Minimum vertex degree conditions for loose Hamilton cycles in 3-uniform hypergraphs

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Abstract

We investigate minimum vertex degree conditions for 3-uniform hypergraphs which ensure the existence of loose Hamilton cycles. A loose Hamilton cycle is a spanning cycle in which consecutive edges intersect in a single vertex. We prove that every 3-uniform n-vertex (n even) hypergraph $\mathcal{H}$ with minimum vertex degree $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} + o(1)\right)\binom{n}{2}$ contains a loose Hamilton cycle. This bound is asymptotically best possible.

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1 Introduction

Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ with vertex set $V = V(\mathcal{H})$ and edge set $E = E(\mathcal{H}) \subseteq \binom{V}{k}$ we denote for a subset $S \in \binom{V}{s}$ by $\deg(S)$ the number of edges of $\mathcal{H}$ containing S and let $\delta_s(\mathcal{H})$ be the minimum s-degree of $\mathcal{H}$, i.e., the minimum of $\deg(S)$ over all s-element sets $S \subseteq V$. For $s = 1$ the corresponding minimum degree $\delta_1(\mathcal{H})$ is referred to as minimum vertex degree whereas for $s = k-1$ we call the corresponding minimum degree $\delta_{k-1}(\mathcal{H})$ the minimum collective degree of $\mathcal{H}$.

We study sufficient minimum degree conditions which enforce the existence of spanning, so-called Hamilton cycles. A k-uniform hypergraph $\mathcal{C}$ is called an *ℓ-cycle* if there is a cyclic ordering of the vertices of $\mathcal{C}$ such that every edge consists of k consecutive vertices, every vertex is contained in an edge and two consecutive edges (where the ordering of the edges is inherited by the ordering of the vertices) intersect in exactly ℓ vertices. For $\ell = 1$ we call the cycle *loose* whereas the cycle is called *tight* if $\ell = k-1$. Naturally, we say that a k-uniform, n-vertex hypergraph $\mathcal{H}$ contains a Hamilton ℓ-cycle if there is a subhypergraph of $\mathcal{H}$ which forms an ℓ-cycle and which covers all vertices of $\mathcal{H}$. Note that a Hamilton ℓ-cycle contains exactly $n/(k-\ell)$ edges, implying that the number of vertices of $\mathcal{H}$ must be divisible by $(k-\ell)$ which we indicate by $n \in (k-\ell)\mathbb{N}$.

Sufficient minimum degree conditions which enforce Hamilton ℓ-cycles were studied extensively. In [9,10] Rödl, Ruciński, and Szemerédi found asymptotically sharp bounds for the existence of tight cycles, proving that such a cycle is guaranteed if $\delta_{k-1}(\mathcal{H}) \geq (1/2 + o(1))n$. Loose cycles were studied in [6,2,3] and it is shown that the appearance of such cycles is forced by the condition $\delta_{k-1}(\mathcal{H}) \geq (\frac{1}{2(k-1)} + o(1))n$ which is asymptotically best possible. Indeed, in [2] the bounds found apply to all $\ell < k/2$. Extending these result asymptotically sharp bounds for the minimum *collective* degree were found in [5] for the remaining ℓ.

We investigate conditions on the minimum *vertex* degree which ensure the existence of Hamilton cycles. For $\delta_1(\mathcal{H})$ very few results on spanning subhypergraph are known (see e.g. [1,7]). Here we give an asymptotically sharp bound on the minimum vertex degree in 3-uniform hypergraphs which enforces the existence of loose Hamilton cycles.

Theorem 1.1 *For all $\gamma > 0$ there exists an n_0 such that every 3-uniform hypergraph $\mathcal{H}$ on $n > n_0$ with $n \in 2\mathbb{N}$ and $\delta_1(\mathcal{H}) > \left(\frac{7}{16} + \gamma\right)\binom{n}{2}$ contains a loose Hamilton cycle.*

Theorem 1.1 is best possible up to the error constant γ as seen by the following 3-uniform hypergraph $\mathcal{H}_3 = (V, E)$ which already appeared in [6].

Let $A \dot\cup B = V$ be a partition of V with $|A| = \frac{n}{4} - 1$ and let E be the set of all triplets from V with at least one vertex in A. Clearly, $\delta_1(\mathcal{H}_3) = \binom{|A|}{2} + |A|(|B| - 1) = \frac{7}{16}\binom{n}{2} - O(n)$. Now consider an arbitrary cycle in $\mathcal{H}_3$. Note that every vertex, in particular every vertex from A, is contained in at most two edges of this cycle. Moreover, every edge of the cycle must intersect A. Consequently, the cycle contains at most $2|A| < n/2$ edges and, hence, cannot be a loose Hamilton cycle.

We note that the construction $\mathcal{H}_3$ satisfies $\delta_2(\mathcal{H}_3) \geq n/4 - 1$ and indeed, the same construction proves that the minimum *collective* degree condition for loose cycle is asymptotically best possible for the case $k = 3$.

This leads to the following conjecture for minimum vertex degree conditions enforcing loose Hamilton cycles in k-uniform hypergraphs. Let $k \geq 3$ and let $\mathcal{H}_k = (V, E)$ be the k-uniform, n-vertex hypergraph on $V = A \dot\cup B$ with $|A| = \frac{n}{2(k-1)} - 1$. Let E consists of all k-sets intersecting A in at least one vertex. Then $\mathcal{H}_k$ does not contain a loose Hamilton cycle and we believe that any k-uniform, n-vertex hypergraph $\mathcal{H}$ which has minimum vertex degree $\delta_1(\mathcal{H}) \geq \delta_1(\mathcal{H}_k) + o(n^2)$ contains a loose Hamilton cycle. Indeed, Theorem 1.1 verifies this for the case $k = 3$.

2 Auxiliary results and outline of the proof

We will "build" the loose Hamilton cycle by connecting loose paths. Such a path (with distinguished ends) is defined similarly to a loose cycle.

Further, the notion of connection is given as follows. We say that a triple system $(x_i, y_i, z_i)_{i\in[k]}$ **connects** $(a_i, b_i)_{i\in[k]}$ if $\left|\bigcup_{i\in[k]}\{a_i, b_i, x_i, y_i, z_i\}\right| = 5k$ (i.e. the pairs and triples are all disjoint), and for all $i \in [k]$ we have $\{a_i, x_i, y_i\}, \{y_i, z_i, b_i\} \in \mathcal{H}$. Suppose that a and b are ends of two different loose paths which do not contain (x, y, z) then the connection (x, y, z) would join these two paths to one loose path.

One can observe that in a 3-uniform hypergraph with sufficiently high minimum vertex degree a linear size family of pairs of vertices can be connected.

Lemma 2.1 (Connecting lemma) *For all $\gamma > 0$ there exists an n_0 such that the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on $n > n_0$ vertices which satisfies $\delta_1(\mathcal{H}) \geq \left(\frac{1}{4} + \gamma\right)\binom{n}{2}$. Let $k \leq \gamma n/12$ and let $(a_i, b_i)_{i\in[k]}$ be a system consisting of k mutually disjoint pairs of vertices. Then there is a system of triples $(x_i, y_i, z_i)_{i\in[k]}$ connecting $(a_i, b_i)_{i\in[k]}$.*

We introduce an Absorbing Lemma (Lemma 2.2). This lemma allows us to relax the problem of ensuring the existence of a loose Hamilton cycle to the problem of ensuring an *almost spanning* loose cycle. It asserts that every 3-uniform hypergraphs $\mathcal{H} = (V, E)$ with sufficiently large minimum vertex degree contains a so-called *absorbing* loose path $\mathcal{P}$, a short but powerful path which can incorporate any set of vertices of linear size.

Lemma 2.2 (Absorbing lemma) *For all $\gamma > 0$ there exist $\beta > 0$ and n_0 such that the following holds. Let $\mathcal{H}$ be a 3-uniform hypergraph on $n > n_0$ vertices which satisfies $\delta_1(\mathcal{H}) \geq \left(\frac{5}{8} + \gamma\right)^2 \binom{n}{2}$. Then there is a loose path $\mathcal{P}$ with $|V(\mathcal{P})| \leq \gamma^7 n$ such that for all subsets $U \subset V \setminus V(\mathcal{P})$ of size at most βn and $|U| \in 2\mathbb{N}$ there exists a loose path $\mathcal{Q} \subset \mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{P}) \cup U$ and $\mathcal{P}$ and $\mathcal{Q}$ have exactly the same ends.*

Rödl, Ruciński, and Szemerédi were the first to use the absorption technique in [9]. This idea has been further refined and applied in [10,8,11,2,1,5,4]. We say that a 7-tuple $(v_1, \ldots, v_7)$ **absorbs** the two vertices $x, y \in V$ if

$$v_1v_2v_3, v_3v_4v_5, v_5v_6v_7 \in \mathcal{H} \quad \text{and} \quad v_2xv_4, v_4yv_6 \in \mathcal{H}$$

are guaranteed. In particular, $(v_1, \ldots, v_7)$ and $(v_1, v_3, v_2, x, v_4, y, v_6, v_5, v_7)$ both form loose paths which, moreover, have the same ends. The main tecnical observation in the proof of Lemma 2.2 is that any pair of vertices is contained in $\Omega(n^7)$ absorbing 7-tuples. The remaining part of the proof relies on a simple application of Chernoff's bound and connecting 7-tuples to a loose path as given in the Connecting Lemma (Lemma 2.1).

The Absorbing Lemma reduces the problem of finding a loose Hamilton cycle to the simpler problem of finding an almost spanning loose cycle, which contains the absorbing path $\mathcal{P}$ and covers at least $(1-\beta)n$ of the vertices. We approach this simpler problem as follows. Let $\mathcal{H}'$ be the induced subhypergraph $\mathcal{H}$, which we obtain after removing the vertices of the absorbing path $\mathcal{P}$ guaranteed by the Absorbing Lemma. We remove from $\mathcal{H}'$ a "small" set R of vertices, called *reservoir*, which has the property that many loose paths can be connected to one loose cycles by using the vertices of R only.

Lemma 2.3 (Reservoir lemma) *For all $1/4 > \alpha > 0$ there exists an n_0 such that for every 3-uniform hypergraph $\mathcal{H}$ on $n > n_0$ vertices satisfying $\delta_1(\mathcal{H}) \geq \left(\frac{1}{4} + \gamma\right) \binom{n}{2}$ there is a set R of size at most γn with the following property: For every system $(a_i, b_i)_{i \in [k]}$ consisting of $k \leq \gamma^3 n/12$ mutually disjoint pairs of vertices from V there is a triple system connecting $(a_i, b_i)_{i \in [k]}$ which, moreover, contains vertices from R only.*

The proof of Lemma 2.3 is a simple application of Chernoff's bound and the Connecting Lemma (Lemma 2.1).

Let $\mathcal{H}''$ be the remaining hypergraph after removing the vertices from R. We will choose $\mathcal{P}$ and R small enough, so that $\delta_1(\mathcal{H}'') \geq (\frac{7}{16} + o(1))|V(\mathcal{H}'')|$. The third auxiliary lemma, the Path-tiling Lemma (Lemma 2.4), asserts that all but $o(n)$ vertices of $\mathcal{H}''$ can be covered by a small family of pairwise disjoint loose paths.

Lemma 2.4 (Path-tiling lemma) *For all $\gamma > 0$ and $\alpha > 0$ there exist integers p and n_0 such that for $n > n_0$ the following holds. Suppose $\mathcal{H}$ is a 3-uniform hypergraph on n vertices with minimum vertex degree $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} + \gamma\right)\binom{n}{2}$. Then there is a family of p disjoint loose paths in $\mathcal{H}$ which covers all but at most αn vertices of $\mathcal{H}$.*

Since the number of paths guaranteed in Lemma 2.4 is constant (independent of n), we can use the Reservoir Lemma (Lemma 2.3) to connect those paths and the absorbing path $\mathcal{P}$ to form a loose cycle by using exclusively vertices from R. This way we obtain a loose cycle in $\mathcal{H}$, which covers all but the $o(n)$ left-over vertices from $\mathcal{H}''$ and some left-over vertices from R. However, we will ensure that the number of those yet uncovered vertices will be smaller than βn and, hence, we can appeal to the absorption property of $\mathcal{P}$ and obtain a Hamilton cycle.

We note that among the auxiliary lemmas stated the Path-tiling Lemma is the only one which requires the minimum degree of $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} + o(1)\right)\binom{n}{2}$. Indeed, we consider this lemma to be the main obstacle to Theorem 1.1. Its proof is based on the *weak regularity lemma for hypergraphs*, a straightforward extension of Szemerédi's regularity lemma for graphs. In the proof of Lemma 2.4 we consider the following hypergraph $\mathcal{M}$. Let $\mathcal{M}$ be the 3-uniform hypergraph defined on the vertex set $[8]$ with the edges $123, 345, 456, 678 \in \mathcal{M}$. It can be shown that bounds on minimum degree guaranteeing an almost perfect $\mathcal{M}$-tiling also guarantee an almost perfect path tiling. In fact, this follows from an application the weak regularity lemma (see e.g. [2] for details). Hence Lemma 2.4 can be deduced from a result, stating that $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} + o(1)\right)\binom{n}{2}$ ensures an almost perfect $\mathcal{M}$-tiling of $\mathcal{H}$. We tackle the problem of $\mathcal{M}$-tiling by considering fractional extension of a given $\mathcal{M}$-tiling combined with the weak regularity lemma. Suppose in the hypergraph $\mathcal{H}$ with $\delta_1(\mathcal{H}) \geq \left(\frac{7}{16} + \gamma\right)\binom{n}{2}$ the maximum $\mathcal{M}$-tiling leaves more than αn vertices uncovered. Then, by applying the weak regularity lemma, one can show that the reduced hypergraph on t vertices contains an $\mathcal{M}$-tiling leaving at least $(\alpha - \varepsilon)t$ vertices uncovered where $\varepsilon \ll \alpha, \gamma$ can be chosen arbitrarily small. Using a fractional extension of $\mathcal{M}$-tiling, which is the main technical part of the proof, one obtains a fractional $\mathcal{M}$-tiling which is substantially larger than the $\mathcal{M}$-tiling guaranteed. By applying a "embedding lemma" to this fractional tiling, we obtain an $\mathcal{M}$-tiling in $\mathcal{H}$ which is larger than the maximum one, which yields the desired contradiction.

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