu\left(\frac{n}{\ell_{\mathrm{reg}}}\right)^{j} \tag{11.12}$$

different j-tuples are in irregular $(n/(\ell_{\rm reg}a_1), j, j)$-complexes of the partition $\mathscr{R}$.

Indeed, every crossing (w.r.t. $\mathscr{R}^{(1)}$) j-tuple which belongs to a

$$((\delta'_2,\ldots,\delta'_{j-1}),(d'_2,\ldots,d'_{j-1}),r')\text{-irregular } (n/(\ell_{\rm reg}a_1),j,j-1)\text{-complex}$$

can be extended to $\binom{\ell_{\rm reg}-j}{k-j}(n/\ell_{\rm reg})^{k-j}$ k-tuples in $\mathcal{K}^{(k)}_{\ell_{\rm reg}}(V_1,\ldots,V_{\ell_{\rm reg}})$ and at most $\binom{k}{j}$ different j-tuples extend to the same k-tuple. Each such k-tuple necessarily belong to a $(\boldsymbol{\delta'},\boldsymbol{d'},r')$-irregular polyad. Due to (i), there are at most $\mu\times(n/\ell_{\rm reg})^k$ such k-tuples.

Since for $j=2,\ldots,k-1$ every edge of $\mathcal{K}^{(j)}_{\ell_{\rm reg}}(V_1,\ldots,V_{\ell_{\rm reg}})$ belongs to a $(n/(\ell_{\rm reg}a_1),j,j-1)$-complex of the partition $\mathscr{R}$ which is either regular or irregular, combining (11.11) and (11.12) yields the following contradiction

$$\left|\mathcal{K}^{(j)}_{\ell_{\rm reg}}(V_1,\ldots,V_{\ell_{\rm reg}})\right|\le\left(\frac{3}{4}+\mu\right)\left(\frac{n}{\ell_{\rm reg}}\right)^j\overset{(11.5)}{\le}\frac{7}{8}\left(\frac{n}{\ell_{\rm reg}}\right)^j.$$

Consequently, the claim follows. $\square$

We close this chapter with the following immediate corollary of Theorem 1.5. Corollary 11.2 will be useful in Chapter 12.

Corollary 11.2. *Let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph on n vertices. Suppose that for each edge H of $\mathcal{H}^{(k)}$ there exists precisely one clique $K^{(k)}_{k+1}$ in $\mathcal{H}^{(k)}$ which contains H. Then $|E(\mathcal{H}^{(k)})|=o(n^k)$ (where $E(\mathcal{H}^{(k)})$ denotes the edge set of the hypergraph $\mathcal{H}^{(k)}$).*

Proof. Since every edge of $\mathcal{H}^{(k)}$ sits in precisely one copy of $K^{(k)}_{k+1}$, the number of copies of $K^{(k)}_{k+1}$ in $\mathcal{H}^{(k)}$ is $|E(\mathcal{H}^{(k)})|/(k+1)\le\binom{n}{k}/(k+1)=o(n^{k+1})$. By Theorem 1.5 (applied with $t=k+1$), we can delete only $o(n^k)$ edges of $\mathcal{H}^{(k)}$ in order to make it $K^{(k)}_{k+1}$-free. On the other hand, we need to remove at least one edge per clique, that is, at least $|E(\mathcal{H}^{(k)})|/(k+1)$ edges. Therefore, $|E(\mathcal{H}^{(k)})|=o(n^k)$. $\square$

Chapter 12

Density Theorems

The results in this Chapter were obtained in collaboration with Vojtěch Rödl, Eduardo Tengan and Norihide Tokushige in [45].

Here we present the proofs of Theorem 1.2–1.4. Consequently we prove Theorem 1.1, as well, as it is a special case of Theorem 1.2. The proofs given here are solely based on Theorem 1.5 and its Corollary 11.2. The arguments given below combined with the combinatorial proof of Theorem 1.5 give the first quantitative proofs of Theorem 1.2–1.4.

The essential part of the reduction of Theorem 1.2 to Corollary 11.2 was already discovered by Solymosi in [53]. We present this proof in Section 12.1 (see also [24]).

Our proof of Theorem 1.3 and Theorem 1.4 extends an idea of Frankl and Rödl from [16].

12.1 Proof of Theorem 1.2

In this section we present a proof of Theorem 1.2. We first prove the special case when the finite configuration T is a subset of the integer lattice (see Lemma 12.1 below). This proof of Lemma 12.1 is based on Corollary 11.2 of Theorem 1.5. This reduction was first considered by Solymosi in [53].

Lemma 12.1. *For all positive integers t, d and every $\delta > 0$, there exists $N_0 = N_0(t, d, \delta)$ such that for $N \geq N_0$ any subset $Z \subset [-N; N]^d$ with $|Z| > \delta(2N + 1)^d$ contains a homothetic copy of $[-t; t]^d$.*

Proof. Suppose, on the contrary, that there exists $Z \subset [-N; N]^d$ with $|Z| > \delta(2N + 1)^d$ which contains no homothetic copy of $[-t; t]^d$. Set $k = (2t + 1)^d - 1$ and $W = Z \times [-N; N]^{k-d}$. Then $|W| > \delta(2N + 1)^k$. We shall show that W contains no homothetic copy of a simplex S defined below, but this contradicts Corollary 11.2 as we will see.

Let $\boldsymbol{e}_0, \boldsymbol{e}_1, \dots, \boldsymbol{e}_k$ denote all the elements of $[-t; t]^d$ where

$$\boldsymbol{e}_0 = (0, 0, \dots, 0),\ \boldsymbol{e}_1 = (1, 0, \dots, 0), \dots,\ \boldsymbol{e}_d = (0, 0, \dots, 1)\,.$$

(We do not need to specify $\boldsymbol{e}_i$ for $i > d$.) For $i = 0, \dots, k-d$, set $(k-d)$-tuples

$$\boldsymbol{f}_0 = (0, 0, \dots, 0),\ \boldsymbol{f}_1 = (1, 0, \dots, 0), \dots,\ \boldsymbol{f}_{k-d} = (0, 0, \dots, 1)\,.$$

Let us define a k-dimensional simplex S with points $\{\boldsymbol{s}_0, \dots, \boldsymbol{s}_k\}$ by

$$\boldsymbol{s}_i = \begin{cases} (\boldsymbol{e}_i, \boldsymbol{f}_0) & \text{if} \quad i = 0, \dots, d\,, \\ (\boldsymbol{e}_i, \boldsymbol{f}_{i-d}) & \text{if} \quad i = d + 1, \dots, k\,. \end{cases}$$

Since Z contains no homothetic copy of $[-t; t]^d$, W contains no homothetic copy of S. Let $\{F_0, \dots, F_k\}$ be the facets of S, and for $i = 0, \dots, k$ let V_i be the set of all hyperplanes in $\mathbb{R}^k$ which are parallel to F_i and intersect $[-N; N]^k$.

Let us show that for every $0 \leq i \leq k$ the cardinality of V_i satisfies $|V_i| = O(N)$. A normal vector $\boldsymbol{w} = (w_1, \dots, w_k)$ of a facet (which is an affine span of k vectors among $\boldsymbol{s}_0, \dots, \boldsymbol{s}_k$) is a non-zero solution of the system

$$\boldsymbol{r}_i \cdot \boldsymbol{w} = 0, \quad i = 1, \dots, k - 1$$

where each $\boldsymbol{r}_i$ is a difference of 2 distinct $\boldsymbol{s}_j$'s and hence is an integer vector whose coordinates have absolute value less than $2t$.

Therefore we may assume that the w_j are given, up to sign, by determinants of integer matrices whose entries are coordinates of $\boldsymbol{r}_i$'s. Hence $|w_j|$ does not exceed $(k-1)!(2t)^{k-1}$. Consequently the hyperplanes of the form

$$\boldsymbol{w} \cdot \boldsymbol{\xi} = w_1\xi_1 + w_2\xi_2 + \cdots + w_k\xi_k = b\,,$$

where b is an integer so that $|b| \leq k!(2t)^{k-1}N$, cover all the points $\boldsymbol{\xi} = (\xi_1, \dots, \xi_k) \in [-N; N]^k$. We conclude that at most $2k!(2t)^{k-1}N + 1 = O(N)$ hyperplanes parallel to the given facet are needed to cover all the points of $[-N; N]^k$.

Next we are going to define a $(k+1)$-partite k-uniform hypergraph $\mathcal{H}^{(k)}$ with vertex partition $V_0 \cup \cdots \cup V_k$. Let H be a set of k vertices of $\mathcal{H}^{(k)}$ with the property $|H \cap V_i| \leq 1$ for all i. Then those k hyperplanes corresponding to H determine a point $\boldsymbol{p} \in \mathbb{R}^k$. We put H in $\mathcal{H}^{(k)}$ if and only if $\boldsymbol{p} \in W$.

Each $H \in E(\mathcal{H}^{(k)})$ determines a point $\boldsymbol{p} \in W$. On the other hand, for each $i = 0, \dots, k$, each point $\boldsymbol{p} \in W$ determines a vertex $v \in V_i$, which corresponds to a hyperplane parallel to F_i and passing through $\boldsymbol{p}$. In this way, $\boldsymbol{p}$ determines the $k+1$ vertices of a clique $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$.

Suppose that $k+1$ hyperplanes determined by a clique $K_{k+1}^{(k)}$ do not meet one point. Then these planes define a simplex homothetic to S in W, which is a contradiction. Thus every clique $K_{k+1}^{(k)}$ must determine a point $\boldsymbol{p} \in W$. This means that for every $H \in E(\mathcal{H}^{(k)})$ there is precisely one clique $K_{k+1}^{(k)}$ which contains H. This implies that $|E(\mathcal{H}^{(k)})| = (k+1)|W|$. Finally, we have $|W| = o(N^k)$ by Corollary 11.2. This contradicts our earlier assumption $|W| > \delta(2N+1)^k$. $\square$

We now deduce Theorem 1.2 from Lemma 12.1.

Proof of Theorem 1.2. Let $\delta > 0$ be given. Let T be a finite subset of $\mathbb{R}^d$. Let $r = r(T)$ be the $\mathbb{Q}$-dimension of T, i.e., the largest number of linearly independent vectors of T over $\mathbb{Q}$. Choose r such vectors $\boldsymbol{\omega}_1, \dots, \boldsymbol{\omega}_r \in \mathbb{R}^d$ so that $T \subset \mathbb{Z}\boldsymbol{\omega}_1 + \cdots + \mathbb{Z}\boldsymbol{\omega}_r$. We define the map $\psi\colon \mathbb{Z}^r \to \mathbb{R}^d$

$$(a_1, \ldots, a_r) \mapsto a_1\boldsymbol{\omega}_1 + \cdots + a_r\boldsymbol{\omega}_r\,.$$

Since $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r$ are linearly independent over $\mathbb{Q}$, the map ψ is injective. Now choose a positive integer t large enough so that $\psi^{-1}(T) \subset [-t;t]^r$ and define $N = N_0(t, r, \delta)$ by Lemma 12.1. Let $C = \psi([-N;N]^r)$; if $Z \subset C$ and $|Z| = |\psi^{-1}(Z)| > \delta|C| = \delta(2N+1)^r$, then $\psi^{-1}(Z)$ contains a homothetic copy of $[-t;t]^r$, say $\boldsymbol{z}' + \lambda[-t;t]^r$ for some $\boldsymbol{z}' \in [-N;N]^r$ and some $\lambda > 0$. Thus $Z \supset \psi(\boldsymbol{z}' + \lambda[-t;t]^r) = \psi(\boldsymbol{z}') + \lambda\psi([-t;t]^r) \supset \psi(\boldsymbol{z}') + \lambda T$, as required. $\square$

12.2 Proof of Theorem 1.3 and Theorem 1.4

The following lemma, Lemma 12.2, is more general than Theorem 1.3 and Theorem 1.4. Its proof elaborates on a construction first considered by Frankl and Rödl [16, Proposition 2.3].

Lemma 12.2. *Let A be a finite, commutative ring with q elements. Then for every $\delta > 0$, there exists $M_0 = M_0(q, \delta)$ such that, for $M \geq M_0$, any subset $Z \subset A^M$ with $|Z| > \delta|A^M| = \delta q^M$ contains a coset of an isomorphic copy (as an A-module) of A.*

In other words, there exist $\boldsymbol{r}$, $\boldsymbol{u} \in A^M$ such that $\boldsymbol{r} + \varphi(A) \subseteq Z$, where $\varphi\colon A \hookrightarrow A^M$, $\varphi(\alpha) = \alpha\boldsymbol{u}$ for $\alpha \in A$, is an injection.

Remark 12.3. *We later only use Lemma 12.2 for a commutative ring A. We remark that commutativity of the ring A is not used in the proof below. In fact, the proof below works verbatim for an arbitrary finite non-commutative ring and left modules, as well.*

Proof of Lemma 12.2. Let $q = |A|$ and let $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2, \ldots, \alpha_{q-1}$ be the elements of the ring A. Let $V = A^m$ and suppose Z is a subset of V which does not contain a coset of an isomorphic copy of A. We shall define a q-partite, q-uniform hypergraph $\mathcal{H}^{(q)}$ whose vertex partition classes $V_0, \ldots, V_{q-1}$ are disjoint copies of V. For $\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{q-1} \in V$ and $\boldsymbol{z} \in Z$, let

$$H(\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{q-1}, \boldsymbol{z}) = (\boldsymbol{h}_0, \ldots, \boldsymbol{h}_{q-1}) \in \prod_{i=0}^{q-1} V_i \,, \quad \text{where}$$

$$\boldsymbol{h}_i = \begin{cases} \boldsymbol{z} + \sum_{j=2}^{q-1} \alpha_j \boldsymbol{v}_j & \text{if} \quad i = 0 \\ \boldsymbol{z} + \sum_{j=2}^{q-1} (\alpha_j - 1) \boldsymbol{v}_j & \text{if} \quad i = 1 \\ \boldsymbol{v}_i & \text{if} \quad i = 2, 3, \ldots, q-1 \,. \end{cases}$$

Set

$$E(\mathcal{H}^{(q)}) = \{H(\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{q-1}, \boldsymbol{z})\colon\ \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{q-1} \in V \text{ and } \boldsymbol{z} \in Z\}. \tag{12.1}$$

(Since, $\mathcal{H}^{(q)}$ is q-partite and q-uniform we may view its edges as ordered q-tuples as defined above.) Clearly, $\mathcal{H}^{(q)}$ has $q|V| = q^{m+1}$ vertices and $q^{m(q-2)}|Z|$ edges. Consequently, Lemma 12.2 follows from Claim 12.4 below. □

Claim 12.4. *Let $\mathcal{H}^{(q)}$ be the hypergraph defined in (12.1), then $|E(\mathcal{H}^{(q)})| = o(q^{m(q-1)})$.*

Proof. First we verify that

$$|H_1 \cap H_2| \le q - 2 \quad \text{for any distinct edges} \quad H_1, H_2 \in E(\mathcal{H}^{(q)}) \,. \tag{12.2}$$

For that let $H_1 = H(\boldsymbol{v}_2, \ldots, \boldsymbol{v}_{q-1}, \boldsymbol{z}_1)$ and $H_2 = H(\boldsymbol{w}_2, \ldots, \boldsymbol{w}_{q-1}, \boldsymbol{z}_2)$ be a pair of distinct edges in $\mathcal{H}^{(q)}$. It is easy to see that if they intersect in $q-1$ points, we must have $\boldsymbol{v}_i = \boldsymbol{w}_i$, $2 \le i \le q-1$, and $\boldsymbol{z}_1 = \boldsymbol{z}_2$, which implies $H_1 = H_2$.

Let $\mathcal{F}(q)$ be the q-uniform hypergraph having $2q$ vertices $a_0, \ldots, a_{q-1}$, $b_0, \ldots, b_{q-1}$ and q edges $F_i = \{a_0, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_{q-1}\}$ for $i = 0, \ldots, q-1$. We shall show that $\mathcal{H}^{(q)}$ contains only "few" copies of $\mathcal{F}(q)$ (see (12.14)).

Suppose that $\boldsymbol{a}_0, \dots, \boldsymbol{a}_{q-1}, \boldsymbol{b}_0, \dots, \boldsymbol{b}_{q-1}$ are the vertices of some copy of $\mathcal{F}(q)$ in $\mathcal{H}^{(q)}$, with $\boldsymbol{a}_i, \boldsymbol{b}_i \in V_i$.

We first consider $F_0 = \{\boldsymbol{b}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{q-1}\}$. There are $\boldsymbol{v}_2, \dots, \boldsymbol{v}_{q-1} \in V$ and $\boldsymbol{z}' \in Z$ such that $F_0 = \{\boldsymbol{b}_0, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{q-1}\} = H(\boldsymbol{v}_2, \dots, \boldsymbol{v}_{q-1}, \boldsymbol{z}')$ and consequently

$$\boldsymbol{b}_0 = \boldsymbol{z}' + \sum_{j=2}^{q-1} \alpha_j \boldsymbol{v}_j \,, \tag{12.3}$$

$$\boldsymbol{a}_1 = \boldsymbol{z}' + \sum_{j=2}^{q-1} (\alpha_j - 1) \boldsymbol{v}_j \,, \tag{12.4}$$

$$\boldsymbol{a}_i = \boldsymbol{v}_i \quad \text{for} \quad i = 2, \dots, q-1 \,. \tag{12.5}$$

Next, we consider $F_1 = \{\boldsymbol{a}_0, \boldsymbol{b}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{q-1}\}$. Since $F_0 \cap F_1 = \{\boldsymbol{a}_2, \dots, \boldsymbol{a}_{q-1}\}$ by (12.5) we have $F_1 = H(\boldsymbol{v}_2, \dots, \boldsymbol{v}_{q-1}, \boldsymbol{z}'')$ for some $\boldsymbol{z}'' \in Z$ such that

$$\boldsymbol{a}_0 = \boldsymbol{z}'' + \sum_{j=2}^{q-1} \alpha_j \boldsymbol{v}_j \,, \tag{12.6}$$

$$\boldsymbol{b}_1 = \boldsymbol{z}'' + \sum_{j=2}^{q-1} (\alpha_j - 1) \boldsymbol{v}_j \,. \tag{12.7}$$

Similarly, for $2 \le i < q$, we see that $F_i = \{\boldsymbol{a}_0, \dots, \boldsymbol{a}_{i-1}, \boldsymbol{b}_i, \boldsymbol{a}_{i+1}, \dots, \boldsymbol{a}_{q-1}\} = H(\boldsymbol{v}_2, \dots, \boldsymbol{v}_{i-1}, \boldsymbol{w}_i, \boldsymbol{v}_{i+1}, \dots, \boldsymbol{v}_{q-1}, \boldsymbol{z}_i)$ for some $\boldsymbol{w}_i \in V$ and $\boldsymbol{z}_i \in Z$ such that

$$\boldsymbol{a}_0 = \boldsymbol{z}_i + \alpha_i(\boldsymbol{w}_i - \boldsymbol{v}_i) + \sum_{j=2}^{q-1} \alpha_j \boldsymbol{v}_j \,, \tag{12.8}$$

$$\boldsymbol{a}_1 = \boldsymbol{z}_i + (\alpha_i - 1)(\boldsymbol{w}_i - \boldsymbol{v}_i) + \sum_{j=2}^{q-1} (\alpha_j - 1) \boldsymbol{v}_j \,, \tag{12.9}$$

$$\boldsymbol{b}_i = \boldsymbol{w}_i \,. \tag{12.10}$$

From (12.6) and (12.8) we infer that for $2 \le i \le q-1$ we have

$$\boldsymbol{z}'' = \boldsymbol{z}_i + \alpha_i(\boldsymbol{w}_i - \boldsymbol{v}_i) \iff \boldsymbol{z}'' - \boldsymbol{z}_i = \alpha_i(\boldsymbol{w}_i - \boldsymbol{v}_i) \,. \tag{12.11}$$

Moreover, comparing (12.4) and (12.9) yields

$$\boldsymbol{z}' = \boldsymbol{z}_i + (\alpha_i - 1)(\boldsymbol{w}_i - \boldsymbol{v}_i) \iff \boldsymbol{z}' - \boldsymbol{z}_i = (\alpha_i - 1)(\boldsymbol{w}_i - \boldsymbol{v}_i) \tag{12.12}$$

for $2 \le i \le q-1$. Combining equation (12.11) and (12.12) for $2 \le i \le q-1$ gives

$$\alpha_i(\boldsymbol{z}' - \boldsymbol{z}_i) = (\alpha_i - 1)(\boldsymbol{z}'' - \boldsymbol{z}_i) \iff \boldsymbol{z}_i = \alpha_i(\boldsymbol{z}' - \boldsymbol{z}'') + \boldsymbol{z}''\,. \tag{12.13}$$

Note that the last equation also holds for $i = 0, 1$ with $\boldsymbol{z}_0 = \boldsymbol{z}''$ and $\boldsymbol{z}_1 = \boldsymbol{z}'$, since $\alpha_0 = 0$ and $\alpha_1 = 1$. We also observe that due to (12.3), (12.6), and $\boldsymbol{a}_0 \neq \boldsymbol{b}_0$ we have $\boldsymbol{z}' \neq \boldsymbol{z}''$.

Now let

$$A_{\mathrm{irreg}} = \{a \in A\colon\ ab = 0 \text{ for some } b \in A, b \neq 0\}$$

be the set of zero-divisors in A and set $t = |A_{\mathrm{irreg}}|$. Since $1 \notin A_{\mathrm{irreg}}$, $t < q$. If $\boldsymbol{u} = \boldsymbol{z}' - \boldsymbol{z}'' \notin A^m_{\mathrm{irreg}}$, then $\varphi\colon A \hookrightarrow A^M$ given by $\varphi(\alpha) = \alpha\boldsymbol{u}$ is an injective A-module homomorphism, and hence (12.13) implies $\boldsymbol{z}''+\varphi(A) \subseteq Z$, which contradicts our assumption on Z. Hence if $\mathcal{H}^{(q)}$ contains some $\mathcal{F}(q) = \{F_0, \dots, F_{q-1}\}$, there exist $\boldsymbol{v}_2, \dots, \boldsymbol{v}_{q-1} \in V$ and $\boldsymbol{z}', \boldsymbol{z}'' \in Z$ with $\boldsymbol{z}' - \boldsymbol{z}'' \in A^m_{\mathrm{irreg}}$ such that (12.3)–(12.10) hold. Conversely, given such quantities, at most one $\mathcal{F}(q)$ is determined: from (12.4) and (12.6), we find $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$; subtracting (12.9) from (12.8), we obtain $\boldsymbol{a}_0 - \boldsymbol{a}_1 = \boldsymbol{w}_i - \boldsymbol{v}_i + \sum_{2\le j\le q-1} \boldsymbol{v}_j$, whence the $\boldsymbol{w}_i$'s are determined. Finally (12.8) determines the $\boldsymbol{z}_i$'s. Hence the number $\#\{\mathcal{F}(q) \subset \mathcal{H}^{(q)}\}$ of copies of $\mathcal{F}(q)$ in $\mathcal{H}^{(q)}$ is bounded by the number of tuples $(\boldsymbol{v}_2, \dots, \boldsymbol{v}_{q-1}, \boldsymbol{z}', \boldsymbol{z}'')$ satisfying the above conditions, and therefore

$$\#\{\mathcal{F}(q) \subset \mathcal{H}^{(q)}\} \le q^{m(q-2)} \times |Z| \times t^m = o(q^{mq})\,, \tag{12.14}$$

where the last assertion used $|Z| \le |V| = |A^m| = q^m$ and $t < q$.

Let $\mathcal{H}^{(q-1)}$ be the $(q-1)$-th shadow of $\mathcal{H}^{(q)}$

$$\mathcal{H}^{(q-1)} = \left\{H'\colon\ |H'| = q-1 \text{ and } H' \subset H \text{ for some } H \in E(\mathcal{H}^{(q)})\right\}.$$

Due to (12.2) for any set Q of q vertices spanning a clique $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ the following holds: either Q is an edge in $\mathcal{H}^{(q)}$ or $Q \subset V(\mathcal{F}(q))$ for some copy of $\mathcal{F}(q)$ in $\mathcal{H}^{(q)}$. Therefore, due to the definition of $\mathcal{H}^{(q)}$ in (12.1) and (12.14), the number of cliques $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ is bounded by $|E(\mathcal{H}^{(q)})| + o(q^{mq}) = |Z|q^{m(q-2)} + o(q^{mq})$. Since $|Z| \leq q^m$ and $|V(\mathcal{H}^{(q-1)})| = |V(\mathcal{H}^{(q)})| = q^{m+1}$, we infer that the number of copies of $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ is $o(q^{mq})$ which is $o(q^{(m+1)q}) = o(|V(\mathcal{H}^{(q-1)})|^q)$.

Hence, Theorem 1.5 (applied for $\ell = q$, $k = q-1$ to $\mathcal{H}^{(q-1)}$ with $\mathcal{F}^{(q-1)} = K_q^{(q-1)}$ and $n = |V(\mathcal{H}^{(q-1)})| = q^{m+1}$) yields that it suffices to delete at most $o(|V(\mathcal{H}^{(q-1)})|^{(q-1)}) = o(q^{(m+1)(q-1)}) = o(q^{m(q-1)})$ edges from $\mathcal{H}^{(q-1)}$ to make it clique free. But due to (12.2) each deleted edge destroys at most one copy of $K_q^{(q-1)}$ in $\mathcal{H}^{(q-1)}$ originating from an edges of $\mathcal{H}^{(q)}$ and, therefore, $|E(\mathcal{H}^{(q)})| = o(q^{m(q-1)})$ as claimed. □

In the rest of this chapter we derive Theorem 1.3 and Theorem 1.4 from Lemma 12.2.

Proof of Theorem 1.3. Consider the ring $A = \mathbb{F}_q \oplus \cdots \oplus \mathbb{F}_q = \bigoplus_{i=1}^d \mathbb{F}_q$. Then $A^m \cong \mathbb{F}_q^{md}$ as an $\mathbb{F}_q$-vector space, and a submodule of A^m isomorphic to A is a d-dimensional subspace of $\mathbb{F}_q^{md}$. Therefore, Lemma 12.2 implies Theorem 1.3 for every sufficiently large $M \equiv 0 \pmod{d}$.

In general, if $M = md + r$, $0 \leq r < d$, $\mathbb{F}_q^M$ is the disjoint union of $|\mathbb{F}_q^M / \mathbb{F}_q^{md}| = q^r$ copies of $\mathbb{F}_q^{md}$; therefore one of these translates, say V, intersects Z in more than $\delta q^M / q^r = \delta q^{md}$ elements and hence $Z \cap V$ (thus Z) will contain a d-dimensional subspace. □

Finally, we close this chapter with the proof of Theorem 1.4.

Proof of Theorem 1.4. Since G is abelian, we may write

$$G \cong \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_r^{e_r}$$

where p_i are (not necessarily distinct) primes and e_i are positive integers. Using this isomorphism, we want to view G as the additive group of the ring $A = \mathbb{Z}/p_1^{e_1} \times \cdots \times \mathbb{Z}/p_r^{e_r}$.

Then A^m and G^m are isomorphic abelian groups. Similarly, a submodule A of A^m is isomorphic to G, also as an abelian group. The theorem then follows from Lemma 12.2. $\square$

Chapter 13

Other Applications

Szemerédi's Regularity Lemma, Theorem 2.1, together with its corresponding Counting Lemma, Fact 2.2, has numerous applications (see [32, 33] for excellent surveys). The FR-Lemma [16] and the companion 3-graph Counting Lemma [36] were exploited in a variety of extremal hypergraph problems (cf. [16, 29, 30, 35, 42, 43, 53]).

We believe that the main theorem of Part I, the Counting Lemma (Theorem 3.6), enables one to apply the RS-Lemma, Theorem 4.11, to a variety of hypergraph problems. Some applications of the Regularity Method for hypergraphs were already discussed in Chapter 11 and Chapter 12. In particular, there we proved Theorem 1.1–1.5 based on the RS-Lemma and the Counting Lemma. Below we briefly discuss some other applications of those theorems.

13.1 Combinatorial Number Theory and Geometry

In [51] Solymosi gives an alternative proof of the Balog–Szemerédi Theorem [2] which implied the affirmative answer to a conjecture of Erdős.

Theorem 13.1 (Balog & Szemerédi [2]). *For every $\delta > 0$ and integer $t > 3$ there is an $n_0 = n_0(\delta, t)$ so that the following holds. If $A \subseteq \mathbb{Z}$ contains*

$\delta|A|^2$ arithmetic progressions of length 3 and $|A| > n_0$, then A contains an arithmetic progression of length t.

Unlike the original proof of Balog and Szemerédi, Solymosi's proof is entirely based on Theorem 1.5 and does not use the well-known theorem of Freiman [17, 18] (see also [49] for a shorter proof of Freiman's Theorem).

In [46] it was shown that Theorem 1.5 also implies the affirmative answer to a geometric problem of Székely [34, p.226], which can be stated as follows.

For a point $\boldsymbol{c} = (c_1, c_2, \ldots, c_k) \in \{1, 2, \ldots, n\}^k$ we define a *jack* $J(\boldsymbol{c})$ with *center* $\boldsymbol{c}$ as the set of points that differ from c in at most one coordinate. For i, $1 \leq i \leq k$, and fixed $c_1, c_2, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k \in \{1, 2, \ldots, n\}$, we also define a *line* as a set of n points of the form

$$\{(c_1, c_2, \ldots, c_{i-1}, x, c_{i+1}, \ldots, c_k), 1 \leq x \leq n\}.$$

Let $LS(n, k)$ be the maximum cardinality of a system J of jacks for which

(*1*) no two distinct jacks share a common line, and

(*2*) $\bigcap_{i=1}^{k} J_i = \varnothing$ for all distinct jacks $J_1, \ldots, J_k \in J$.

Clearly $LS(n, k) \leq n^{k-1}$. Székely conjectured that $LS(n, k)/n^{k-1}$ tends to 0 as $n \to \infty$. In [46] this conjecture was verified.

Theorem 13.2 (Rödl & Skokan [46])**.** *For every positive integer k and every $\varepsilon > 0$ there exist an $n_0 = n_0(k, \varepsilon)$ such that for every $n \geq n_0$*

$$LS(n, k) \leq \varepsilon n^{k-1}.$$

In [52] Solymosi applies Theorem 1.5 to a geometric problem. Roughly speaking he proves that if the number of incidences between hyperplanes and points in dimension d is "close" to the maximum possible, then there are always "dense" subsets, i.e., large point sets such that any d of them are incident to a hyperplane from the arrangement.

13.2 Extremal Hypergraph Results

In [37] we give a few applications of the Regularity Method for Hypergraphs.

We give an alternative proof of the following Ramsey-type theorem due to Nešetřil and Rödl.

Theorem 13.3 (Nešetřil & Rödl [39])**.** *For every integer $\chi \geq 2$ and every fixed k-uniform hypergraph $\mathcal{F}^{(k)}$ there exists a k-uniform hypergraph $\mathcal{H}^{(k)}$ such that every χ-coloring of the edges of $\mathcal{H}^{(k)}$ yields a monochromatic and induced copy of $\mathcal{F}^{(k)}$.*

We also extend Turán-type results from [12, 14, 35] concerning the asymptotic number of labeled hypergraphs not containing any copy of a hypergraph from a fixed family. Let $\mathscr{F}^{(k)} = \{\mathcal{F}_1^{(k)}, \ldots, \mathcal{F}_s^{(k)}\}$ be a fixed family of k-uniform hypergraphs. For an integer n we denote by $\mathrm{Forb}(n, \mathscr{F}^{(k)})$ the family of all distinct labeled k-uniform hypergraphs on $[n]$ which are $\mathcal{F}_t^{(k)}$-free for every $t \in [s]$. We set

$$\mathrm{ex}\left(n, \mathscr{F}^{(k)}\right) = \max\left\{\left|\mathcal{H}^{(k)}\right| \colon \mathcal{H}^{(k)} \in \mathrm{Forb}\left(n, \mathscr{F}^{(k)}\right)\right\}.$$

Theorem 13.4 (Nagle, Rödl & Schacht [37])**.** *For every integer $k \geq 2$, every positive real ε and every finite family $\mathscr{F}^{(k)}$, there exist an integer $n_0 = n_0(k, \varepsilon, \mathscr{F}^{(k)})$ such that for every $n \geq n_0$*

$$\left|\,\mathrm{Forb}\left(n, \mathscr{F}^{(k)}\right)\right| = 2^{\mathrm{ex}(n, \mathscr{F}^{(k)}) + \varepsilon n^k}.$$

We also discuss an induced version of Theorem 13.4. This was done for graphs in [1, 5, 41] and for 3-uniform hypergraphs in [30].

We hope that there will be further applications of the Regularity Method for hypergraphs in the future.

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