Ramsey-type numbers involving graphs and hypergraphs with large girth

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Abstract

For a set of integers S, define $\binom{S}{AP_k}$ to be the k-uniform hypergraph with vertex set S and hyperedges corresponding to the set of all arithmetic progression of length k in S. Similarly, for a graph H, define $\binom{H}{K_k}$ to be the $\binom{k}{2}$-uniform hypergraph on the vertex set $E(H)$ with hyperedges corresponding to the edge sets of all copies of K_k in H. Also, we say that a k-uniform hypergraph has girth at least g if any h edges $(1 \leq h < g)$ span at least $(k-1)h+1$ vertices.

For all integers k and ℓ, we establish the existence of a relatively small graph H having girth k and the property that every ℓ-coloring of the edges of H yields a monochromatic copy of C_k. We also show that for all integers k, ℓ, and g, there exists a relatively small set $S \subset \mathbb{N}$ such that the related hypergraph $\binom{S}{AP_k}$ has girth g and each ℓ-coloring of S yields a monochromatic arithmetic progression of length k. Finally, for all integers k, ℓ, and g, we establish the existence of a relatively small graph H such that the associated hypergraph $\binom{H}{K_k}$ has girth g and each ℓ-coloring of the edges of H yields a monochromatic copy of K_k. Our proofs give improved (and the first explicit) numerical bounds on the size of these objects.

1 Introduction

We say that a graph H is *Ramsey* to a graph G for ℓ colors if every ℓ-coloring of the edges of H yields a monochromatic copy of G; we represent this by writing $H \to (G)_\ell$. For any graph G, let $R_\ell(G)$ denote the least integer n such that $K_n \to (G)_\ell$. Thus $R_\ell(K_k)$ are the well known *Ramsey numbers* for K_k and ℓ colors. In this paper we discuss three quantitative Ramsey-type results all related to cycles in graphs and hypergraphs.

The first result relates to a question of Erdős [2], who asked if for every pair of integers ℓ and k, there exists a graph H having $\mathrm{girth}(H) = k$ and the property $H \to (C_k)_\ell$. The existence of graphs with this property was first established in [10]. We give a new proof of this result which yields an explicit upper bound on the order of the smallest graph H with these properties.

Theorem 1.1 *For every pair of integers $k \geq 4$, $\ell \geq 2$, and $R := R_\ell(C_k)$, there exists a graph H satisfying*

$$\mathrm{girth}(H) = k, \quad H \to (C_k)_\ell, \quad \textit{and} \quad |V(H)| \leq R^{40k^2} k^{40k^3}.$$

Note that an exponential dependency for $|V(H)|$ on k as in Theorem 1.1 is unavoidable. This follows from the observation that a minimal graph H with the desired properties must have minimum degree greater than ℓ and girth k. Also note that the bound in Theorem 1.1 involves $R_\ell(C_k)$ which for fixed even k is known to be polynomial in ℓ and for fixed odd k satisfies the exponential relation $c_1^\ell \leq R_\ell(C_k) \leq c_2^{\ell \log \ell}$ for some $c_1, c_2 > 1$. This dichotomy between even and odd k leads to the following corollary.

Corollary 1.2 *For every integer $k \geq 3$, there exist constants C_{odd}, C_{even} such that for every integer $\ell \geq 2$, there exists a graph H with $\mathrm{girth}(H) = k$ such that $H \to (C_k)_\ell$. Moreover, $|V(H)| \leq \ell^{C_{even}}$ if k is even and $|V(H)| \leq C_{odd}^{\ell \log \ell}$ if k is odd.*

Our next two theorems have their roots in a classical result of Erdős and Hajnal who, in 1966, established the existence of hypergraphs having both large chromatic number and large girth (a k-uniform hypergraph has girth at least g if any h edges $(1 \leq h < g)$ span at least $(k-1)h+1$ vertices). Our second result extends this in the context of arithmetic progressions. Specifically for a set S of integers we write $S \to (AP_k)_\ell$ if every ℓ-coloring of the elements of S yields a monochromatic arithmetic progression of length k (AP_k). The *van der Waerden number* $W_\ell(k)$ is then the least integer N such that $[N] \to (AP_k)_\ell$ (where $[N] = \{1, 2, \ldots, N\}$), and van der Waerden's Theorem states that $W_\ell(k) < \infty$ for all k and ℓ. One of many generalizations of this theorem asked by Erdős in [3] is, if for all integers k and ℓ there exists a set S not containing an AP_{k+1} and still having the

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property $S \to (AP_k)_\ell$. This was answered independently by Spencer [15] and by Nešetřil and Rödl [8]. Moreover, for a set $S \subset \mathbb{N}$ define $\binom{S}{AP_k}$ to be the k-uniform hypergraph having vertex set S and hyperedges corresponding to arithmetic progressions of length k in S. Thus $S \to (AP_k)_\ell$ if and only if the chromatic number of $\binom{S}{AP_k}$ satisfies $\chi\binom{S}{AP_k} > \ell$; here we write $\chi\binom{S}{AP_k}$ for $\chi\left(\binom{S}{AP_k}\right)$, a convention which we will keep. In [10] and [11], Rödl and Ruciński established that for every $k \geq 3$ and $g \geq 2$, there exists a set $S \subset \mathbb{N}$ with both $\chi\binom{S}{AP_k} > \ell$ and $\mathrm{girth}\binom{S}{AP_k} \geq g$. Note that this extends the question asked by Erdős in [3] since if the girth of $\binom{S}{AP_k}$ is greater than two, any two arithmetic progressions of length k in S can share at most one element, and thus S cannot contain an AP_{k+1}. We provide a new proof of this result, giving the first explicit upper bound on the size of a minimum set S with this desired property.

Theorem 1.3 *For all integers $k \geq 3$, $\ell \geq 2$, $g \geq 2$, and $W := W_\ell(k)$, there exists a set $S \subset \mathbb{N}$ such that*

$$\chi\binom{S}{AP_k} > \ell, \quad \mathrm{girth}\binom{S}{AP_k} \geq g, \quad \textit{and} \quad |S| \leq k^{32k^2(k+g)} W^{10k(k+g)} g^{4kg}.$$

To illustrate the result, consider the special case $k = 3$. In this case, a result of Sanders [13] implies that $W_\ell(3) \leq \exp\left(\ell^{1+o(1)}\right)$ where the error term $o(1) \to 0$ as $\ell \to \infty$. For any $g \geq 2$, our result yields the existence of a set S of size at most $\exp\left(\ell^{1+o(1)}\right)$ such that $S \to (AP_3)_\ell$ and $\binom{S}{AP_3} \geq g$. Hence, our bound asymptotically matches Sanders' bound.

Our next result extends the result of Erdős and Hajnal on hypergraphs with large chromatic number and large girth in the context of a Ramsey-type problem. The origins of the problem relate to yet another question of Erdős and Hajnal, who asked if for every pair of integers k and ℓ, there exists a K_{k+1}-free graph H with $H \to (K_k)_\ell$. The case $\ell = 2$ was answered by Folkman and the case $\ell > 2$ was later answered by Nešetřil and Rödl [9]. In response to these developments, Erdős subsequently asked about a strengthened form of this result, namely the existence of a graph H with $H \to (K_k)_\ell$ in which no two copies of K_k share more than one edge. This was established in [6]. Analogous to the case for sets and arithmetic progressions, for graphs H and G, define the $e(G)$-uniform hypergraph $\binom{H}{G}$ to have vertex set $E(H)$ and hyperedges correspond to the edges of copies of G in H. Consequently, the statement $H \to (G)_\ell$ can be expressed equivalently in terms of chromatic number by $\chi\binom{H}{G} > \ell$. Also, the statement that any two copies of G in H share at most one edge is equivalent to $\binom{H}{G}$ being a (simple) linear hypergraph. In [7], it was shown that for every graph G and pair of integers ℓ and g, there exists a graph H such that $\chi\binom{H}{G} > \ell$ and $\mathrm{girth}\binom{H}{G} \geq g$.

We give a new proof of this result that yields an explicit numerical upper bound on the size of the smallest such H for $G = K_k$.

Theorem 1.4 *For all integers $k \geq 3$, $\ell \geq 2$, $g \geq 2$, and $R := R_\ell(k)$, there exists a graph H such that*

$$\chi\binom{H}{K_k} > \ell, \quad \mathrm{girth}\binom{H}{K_k} \geq g, \quad \textit{and} \quad |V(H)| \leq k^{2^{10}k^4g^2} R^{8k^2g}.$$

By reversing the dependency between g and $|V(H)|$ we obtain the following corollary.

Corollary 1.5 *For all $k \geq 3$ and $\ell \geq 2$, there exist $c, n_0 > 0$ such that for all $n \geq n_0$, there exists a graph H on n vertices with $H \to (K_k)_\ell$ and with* $\mathrm{girth}\binom{H}{K_k} \geq c\sqrt{\log n}$.

For a fixed ℓ and k, any graph H on n vertices satisfying $H \to (K_k)_\ell$ must also satisfy $\mathrm{girth}\binom{H}{K_k} = O(\log n)$; this follows from the fact that in an edge minimal graph H with the property $H \to (K_k)_\ell$, the hypergraph $\binom{H}{K_k}$ must have minimum degree at least ℓ. It would be interesting to close the gap between this bound and the one in Corollary 1.5.

The proofs of all three theorems are similar in the sense that they all analyze random constructions by way of a recent result of Saxton and Thomason on hypergraph containers [14]. Alternatively, a result due to Balogh, Morris, and Samotij in [1] could likely be used in place of [14] to obtain similar estimates. This method is similar to a recent paper of Rödl, Ruciński, and Schacht [12] which used some ideas of Nenadov and Steger [5].

2 Sketch of the proofs

Since the three proofs are similar, we will only sketch the one of Theorem 1.4. The main step of this proof establishes the following strong Ramsey property for the random graph $G(n,p)$. This property essentially says that for suitable parameters the random graph $G(n,p)$ is Ramsey to K_k even after removal of any not too large subset of edges.

Lemma 2.1 *For given $k \geq 3$, $\ell \geq 2$, and $g \geq 2$, let $R := R(k,\ell)$ and let*

$$n := k^{2^{10}k^4g^2} R^{8k^2g}, \quad c_p := 2^{5\sqrt{\log n \log k}} R^{16}, \quad p := c_p n^{\frac{-2}{k+1}}, \qquad t := \frac{p\binom{n}{2}}{2R^2}.$$

Then $p < 1$ and with probability at least $1 - 2\exp\left\{-p\binom{n}{2}/24R^2\right\}$ we have

$$\big(G(n,p) \setminus T\big) \to (K_k)_\ell \qquad \textit{for any edge set } T \subset G(n,p) \textit{ of size } |T| \leq t.$$

With this lemma established, Theorem 1.4 follows. Indeed, let X_2 represent the expected number of 2-cycles in $\binom{G(n,p)}{K_k}$, i.e. pairs of copies of K_k in $G(n,p)$ that intersect in more than two edges in $G(n,p)$, and let X_j represent the expected number of j-cycles in

$\binom{G(n,p)}{K_k}$ for $2 < j < g$, then we have that

$$X_2 \leq \sum_{i=3}^{k-1} n^{2k-i} p^{2\binom{k}{2}-\binom{i}{2}} \quad \text{and} \quad X_j \leq n^{(k-2)j} p^{j\binom{k}{2}-j}. \tag{1}$$

With the parameters from Lemma 2.1 and using Markov's inequality it follows from (1) that with probability at least $1/2$ the total number of cycles of length less than g in $\binom{G(n,p)}{K_k}$ is at most $p\binom{n}{2}/2R^2 = t$. Consequently, since $1 - \exp\left\{-p\binom{n}{2}/24R^2\right\} - \frac{1}{2} > 0$, we conclude that there exists a graph H' on n vertices which satisfies the conclusion of Lemma 2.1 and for which the associated hypergraph $\binom{H'}{K_k}$ contains at most t cycles of length less than g. By removing at most one vertex from each cycle of length less than g in $\binom{H'}{K_k}$, we obtain a graph $H \subset H'$ having at most t fewer edges than H' for which $\mathrm{girth}\binom{H}{K_k} \geq g$. Furthermore, as H is obtained from H' by removing at most t edges, $H \to (K_k)_\ell$. This establishes Theorem 1.4.

Proof Sketch of Lemma 2.1: The crux of the proof is applying the container lemma in [14] to the complete graph on the labeled vertex set $[n]$. A somewhat technical application of this lemma yields a family $\mathcal{C}$ of graphs on $[n]$ such that

(i) each K_k-free graph on $[n]$ is a subgraph of some $C \in \mathcal{C}$,

(ii) each $C \in \mathcal{C}$ has at most $\binom{n}{k}/(\ell R^2)$ copies of K_k, and

(iii) $|\mathcal{C}| \leq \exp\left\{p\binom{n}{2}/24\ell R^2\right\}$.

Define the family $\mathcal{B}$ to be the set of all graphs B on $[n]$ with the property that there exists a subgraph $T \subset B$ of size $p\binom{n}{2}/2R^2 = t$ such that $(B \setminus T) \not\to (K_k)_\ell$ i.e. there exists an ℓ-coloring of the edges of the graph $B \setminus T$ that does not contain a monochromatic K_k. Hence $\mathcal{B}$ is the set of all ('bad') graphs on n vertices that do not have the desired strong Ramsey property. To establish the lemma, we need to bound $\mathbb{P}(G(n,p) \in \mathcal{B})$ from above.

Consider any fixed graph $B \in \mathcal{B}$. By definition there is a set T with $|T| \leq t$ and a partition $(B \setminus T) = G_1 \cup \cdots \cup G_\ell$ such that that each of the subgraphs G_i contains no K_k. Hence, there is an associated set of containers $(C_1, C_2, \ldots, C_\ell)$ such that $G_i \subset C_i \in \mathcal{C}$ for all i. For a given B, define the associated graph $D := K_n \setminus \bigcup_{i\in[\ell]} C_i$. Observe that $|B \cap D| \leq |T|$, since $(B \setminus T) \subset G_1 \cup \cdots \cup G_\ell \subset (K_n \setminus D)$. Let $\mathcal{D}$ denote the family of all graphs D which arise this way by considering all $B \in \mathcal{B}$.

We have argued that for every $B \in \mathcal{B}$, there exists a $D \in \mathcal{D}$ such that $|B \cap D| \leq t$. It follows that if $G(n,p) \in \mathcal{B}$, then for some $D \in \mathcal{D}$ we have $|G(n,p) \cap D| \leq t$. The union bound now yields:

$$\begin{aligned}\mathbb{P}(G(n,p) \in \mathcal{B}) &\leq \mathbb{P}(\exists D \in \mathcal{D} : |G(n,p) \cap D| \leq t) \\ &\leq |\mathcal{C}|^\ell \cdot \max\{\mathbb{P}(|G(n,p) \cap D| \leq t) : D \in \mathcal{D}\}.\end{aligned} \tag{2}$$

Using (ii) it can be established that each $D \in \mathcal{D}$ has at least size at least $\frac{1}{R^2}\binom{n}{2}$, and thus the probability that the random graph $G(n,p)$ contains less than $t = \frac{p\binom{n}{2}}{2R^2}$ edges of D is small. Specifically, using Chernoff's bound we obtain

$$\max\{\mathbb{P}(|G(n,p) \cap D| \leq t) : D \in \mathcal{D}\} \leq 2\exp\left(-\frac{p\binom{n}{2}}{12R^2}\right)$$

Combined with (iii) which established that $|\mathcal{C}|$ is small, equation (2) becomes:

$$\mathbb{P}(G(n,p) \in \mathcal{B}) \leq \exp\left(\frac{p\binom{n}{2}}{24R^2}\right) \cdot 2\exp\left(-\frac{p\binom{n}{2}}{12R^2}\right) \leq 2\exp\left(-\frac{p\binom{n}{2}}{24R^2}\right),$$

which completes the proof sketch of Lemma 2.1.

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