On the bandwidth conjecture for 3-colourable graphs

Julia Böttcher* Mathias Schacht† Anusch Taraz‡

September 25, 2006

Abstract

A conjecture by Bollobás and Komlós states that for every $\gamma > 0$ and integers $r \geq 2$ and Δ, there exists $\beta > 0$ such that for sufficiently large n the following holds: If G is a graph on n vertices with minimum degree at least $(\frac{r-1}{r}+\gamma)n$ and H is an r-chromatic graph on n vertices with bandwidth at most βn and maximum degree at most Δ, then G contains a copy of H. This conjecture generalises several results concerning sufficient degree conditions for the containment of spanning subgraphs. We prove the conjecture for the case $r = 3$. Our proof yields a polynomial time algorithm for embedding H into G if H is given together with a 3-colouring and vertex labelling respecting the bandwidth bound.

1 Introduction and results

The study of sufficient degree conditions which imply that a given graph G satisfies a certain property is one of the central themes in extremal graph theory. In this paper we are concerned with conditions on the minimum degree of G which guarantee that G contains a copy of a particular spanning subgraph H.

A well known example of such a result is Dirac's theorem [13]. It asserts that any graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains a spanning, so called Hamiltonian, cycle. Another classical result of that type by Corrádi and Hajnal [9] states that every graph G with n vertices and $\delta(G) \geq 2n/3$ contains $\lfloor n/3 \rfloor$ vertex disjoint triangles. This was generalised by Hajnal and Szemerédi [19], who proved that every graph G with $\delta(G) \geq (r-1)n/r$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r.

Pósa (see, e.g., [15]) and Seymour [35] indicated how these theorems could actually fit into a common framework. They conjectured that, at the same threshold $\delta(G) \geq (r-1)n/r$, one can in fact ask for 'well-connected' cliques, more precisely that such a graph G contains a copy of the $(r-1)$-st power of a Hamiltonian cycle (where the $(r-1)$-st power of an arbitrary graph is obtained by inserting an edge between every two vertices of distance at most $r-1$ in the original graph). The following approximate version of this conjecture for the case $r = 3$ was proved by Fan and Kierstead [17], and independently, by Komlós, Sárközy, and Szemerédi [26].

Theorem 1.1. ([17, 26]) *For every constant $\gamma > 0$ there is a constant n_0 such that every graph G on $n \geq n_0$ vertices with $\delta(G) \geq (2/3+\gamma)n$ contains the square of a Hamiltonian cycle.*

Fan and Kierstead [18] also gave a proof for the exact statement (i.e., with $\gamma = 0$ and $n_0 = 1$) for the square of a Hamiltonian path.[1] Moreover, Komlós, Sárközy, and Szemerédi [26] proved the approximate version concerning the $(r-1)$-st power of a Hamiltonian cycle. Finally, the same authors [23, 27] gave a proof of the sharp Pósa–Seymour conjecture for sufficiently large graphs G and general r.

Recently, several other results of a similar flavour have been obtained which deal with a variety of spanning subgraphs H, such as, e.g., trees, F-factors, and planar graphs [3, 5, 6, 7, 10, 11, 22, 28, 29, 30, 31, 32, 36].

Facing this wealth of results, there seems to be a need for a unifying generalisation. Which parameter(s) of H determine the minimum degree threshold for G to guarantee a spanning copy of H as a subgraph? The results above indicate that the chromatic number of H plays a crucial rôle.

Obviously, by the classical results of Turán [38] and Erdős, Stone and Simonovits [16, 14], any graph H of constant size with $\chi(H) = r$, is forced to appear as a subgraph in any sufficiently large graph G if $\delta(G) \geq (\frac{r-2}{r-1}+\gamma)n$. However, if H has as many vertices as G and if in every r-colouring of H the colour classes are of the same size, then it is clear that we do indeed

*Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D-85747 Garching bei München, Germany, boettche@ma.tum.de

†Supported by DFG grant SCHA 1263/1-1. Institut für Informatik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany, schacht@informatik.hu-berlin.de

‡Zentrum Mathematik, Technische Universität München, Boltzmannstraße 3, D-85747 Garching bei München, Germany, taraz@ma.tum.de

[1] In fact, Fan and Kierstead [18] showed that for the existence of a square of a Hamiltonian path $\delta(G) \geq (2n-1)/3$ is a sufficient and sharp minimum degree condition.

need $\delta(G) \geq \frac{r-1}{r}n$. For example, let G be the complete r-partite graph with partition classes almost, but not exactly, of the same size and let H be the union of vertex disjoint r-cliques. (See, e.g., [22, 31, 36] for a more detailed discussion how a less balanced r-colouring of H can lead to a smaller minimum degree threshold between $\frac{r-2}{r-1}n$ and $\frac{r-1}{r}n$.)

Thus, in an attempt to move away from results that concern only graphs H with a special, rigid structure, a naïve conjecture could be that $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$ suffices to guarantee that G contains a spanning copy of any r-chromatic graph H of bounded maximum degree. While the results mentioned above are in accordance with this idea, it is known that it fails in general as the following simple example shows.

Let H be a random bipartite graph with bounded maximum degree and partition classes of size $n/2$ each, and let G be the graph formed by two cliques of size $(1/2 + \gamma)n$ each, which share exactly $2\gamma n$ vertices. It is then easy to see that G cannot contain a copy of H, since in H every set of vertices of size $(1/2 - \gamma)n$ has more than $2\gamma n$ neighbours.

One way to rule out such expansion properties for H, is to restrict the *bandwidth* of H. A graph is said to have bandwidth at most b, if there exists a labelling of the vertices by numbers $1, \ldots, n$, such that for every edge $\{i, j\}$ of the graph we have $|i - j| \leq b$. Bollobás and Komlós [21, Conjecture 16] conjectured that every r-chromatic graph on n vertices of bounded degree and bandwidth limited by $o(n)$, can be embedded into any graph G on n vertices with $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$. In this paper we give a proof of this conjecture for the case $r = 3$.

Theorem 1.2. *For all $\Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds.*

If H is a 3-chromatic graph on n vertices with $\Delta(H) \leq \Delta$, and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (2/3 + \gamma)n$, then G contains a copy of H.

Moreover, such an embedding of H can be found in $O(n^{3.376})$ if H is given together with a valid 3-colouring and a labelling of the vertices respecting the bandwidth bound βn.

This theorem embraces a fairly large class of 3-chromatic graphs H – the structural requirement of a $o(n)$-bandwidth is less restrictive than, e.g., being a particular spanning subgraph. In fact, most graphs H considered so far were of constant bandwidth, whereas Theorem 1.2 includes (higher dimensional) grid graphs as possible graphs H.

The analogue of Theorem 1.2 for bipartite H was announced by Abbasi [1] in 1998 and a proof can be found in [20]. In [2] it is shown that in this case no sharp version of Theorem 1.2 (with $\gamma = 0$) is possible. More precisely, it is shown that if $\gamma \to 0$ and $\Delta \to \infty$ then β must tend to 0 in Theorem 1.2. However, the bound on β coming from our proof is rather poor, having a tower-type dependence on $1/\gamma$.

The proof of Theorem 1.2 is based on the *regularity method* and uses, in particular, the regularity lemma [37] and the blow-up lemma [24] together with Theorem 1.1. There is a well established strategy for proofs of this kind, which, as described by Komlós in his survey [21], proceeds in several steps:

First, prepare the graph H by dividing it into a constant number of smaller pieces, which is usually possible and not too difficult by calling upon the structural properties guaranteed for H. Secondly, prepare the graph G by applying the regularity lemma and thus obtaining a sufficiently regular vertex partition. Thirdly, find an assignment that maps vertices of H to the partition classes of G. Fourthly, ensure that the edges between the different parts of H are mapped to edges in G. Finally, complete the embedding by applying the blow-up lemma to the individual pieces of H and their counterparts in G.

Although steps 2, 3, and 5 have been standardised by the use of the powerful tools mentioned above, the proofs are still technically rather involved: although H and G have been 'prepared' roughly for each other, there is still a great deal of details that have to be carefully adjusted and fitted, especially in step 4. Since, in our case, we have very little control about the structure of H, this difficulty becomes particularly pressing. In order to avoid the looming threat of many cases, we have pushed the agenda described above a bit further.

We will prove two main lemmas. While they will deal exclusively with the graph G and the graph H respectively, they are linked to each other in the following way: the lemma for G (Lemma 2.2) will suggest a partition of G and communicate the structure of this partition (but not the graph G) to the lemma for H. The lemma for H (Lemma 2.3) will then try to find a partition of H with a very similar structure, and return the sizes of the partition classes to the lemma for G. The latter will then adjust its partition classes by shifting a few vertices of G, until they fit exactly class sizes of H. The embedding of H into G can then be found using (a slight variant of) the embedding lemma (Lemma 2.4) first used by Chvátal et al. for step 4 and the blow-up lemma (Lemma 2.1) for step 5.

This approach provides a very modular proof strategy that can easily be checked and may be of further use for other similar problems. For example, our current work-in-progress indicates that a proof of the Bollobás-Komlós conjecture for general r-chromatic graphs H is now within reach.

This extended abstract is organised as follows. In § 2, we state and explain our two main lemmas, Lemma 2.2 and 2.3, together with the two embedding lemmas mentioned above. Here we also *outline* how Theorem 1.2 can be deduced from these lemmas. This deduction is given in § 3. We conclude by briefly sketching the proofs of Lemma 2.2 and 2.3 in § 4 and § 5.

2 Main lemmas and outline of the proof

In this section we introduce the central lemmas that are needed for the proof of our main theorem. Our emphasis in this section is to explain how they work together to give the proof of Theorem 1.2, which itself is then presented in full detail in the subsequent section, § 3.

We start with some basic definitions. Our general aim is to find a *copy* of some graph H in some other graph G, by which we mean that G contains a subgraph which is isomorphic to H. In other words, we are looking for an *embedding* of H into G, i.e., an injective function $f\colon V(H) \to V(G)$ such that for every edge $\{u, v\} \in E(H)$ we have $\{f(u), f(v)\} \in E(G)$.

One of the main tools in our proof is Szemerédi's regularity lemma [37], which pivots around the concept of an ε-regular pair. Let $G = (V, E)$ be a graph and A, $B \subseteq V$ be disjoint vertex sets. The ratio $d(A, B) = e(A, B)/(|A||B|)$ is called the *density* of (A, B). The pair (A, B) is *ε-regular*, if for all $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ it is true that $|d(A, B) - d(A', B')| < \varepsilon$. An ε-regular pair (A, B) is called (ε, d)-regular if it has density at least d. If, in addition, every $v \in A$ has at least $d|B|$ neighbours in B and every $u \in B$ has at least $d|A|$ neighbours in A, the pair (A, B) is called *(ε, d)-super-regular*.

It is easy to check that every (ε, d)-regular pair (A, B) contains an $(2\varepsilon, d - 2\varepsilon)$-super-regular sub-pair (A', B') with $|A'| \geq (1 - \varepsilon)|A|$ and $|B'| \geq (1 - \varepsilon)|B|$. An exciting feature about super-regular pairs is that a powerful theorem, the so-called *blow-up lemma* proven by Komlós, Sárközy and Szemerédi [24] (see also [33] for an alternative proof), guarantees that bipartite spanning graphs of bounded degree can be embedded into sufficiently super-regular pairs. In fact, the statement is more general and allows the embedding of r-chromatic graphs into the union of r vertex classes that form $\binom{r}{2}$ super-regular pairs, but we will only use this lemma in the following restricted form for 3-chromatic graphs.

Lemma 2.1. (Blow-up lemma [24]) *For every d, Δ, $c > 0$ there exist constants $\varepsilon_{\mathrm{BL}} = \varepsilon_{\mathrm{BL}}(d, \Delta, c)$ and $\alpha_{\mathrm{BL}} = \alpha_{\mathrm{BL}}(d, \Delta, c)$ such that the following holds.*

Let n_1, n_2, and n_3 be arbitrary positive integers, $0 < \varepsilon < \varepsilon_{\mathrm{BL}}$, and $G = (V_1 \dot\cup V_2 \dot\cup V_3, E)$ be a 3-partite graph with $|V_i| = n_i$ for $i \in [3]$ and with all pairs (V_i, V_j) being (ε, d)-super-regular for $1 \leq i < j \leq 3$. Suppose H is a 3-partite graph on vertex classes $W_1 \dot\cup W_2 \dot\cup W_3$ of sizes n_1, n_2, and n_3 with $\Delta(H) \leq \Delta$. Moreover, suppose that in each class W_i there is a set of at most $\alpha_{\mathrm{BL}} n_i$ special *vertices y, each of them equipped with a set $C_y \subseteq V_i$ with $|C_y| \geq cn_i$.*

Then there is an embedding of H into G such that every special vertex y is mapped to a vertex in C_y.

We say that the special vertices y in Lemma 2.1 are *image restricted to C_y*.

Of course a comfortable embedding tool like the blow-up lemma only makes sense, if we can be sure to find super-regular (or regular) pairs in G. This was proven much earlier by Szemerédi [37] in 1978. Our next lemma incorporates the regularity lemma and is derived from it, but before we can state it we will need some more definitions.

Let $G = (V, E)$ be a graph, let $V_1 \dot\cup \cdots \dot\cup V_k$ be a partition of V, and let R_k be a graph on the vertex set $[k]$. We say that $V_1 \dot\cup \cdots \dot\cup V_k$ is *(ε, d)-regular on R_k* if (V_i, V_j) is (ε, d)-regular for every $\{i, j\} \in E(R_k)$. R_k is also called the *reduced* graph for $V_1 \dot\cup \cdots \dot\cup V_k$. Similarly, $V_1 \dot\cup \cdots \dot\cup V_k$ is *(ε, d)-super-regular on R_k* if (V_i, V_j) is (ε, d)-super-regular for every $\{i, j\} \in E(R_k)$. For all $n, k \in \mathbb{N}$ with k divisible by 3, we call an integer partition $n_1 + \cdots + n_k = n$ (with $n_i \in \mathbb{N}$ for all $i \in [k]$) *equitriangular*, if $|n_{3(j-1)+l} - n_{3(j-1)+l'}| \leq 1$ for all $j \in [k/3]$ and $l, l' \in [3]$. We denote by $R^*_k = ([k], E(R^*_k))$ the square of the Hamiltonian cycle with edges $\{\{i, i+1\}\colon i = 1, \ldots, k-1\} \cup \{\{1, k\}\}$. Moreover, we write R^{**}_k for the subgraph of R^*_k consisting of the family of $k/3$ vertex disjoint triangles in R^*_k with vertex sets $3(j-1)+1$, $3(j-1)+2$, and $3(j-1)+3$ for $j \in [k/3]$.

We can now state (and then explain) our first main lemma which 'prepares' the graph G for the embedding of H into G.

Lemma 2.2. (Lemma for G) *For all $\gamma > 0$ there exist d and $\varepsilon_0 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there exist K_0 and $\xi_0 > 0$ such that for all $n \geq K_0$ and for every graph G on vertex set $[n]$ with $\delta(G) \geq (2/3 + \gamma)n$ there exist $k \in \mathbb{N} \setminus \{0\}$ and a graph R_k on vertex set $[k]$ with*

- *(R1) $k \leq K_0$ and $3|k$,*
- *(R2) $\delta(R_k) \geq (2/3 + \gamma/2)k$,*
- *(R3) $R^{**}_k \subseteq R^*_k \subseteq R_k$, and*
- *(R4) there is an equitriangular integer partition $m_1 + \cdots + m_k$ of n with $m_i \geq (1 - \varepsilon)n/k$ such that the following holds.*

For every partition $n = n_1 + \cdots + n_k$ with $m_i - \xi_0 n \leq n_i \leq m_i + \xi_0 n$ there exists a partition $V_1 \dot\cup \cdots \dot\cup V_k$ of V with

(*V1*) $|V_i| = n_i$,

(*V2*) $V_1 \dot\cup \cdots \dot\cup V_k$ *is (ε, d)-regular on R_k, and*

(*V3*) $V_1 \dot\cup \cdots \dot\cup V_k$ *is (ε, d)-super-regular on R_k^{**}.*

In order to understand what this lemma says, let us first ignore property (*R4*), the two lines thereafter, and property (*V1*), and instead propose that the sizes $|V_i|$ form an equitriangular partition of n. In this case, Lemma 2.2 could be considered a standard corollary of the regularity lemma for graphs G with $\delta(G) \geq (2/3 + \gamma)n$ (see, e.g., [32, Proposition 9]). Here it would guarantee a partition of $V(G)$ in such a way that the partition classes form many (super-)regular pairs, and that these pairs are organised in a sort of backbone, namely in the form of a square of a Hamiltonian cycle R_k^* for the regular pairs, and, contained therein, a spanning family R_k^{**} of disjoint triangles for the super-regular pairs.

However, the lemma says more. When we come to the point (*R4*), the lemma 'has in mind' the partition we just described, but doesn't exhibit it. Instead, it only discloses the sizes m_i and allows us to wish for small amendments: for every $i \in [k]$, we can now look at the value m_i and ask for the size of the i-th partition class to be adjusted to a new value n_i, differing from m_i by at most $\xi_0 n$.

When proving Lemma 2.2, one needs to alter the partition by shifting a few vertices. Note that while (ε, d)-regularity is very robust towards such small alterations, (ε, d)-*super*-regularity is not, so this is where the main difficulty lies. We sketch the proof of Lemma 2.2, which borrows ideas from [29], in § 4.

Now we come to the second main lemma which prepares the graph H so that it can be embedded into G. This is exactly the place which, given the values m_i, specifies the new values n_i in the setting described above.

Lemma 2.3. (Lemma for H) *Let $k \geq 1$ be an integer and let β, $\xi > 0$ satisfy $\beta \leq \xi^2/10^4$. Let H be a 3-colourable graph on n vertices with bandwidth at most βn and let R_k be a graph with $V(R_k) = [k]$ such that $\delta(R_k) > 2k/3$ and $R_k^{**} \subseteq R_k^* \subseteq R_k$. Furthermore, suppose $m_1 + \cdots + m_k$ is an equitriangular integer partition of n with $m_i \geq \beta n$ for every $i \in [k]$.*

Then there exists a mapping $f\colon V(H) \to [k]$ and a set of special vertices $X \subseteq V(H)$ with the following properties

(*a*) $|X| \leq k\xi n$,

(*b*) $m_i - \xi n \leq |W_i| := |f^{-1}(i)| \leq m_i + \xi n$ *for every* $i \in [k]$,

(*c*) *for every* $\{u, v\} \in E(H)$ *we have* $\{f(u), f(v)\} \in E(R_k)$, *and*

(*d*) *if* $\{u, v\} \in E(H)$ *and, moreover, u and v are both in* $V(H) \setminus X$, *then* $\{f(u), f(v)\} \in E(R_k^{**})$.

In other words, Lemma 2.3 receives a graph H as input and, from Lemma 2.2, a reduced graph R_k (with $R_k^{**} \subseteq R_k^* \subseteq R_k$), an equitriangular partition $n = m_1 + \cdots + m_k$, and a parameter ξ.

Again we emphasise that this is all what Lemma 2.3 needs to know about G. It then provides us with a function f which maps the vertices of H onto the vertex set $[k]$ of R_k in such a way that $i \in [k]$ receives $n_i := |W_i|$ vertices from H, with $|n_i - m_i| \leq \xi n$. Although the vertex partition of G is not known exactly at this point, we already have its reduced graph R_k. Lemma 2.3 guarantees that the endpoints of an edge $\{u, v\}$ of H get mapped into vertices $f(u)$ and $f(v)$ of R_k, representing future partition classes $V_{f(u)}$ and $V_{f(v)}$ in G which will form a *super-regular* pair (see (*d*)) – except for those few edges with one or both endpoints in some small special set X. But even these edges will be mapped into pairs of classes in G that will form at least *regular* pairs (see (*c*)). Lemma 2.3 will then return the values n_i to Lemma 2.2, which will finally produce a corresponding partition of the vertices of G.

If we consider the triangles $3(j-1)+1$, $3(j-1)+2$, and $3(j-1)+3$ for every $j \in [k/3]$ that form the edge set of R_k^{**}, then Lemma 2.1 yields an embedding of $H[W_{3(j-1)+1}, W_{3(j-1)+2}, W_{3(j-1)+3}]$ into $G[V_{3(j-1)+1}, V_{3(j-1)+2}, V_{3(j-1)+3}]$ that takes care of all edges of $H[V(H) \setminus X]$.

Edges of H with one or both vertices in the special set X will need some special treatment. However, due to part (*a*) of Lemma 2.3 the size of X is quite small. In particular we will be able to ensure that $|X| \ll n/k$. Our strategy will be first to find an embedding g of the vertices of X into $V(G)$ such that for every $y \in N_H(X) := \{y \in V(H) \setminus X \colon \exists xy \in E(H) \text{ s.t. } x \in X\}$ the set $C_y := V_{f(y)} \cap \bigcap_{x \in N_H(y) \cap X} N_G(g(x))$ is sufficiently large. The following lemma guarantees the existence of such an embedding g of X. Once we have applied it, we can complete the partial embedding g with the blow-up lemma, which will 'respect' the image restriction to C_y for every $y \in N_H(X)$.

Lemma 2.4 is in fact very similar to the embedding lemma of Chvátal, Rödl, Szemerédi, and Trotter [8] (see also [12, Lemma 7.5.2]) and hence we omit its proof here. The only difference between Lemma 2.4 and their embedding lemma is that we only embed *some* of the vertices of a given graph B into G and reserve sufficiently many places in G for a future embedding of the remaining vertices of B.

LEMMA 2.4. (PARTIAL EMBEDDING LEMMA) *For every integer $\Delta \geq 1$ and every $d \in (0,1]$ there exist constants $c = c(\Delta, d)$ and $\varepsilon_{\mathrm{PEL}} = \varepsilon_{\mathrm{PEL}}(\Delta, d)$ such that for all $\varepsilon \leq \varepsilon_{\mathrm{PEL}}$ the following is true.*

Let R_k be a graph with $V(R_k) = [k]$ and G be a graph with $V(G) = V_1 \dot{\cup} \cdots \dot{\cup} V_k$, such that $|V_i| \geq (1-\varepsilon_{\mathrm{PEL}})n/k$ for all $i \in [k]$ and $V_1 \dot{\cup} \cdots \dot{\cup} V_k$ is (ε, d)-regular on R_k. Let, furthermore, B be a graph with $V(B) = X \dot{\cup} Y$ and $f\colon V(B) \to V(R_k)$ be a mapping with $\{f(b), f(b')\} \in E(R_k)$ for all $\{b, b'\} \in E(B)$.

If $|V(B)| \leq \varepsilon_{\mathrm{PEL}} n/k$ and $\Delta(B) \leq \Delta$, then there exists an injective mapping $g\colon X \to V(G)$ with $g(x) \in V_{f(x)}$ for all $x \in X$ such that for all $y \in Y$ there exist sets $C_y \subseteq V_{f(y)} \setminus g(X)$ such that

(*i*) *If x and $x' \in X$ and $\{x, x'\} \in E(B)$ then $\{g(x), g(x')\} \in E(G)$,*

(*ii*) *for all $y \in Y$ we have $C_y \subseteq N_G(g(x))$ for all $x \in N_B(y) \cap X$, and*

(*iii*) *$|C_y| \geq c|V_{f(y)}|$.*

In the next section we give the precise details how Theorem 1.2 can be deduced from the lemmas just presented.

3 Proof of Theorem 1.2

In this section we give the proof of Theorem 1.2 based on Lemmas 2.1–2.4 from § 2. In particular, we will use Lemma 2.2 for partitioning G, and Lemma 2.3 for assigning the vertices of H to the parts of G. For this, it will be necessary, to split the application of Lemma 2.2 into two phases. The first phase is used to set up the parameters for Lemma 2.3. With this input, Lemma 2.3 then defines the sizes of the parts of G that are constructed during the execution of the second phase of Lemma 2.2.

Finally, H is embedded into G by using the blow-up lemma, Lemma 2.1, on the partition of G and by treating the special vertices $X \subseteq V(H)$ from Lemma 2.3 with the help of the partial embedding lemma, Lemma 2.4.

Here is how the constants that appear in the proof are related:

$$\frac{1}{\Delta}, \gamma \gg d \gg \varepsilon \gg \frac{1}{K_0} \gg \xi \gg \beta \quad \text{and} \quad c \gg \varepsilon \gg \alpha\,.$$

Proof. Given Δ and γ, let ε_0 and d be as asserted by Lemma 2.2 for input γ. Let $c = c(\Delta, d)$ and $\varepsilon_{\mathrm{PEL}} = \varepsilon_{\mathrm{PEL}}(\Delta, d)$ be as given by Lemma 2.4, and $\varepsilon_{\mathrm{BL}} = \varepsilon_{\mathrm{BL}}(d/2, \Delta, c)$ and $\alpha_{\mathrm{BL}} = \alpha_{\mathrm{BL}}(d/2, \Delta, c)$ as given by Lemma 2.1. Set

$$\varepsilon := \min\{\varepsilon_0, \varepsilon_{\mathrm{PEL}}/2, \varepsilon_{\mathrm{BL}}/2, d/4\}\,. \tag{3.1}$$

Then, the lemma for G (Lemma 2.2) provides constants K_0 and ξ_0 for this ε. We define

$$\xi := \min\left\{\xi_0, \frac{1}{4K_0}, \frac{\varepsilon}{K_0^2(\Delta+1)}, \frac{\alpha_{\mathrm{BL}}}{2K_0^2(\Delta+1)}\right\} \tag{3.2}$$

as well as $n_0 := K_0$, $\beta := \min\{\xi^2/10^4, (1-\varepsilon)/K_0\}$ and consider arbitrary graphs H and G on $n \geq n_0$ vertices that meet the conditions of Theorem 1.2.

Applying Lemma 2.2 to G we get an integer k with $0 < k \leq K_0$, graphs $R_k^{**} \subseteq R_k^* \subseteq R_k$ on vertex set $[k]$, and an equitriangular partition $m_1 + \cdots + m_k$ of n such that (*R1*)–(*R4*) are satisfied.

Before continuing with Lemma 2.2, we will now apply the lemma for H (Lemma 2.3). Note that due to (*R4*) and the choice of β above, we have $m_i \geq (1-\varepsilon)n/k \geq \beta n$ for every $i \in [k]$. Consequently, for constants k, β, and ξ, graphs H and $R_k^{**} \subseteq R_k^* \subseteq R_k$, and the equitriangular integer partition $m_1 + \cdots + m_k = n$ we can apply Lemma 2.3. This yields a mapping $f\colon V(H) \to [k]$ and a set of special vertices $X \subseteq V(H)$. These will be needed later. For the moment we are only interested in the sizes $n_i := |W_i| = |f^{-1}(i)|$ for $i \in [k]$. Condition (*b*) of Lemma 2.3 and the choice of $\xi \leq \xi_0$ in (3.2) imply that the partition $n = n_1 + \cdots + n_k$ satisfies $m_i - \xi_0 n \leq m_i - \xi n \leq n_i \leq m_i + \xi n \leq m_i + \xi_0 n$ for every $i \in [k]$. Accordingly, we can continue with Lemma 2.2 to obtain a partition $V = V_1 \dot{\cup} \cdots \dot{\cup} V_k$ with $|V_i| = n_i$ that satisfies conditions (*V1*)–(*V3*) of Lemma 2.2. Note that

$$\begin{aligned} |V_i| = n_i \geq m_i - \xi n &\overset{(R4)}{\geq} (1-\varepsilon)\frac{n}{k} - \xi n \\ &\geq (1-\varepsilon_{\mathrm{PEL}})\frac{n}{k} \geq \frac{1}{2}\frac{n}{k}. \end{aligned} \tag{3.3}$$

Now, we have partitions $W_1 \dot{\cup} \cdots \dot{\cup} W_k$ of H and $V_1 \dot{\cup} \cdots \dot{\cup} V_k$ of G with $|W_i| = |V_i| = n_i$ for all $i \in [k]$. We will build the embedding of H into G such that each vertex $v \in W_i \subseteq V(H)$ will be embedded into the corresponding set $V_i \subseteq V(G)$ for $i \in [k]$.

For embedding the special vertices X of H in G, we use the partial embedding lemma (Lemma 2.4). We provide Lemma 2.4 with constants Δ, d, R, and k, the graph G with vertex partition $V_1 \dot{\cup} \cdots \dot{\cup} V_k = V(G)$, the graph $B := H[X \cup Y]$ where $Y := N_H(X)$ consists of the neighbours of vertices of X outside X, and the mapping f restricted to $X \dot{\cup} Y$. By (*V2*) of Lemma 2.2 and (*c*) of Lemma 2.3, G and f fulfil the requirements of Lemma 2.4. Moreover, since $\Delta(B) \le \Delta(H) \le \Delta$

$$\begin{aligned} |X| + |Y| &= |V(B)| \le (\Delta+1)|X| \\ &\le (\Delta+1)k\xi n \overset{(3.2)}{\le} \varepsilon\frac{n}{k} \end{aligned} \tag{3.4}$$

by (a) of Lemma 2.3. Accordingly, since $\varepsilon \le \varepsilon_{\mathrm{PEL}}$ we can apply Lemma 2.4 for obtaining an embedding g of the vertices in X, and for every $y \in Y$ sets C_y such that $C_y \subseteq V_{f(y)} \setminus g(X)$ and

$$|C_y| \ge c|V_{f(y)}| \ge c|V_{f(y)} \setminus g(X)|\,.$$

The sets C_y will be used in the blow-up lemma for the image restriction of the vertices in $Y = N_H(X)$. We first check that there are not too many of these vertices. Let $W_i' := W_i \setminus X$, $V_i' := V_i \setminus g(X)$ and $n_i' := |W_i'| = |V_i'|$ for each $i \in [k]$. Observe that

$$|X| + |Y| \overset{(3.4)}{\le} (\Delta+1)k\xi n \overset{(3.2)}{\le} \frac{\alpha_{\mathrm{BL}}}{2k} n \overset{(3.3)}{\le} \alpha_{\mathrm{BL}} n_i,$$

and, hence,

$$\begin{aligned} |N_H(X)| = |Y| &\le \alpha_{\mathrm{BL}} n_i - |X| \\ &\le \alpha_{\mathrm{BL}}(n_i - |X|) \le \alpha_{\mathrm{BL}} n_i'. \end{aligned}$$

For all $j \in [k/3]$ we apply Lemma 2.1 and find an embedding of $H[W'_{3(j-1)+1}, W'_{3(j-1)+2}, W'_{3(j-1)+3}]$ into $G[V'_{3(j-1)+1}, V'_{3(j-1)+2}, V'_{3(j-1)+3}]$ in such a way that every $y \in N_H(X)$ will be embedded into C_y. It is easy to check the the respective conditions are satisfied. Indeed, recall that by $(V3)$ the pair $(V_{3(j-1)+l}, V_{3(j-1)+l'})$ is (ε, d)-super-regular and that $V_i' = V_i \setminus g(X)$ for every $i \in [k]$. It follows directly from the definition of a super-regular pair and (3.3), (3.4), and $\varepsilon \le d/4$, that $(V'_{3(j-1)+l}, V'_{3(j-1)+l'})$ is $(2\varepsilon, d/2)$-super-regular with $\varepsilon \le \varepsilon_{\mathrm{BL}}/2$ (see (3.1)).

Having applied the blow-up lemma for every $j \in [k/3]$, we have obtained a bijection

$$h\colon\ W_1' \dot\cup \cdots \dot\cup W_k' \to V_1' \dot\cup \cdots \dot\cup V_k'$$

with

$$h(W_i') = V_i' \text{ for every } i \in [k]$$

such that

$$h(y) \in C_y \quad \text{for every } y \in N_H(X) \tag{3.5}$$

and

$$H[W_1' \dot\cup \cdots \dot\cup W_k'] \subseteq G[h(W_1') \dot\cup \cdots \dot\cup h(W_k')].$$

Now we finish the proof by checking that the united embedding $\bar h\colon V(H) \to V(G)$ defined by

$$v \mapsto \bar h(v) := \begin{cases} h(v) & \text{if } v \in V(H) \setminus X \\ g(v) & \text{if } v \in X \end{cases}$$

is indeed an embedding of H into G. Let $e = \{u, v\}$ be an edge of H. We distinguish three cases.

If $u, v \in X$, then $\{\bar h(u), \bar h(v)\} = \{g(u), g(v)\}$, which is an edge in G since g is an embedding of $H[X]$ into G by the partial embedding lemma.

If $u \in X$ and $v \in V(H) \setminus X$, then $v \in N_H(u) \subseteq N_H(X)$, so we have $h(v) \in C_v \subseteq N_G(g(u))$ by (3.5), (3.3), and part (ii) of Lemma 2.4, thus $\{\bar h(u), \bar h(v)\} = \{g(u), h(v)\} \in E(G)$.

If, finally, $u, v \in V(H) \setminus X$, then by part (d) of Lemma 2.3, $\{f(u), f(v)\} \in E(R_k^{**})$. In other words, there exists a $j \in [k/3]$, such that $\{u, v\}$ is contained in $H[W'_{3(j-1)+1}, W'_{3(j-1)+2}, W'_{3(j-1)+3}]$ and hence $\{\bar h(u), \bar h(v)\} = \{h(u), h(v)\} \in E(G)$ by (3.5).

Finally, we note that this proof yields an algorithm, which finds an embedding of H in G, if H is given along with a valid 3-colouring and a labelling of the vertices respecting the bandwidth bound βn. This follows from the observation that the proof above is constructive, and all the lemmas used in the proof (Lemma 2.1–2.4) have algorithmic proofs. Algorithmic versions of the blow-up lemma, Lemma 2.1, were obtained in [25, 34]. In [25] a running time of order $O(\max\{n_1, n_2, n_3\}^{3.376})$ was proved. The key ingredient of Lemma 2.2 is Szemerédi's regularity lemma for which a $O(n^{2.376})$ algorithm exists due to [4]. All other arguments in the proof of Lemma 2.2 can be done algorithmically in $O(n^2)$ (see § 4). Similarly, the proof of Lemma 2.3 is constructive if a 3-colouring of H and a bandwidth ordering is given (see § 5). Finally, we note that the proof of Lemma 2.4 (following along the lines of [8]) gives rise to a $O(n^3)$ algorithm. Thus there is a

$$O\big(k \times ((1/k + \xi_0)n)^{3.376} + n^{2.376} + n^2 + n^3\big) = O\big(n^{3.376}\big)$$

embedding algorithm, where the implicit constant depends on γ and Δ only.

4 Lemma for G

The main ingredients for the proof of Lemma 2.2 are Szemerédi's regularity lemma which provides a reduced graph R_k for G and a partition of $V(G)$, Theorem 1.1 which guarantees the square of a Hamiltonian cycle in R_k, and a strategy for moving vertices between the partition classes of G in order to adjust the sizes of these classes. In the following, we will sketch the main ideas of this proof.

Let us first assume that we want to prove Lemma 2.2 only for the special case that $n_i = m_i$ for all $i \in [k]$. Hence we are interested in finding graphs $R_k^{**} \subseteq R_k^* \subseteq R_k$ on k vertices that satisfy $\delta(R_k) \ge (2/3 + \gamma/2)k$ and a partition $V(G) = V_1 \dot\cup \cdots \dot\cup V_k$ with $|V_i| \ge (1-\varepsilon)n/k$ for all $i \in [k]$ that is regular on R_k

and super-regular on R_k^{**}. For this, we proceed in three steps.

We apply the regularity lemma to construct a partition $V_0'\dot\cup V_1'\dot\cup\cdots\dot\cup V_{k'}'$ of $V(G)$ with reduced graph $R'=([k'],E(R'))$ such that $G[V_i'\cup V_j']$ is (ε',d')-regular for some suitable constants $\varepsilon'<\varepsilon$ and $d'>d$ whenever $\{i,j\}\in E(R')$. Since $\delta(G)\geq(2/3+\gamma)n$, it can easily be assured that there exists a subgraph $R_k\subseteq R'$ on k vertices such that $\delta(R_k)\geq(2/3+\gamma/2)k$ and $3|k$ by deleting some few sets V_i' from R' and adding them to V_0'. Let $V_0''\dot\cup V_1''\dot\cup\cdots\dot\cup V_k''$ be the resulting partition of $V(G)$.

From Theorem 1.1 it follows that subgraphs $R_k^{**}\subseteq R_k^*\subseteq R_k$ exist (provided that k' and thus k was chosen sufficiently large). Next, we modify the partition $V_0''\dot\cup V_1''\dot\cup\cdots\dot\cup V_k''$ in order to obtain super-regularity on R_k^{**}. This is achieved by deleting those vertices from each V_i'' with $i\in[k]$ that violate the super-regularity on R_k^{**} and adding them to V_0'', i.e., we delete those vertices v from V_i'' for which $|N(v)\cap V_j''|$ is too small for some j with $\{i,j\}\in R_k^{**}$. In addition we remove some more vertices such that the resulting partition classes are of equal size. Since most vertices in a regular pair have high degree, not many vertices are moved in this process.

In a last step we redistribute the vertices of the new exceptional class V_0''' among the other classes of the new partition $V_0'''\dot\cup V_1'''\dot\cup\cdots\dot\cup V_k'''$, while maintaining super-regularity. Here the following problem occurs. Although a pair remains almost as *regular* as before when a few vertices leave or enter a partition class, the property of being *super-regular* is not that robust: *every* vertex that is moved to a new class which is part of a super-regular triangle (of R_k^{**}) must make sure that it has sufficiently many neighbours inside the neighbouring classes within the triangle.

For this purpose, let u be a vertex in V_0'''. A triangle $i+1$, $i+2$, $i+3$ of R_k^{**} is called *u-friendly*, if u has at least dn/k neighbours in each of V_{i+1}''', V_{i+2}''', and V_{i+3}'''. Note that we can move u to any of the classes V_{i+1}''', V_{i+2}''', and V_{i+3}''' without compromising super-regularity. Since $\delta(G)\geq(2/3+\gamma)n$, it follows that each $u\in V_0'''$ has at least $\gamma k/3$ u-friendly triangles. When we distribute the vertices $u\in V_0'''$ to classes of u-friendly triangles as evenly as possible, we therefore add at most $|V_0'''|/(\gamma k)$ vertices to each V_i''' with $i\in[k]$. If we chose ε' small enough, we will still have (ε,d)-super-regularity after these changes. The resulting partition is the desired equitriangular partition $V_1\dot\cup\cdots\dot\cup V_k$.

This proves the special case of Lemma 2.2 with $n_i=m_i$ for every $i\in[k]$. For the general version, it remains to show that the sizes of the classes V_i can be slightly changed from m_i to n_i by moving some vertices without destroying any of the achieved properties. At this point we use the structure of the graph R_k^*.

Let σ be the unique 3-colouring of R_k^* with $\sigma(3j+c)=c$ for all $0\leq j<k/3$ and $c\in[3]$. We also say that the class V_{3j+c} is of colour c. Let $i+1,i+2,i+3$ be a triangle in R_k^{**} with $i=3j$ for some $j\in\{0,\dots,k/3-1\}$. Observe that $\{i+4,i+2\}$ and $\{i+4,i+3\}$ are edges in R_k^*. Since $V_1\dot\cup\cdots\dot\cup V_k$ is regular on $R_k\supseteq R_k^*$ it follows that typical vertices in V_{i+4} have many neighbours in V_{i+2} and V_{i+3}. Thus we can move such a typical vertex from V_{i+4} to V_{i+1} without violating super-regularity. Since R_k^* is the square of a Hamiltonian *cycle* this can be applied repeatedly and we can move vertices of any class of colour 1 to any other class of colour 1. We call this procedure *method 1*. Similarly, vertices can be moved from V_{i+3} to V_{i+6}, and, repeating the argument, to any other class of colour 3. The classes of colour 2 however need special treatment. Consider e.g. V_{i+2}. Unfortunately we have no other vertex in R_k^* that is adjacent to $i+1$ and $i+3$. Notice though, that there are more than $k/3$ vertices x in R_k that are adjacent to $i+1$ and $i+3$ because $\delta(R_k)>2k/3$. All of them can be used to move vertices from V_x to V_{i+2}. In particular, we can find such an x that is not of colour 2 (*method 2*). Moreover, by an easy counting argument, for each $x\in[k]$ we can find a triangle $i'+1,i'+2,i'+3$ in R_k^{**} such that $i'+1,i'+2,i'+3\in N_{R_k}(x)$. This fact can be used for moving vertices out of an arbitrary class V_x into any of the classes $V_{i'+1}$, $V_{i'+2}$, or $V_{i'+3}$ and thus into a class of a different colour (*method 3*).

Combining these ideas, we get the following strategy. As long as $|V_i|<n_i$ ($|V_i|>n_i$), we say that the class V_i is *deficient* (*excessive*). We start by eliminating all deficient classes of colour 2 by repeatedly applying method 2. Then, we take one deficient class V_i and one excessive class V_j at a time. Note, that $\sigma(i)\neq 2$. If $\sigma(i)=\sigma(j)$, we can therefore use method 1 for moving a vertex from V_j to V_i. In the case that $\sigma(i)\neq\sigma(j)$, we first make use of method 3, for moving a vertex from V_j to a class of colour $\sigma(i)$ and then proceed as before. We repeat these steps until no deficient and excessive classes are left. Since $|n_i-m_i|$ is small, not many vertices get moved during this process and so the adjusted vertex partition is still (ε,d)-regular on R_k and (ε,d)-super-regular on R_k^{**} provided that we chose $\varepsilon'\ll\varepsilon$ and $d'\gg d$.

Finally, recall that in the outline above we could 'freely' shift only vertices of classes with colour 1 or 3 (see method 1). Classes of colour 2 needed some special treatment (first addressing all deficiencies with method 2). This is the point where our argument breaks down for the case $r>3$ of the Bollobás–Komlós conjecture. For $r>3$, our approach would, due to [27], yield $R_k^{**} \subseteq R_k^* \subseteq R_k$, with R_k^{**} being k/r disjoint copies of K_r and R_k^* being the $(r-1)$-st power of a Hamiltonian cycle. However, we would only be able to 'freely' move vertices from classes of colour 1 and colour r. In particular, we would not be able to first address all deficiencies of classes with colours $2, \ldots, r-1$ by only using vertices from classes with colour 1 or r.

5 Lemma for H

In this section we sketch the proof of Lemma 2.3. Recall that for this lemma we are given graphs H and $R_k^{**} \subseteq R_k^* \subseteq R_k$, and an equitriangular integer partition $m_1 + \cdots + m_k = n$. The task now is to determine a small set X and a mapping f that sends roughly m_i vertices of H to vertex i of R_k^*. At the same time, f needs to make sure that every edge $\{u, v\}$ of H gets mapped to an edge $\{f(u), f(v)\} \in E(R_k^{**})$, unless u or v lie in X, in which case we still need to guarantee that $\{f(u), f(v)\} \in E(R_k)$.

Suppose that the vertices of H are labelled by numbers $1, \ldots, n$, such that the limited bandwidth guarantees that every edge $\{u, v\}$ satisfies $|u - v| \leq \beta n$. In a first step we follow this ordering and cut H into *segments* S_j of size $m_{3j+1} + m_{3j+2} + m_{3j+3}$, where $j = 0, \ldots, k/3 - 1$.

The idea is to map almost all vertices of S_j to the triangle $\{3j+1, 3j+2, 3j+3\}$ in R_k^{**}. Since H is 3-colourable, it seems tempting to try to map all vertices in S_j of colour $c \in [3]$ to vertex $3j + c$, thereby guaranteeing that all edges of $H[S_j]$ will be mapped to the respective edges of the triangle in R_k^{**}. However, the problem is that the m_i are equitriangular, i.e. almost identical, but the three colour classes of $H[S_j]$ may vary in size. Hence we will have to re-balance the colouring in the following way.

Take an arbitrary vertex $s \in H$ and two colours $l, l' \in [3]$. It is not difficult to check that switching the colours l and l' for all vertices $v > s$ and assigning the new colour 0 to all originally l-coloured vertices in the interval $[s - \beta n, s + \beta n]$ yields a new proper 4-colouring. By repeating this colour–switch after (roughly) every ξn vertices of H, each time with appropriate colours l, l', we can obtain a proper colouring σ of $H[S_j]$ with colour classes 1, 2 and 3 of almost equal size (up to roughly ξn), and with only a few occurrences of colour 0, concentrated around the pivotal vertices s. Denote the set of vertices with colour 0 in S_j by X_j.

The next thing we need to take care of are edges between S_j and S_{j+1}. Let L_j be the last βn vertices from S_j and F_{j+1} the first βn vertices from S_{j+1}. With a little bit of care in choosing the pivotal vertices we can make sure that the F_j, X_j, and L_j are pairwise disjoint, have no edges between each other, and $[s - \beta n, s + \beta n]$ lie well in S_j. Set

$$X := \bigcup_{j=0}^{k/3-1} F_j \dot\cup X_j \dot\cup L_j.$$

Now for all $j = 0, \ldots, k/3 - 1$, we map all vertices in $H[S_j \setminus X]$ of colour $c \in [3]$ to vertex $3j + c$ in R_k^{**}, as originally planned, thus satisfying claim (d) in the Lemma. This defines f on $V_H \setminus X$. For the other vertices, we need to extend f such that claim (c) is fulfilled.

We first consider the vertices in the sets X_j. Due to the bandwidth constraint, these vertices have no neighbours outside S_j. Since X_j forms an independent set in H (the vertices were all of colour 0 before) and has no edges to vertices in L_j or F_j, we therefore have $N_H(X_j) \subseteq S_j \setminus X$, hence $f(N_H(X_j)) \subseteq \{3j+1, 3j+2, 3j+3\}$. Thus it suffices to find a vertex r_j in R_k with $\{3j+1, 3j+2, 3j+3\} \subseteq N_{R_k}(r_j)$ and set $f(v) = r_j$ for all $v \in X_j$. Such a vertex r_j clearly exists because $\delta(R_k) > 2k/3$.

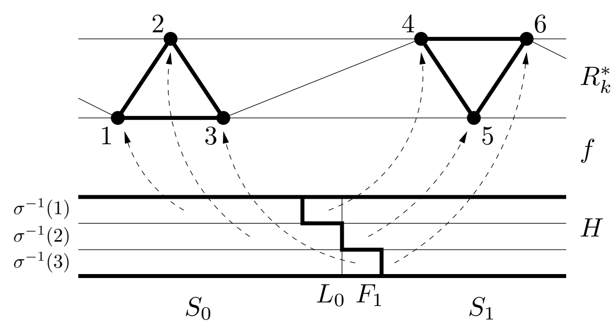

Figure 1: The mapping f from H to R_k^*.

Finally we deal with the vertices in the sets F_j and L_j. For the edges between these sets we make use of the special structure of R_k^*. We define f as follows. For all $v \in L_j$ set

$$f(v) = \begin{cases} 3(j+1)+1 & \text{if } \sigma(v) = 1, \\ 3j + \sigma(v) & \text{if } \sigma(v) \in \{2, 3\}, \end{cases}$$

and for $v \in F_{j+1}$ set

$$f(v) = \begin{cases} 3(j+1) + \sigma(v) & \text{if } \sigma(v) \in \{1, 2\}, \\ 3j + 3 & \text{if } \sigma(v) = 3\,. \end{cases}$$

It is easy to check that the mapping f defined in this way has all the required properties.

References

[1] S. Abbasi, *Embedding low bandwidth bipartite graphs*, unpublished. 1

[2] ———, *How tight is the Bollobás-Komlós conjecture?*, Graphs Combin. **16** (2000), no. 2, 129–137. 1

[3] M. Aigner and S. Brandt, *Embedding arbitrary graphs of maximum degree two*, J. London Math. Soc. (2) **48** (1993), no. 1, 39–51. 1

[4] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster, *The algorithmic aspects of the regularity lemma*, J. Algorithms **16** (1994), no. 1, 80–109. 3

[5] N. Alon and E. Fischer, *2-factors in dense graphs*, Discrete Math. **152** (1996), no. 1–3, 13–23. 1

[6] N. Alon and R. Yuster, *Almost H-factors in dense graphs*, Graphs Combin. **8** (1992), no. 2, 95–102. 1

[7] ———, *H-factors in dense graphs*, J. Combin. Theory Ser. B **66** (1996), no. 2, 269–282. 1

[8] V. Chvátal, V. Rödl, E. Szemerédi, and W. T. Trotter, Jr., *The Ramsey number of a graph with bounded maximum degree*, J. Combin. Theory Ser. B **34** (1983), no. 3, 239–243. 2, 3

[9] K. Corradi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta Math. Acad. Sci. Hungar. **14** (1963), 423–439. 1

[10] B. Csaba, A. Shokoufandeh, and E. Szemerédi, *Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three*, Combinatorica **23** (2003), no. 1, 35–72, Paul Erdős and his mathematics (Budapest, 1999). 1

[11] A. Czygrinow and H. A. Kierstead, *2-factors in dense bipartite graphs*, Discrete Math. **257** (2002), no. 2–3, 357–369, Kleitman and combinatorics: a celebration (Cambridge, MA, 1999). 1

[12] R. Diestel, *Graph theory*, third ed., Graduate Texts in Mathematics, vol. 173, Springer-Verlag, Berlin, 2005. 2

[13] G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3) **2** (1952), 69–81. 1

[14] P. Erdős and A. H. Stone, *On the structure of linear graphs*, Bull. Amer. Math. Soc. **52** (1946), 1087–1091. 1

[15] P. Erdős, *Problem 9*, Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963) (M. Fiedler, ed.), Publ. House Czechoslovak Acad. Sci., Prague, 1964, p. 159. 1

[16] P. Erdős and M. Simonovits, *A limit theorem in graph theory*, Studia Sci. Math. Hungar **1** (1966), 51–57. 1

[17] G. Fan and H. A. Kierstead, *The square of paths and cycles*, J. Combin. Theory Ser. B **63** (1995), no. 1, 55–64. 1, 1.1

[18] ———, *Hamiltonian square-paths*, J. Combin. Theory Ser. B **67** (1996), no. 2, 167–182. 1, 1

[19] A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623. 1

[20] H. Hàn, *Bipartite Graphen mit kleiner Bandbreite in dichten Graphen*, Master's thesis, Humboldt-Universität zu Berlin, Institut für Informatik, Summer 2006. 1

[21] J. Komlós, *The blow-up lemma*, Combin. Probab. Comput. **8** (1999), no. 1–2, 161–176, Recent trends in combinatorics (Mátraháza, 1995). 1, 1

[22] ———, *Tiling Turán theorems*, Combinatorica **20** (2000), no. 2, 203–218. 1

[23] J. Komlós, G. N. Sárközy, and E. Szemerédi, *On the square of a Hamiltonian cycle in dense graphs*, Random Structures Algorithms **9** (1996), no. 1–2, 193–211. 1

[24] ———, *Blow-up lemma*, Combinatorica **17** (1997), no. 1, 109–123. 1, 2, 2.1

[25] ———, *An algorithmic version of the blow-up lemma*, Random Structures Algorithms **12** (1998), no. 3, 297–312. 3

[26] ———, *On the Pósa-Seymour conjecture*, J. Graph Theory **29** (1998), no. 3, 167–176. 1, 1.1, 1

[27] ———, *Proof of the Seymour conjecture for large graphs*, Ann. Comb. **2** (1998), no. 1, 43–60. 1, 4

[28] ———, *Proof of the Alon-Yuster conjecture*, Discrete Math. **235** (2001), no. 1–3, 255–269. 1

[29] ———, *Spanning trees in dense graphs*, Combin. Probab. Comput. **10** (2001), no. 5, 397–416. 1, 2

[30] D. Kühn and D. Osthus, *Spanning triangulations in graphs*, J. Graph Theory **49** (2005), no. 3, 205–233. 1

[31] ———, *Critical chromatic number and the complexity of perfect packings in graphs*, Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2006, Miami, Florida, USA, January 22–26, 2006, ACM Press, 2006, pp. 851–859. 1

[32] D. Kühn, D. Osthus, and A. Taraz, *Large planar subgraphs in dense graphs*, J. Combin. Theory Ser. B **95** (2005), no. 2, 263–282. 1, 2

[33] V. Rödl and A. Ruciński, *Perfect matchings in ϵ-regular graphs and the blow-up lemma*, Combinatorica **19** (1999), no. 3, 437–452. 2

[34] V. Rödl, A. Ruciński, and M. Wagner, *Matchings meeting quotas and their impact on the blow-up lemma*, SIAM J. Comput. **31** (2001), no. 2, 428–446. 3

[35] P. Seymour, *Problem section*, Combinatorics (Proc. British Combinatorial Conf. 1973) (T. P. McDonough and V. C. Mavron, eds.), Cambridge Univ. Press, London, 1974, pp. 201–204. London Math. Soc. Lecture Note Ser., No. 13. 1

[36] A. Shokoufandeh and Y. Zhao, *Proof of a tiling conjecture of Komlós*, Random Structures Algorithms **23** (2003), no. 2, 180–205. 1

[37] E. Szemerédi, *Regular partitions of graphs*, Problèmes combinatoires et théorie des graphes (Orsay, 1976), Colloques Internationaux CNRS, vol. 260, CNRS, 1978, pp. 399–401. 1, 2, 2

[38] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452. 1