Embedding spanning subgraphs of small bandwidth

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Abstract

In this paper we prove the following conjecture by Bollobás and Komlós: *For every $\gamma > 0$ and positive integers r and Δ, there exists $\beta > 0$ with the following property. If G is a sufficiently large graph with n vertices and minimum degree at least $(\frac{r-1}{r}+\gamma)n$ and H is an r-chromatic graph with n vertices, bandwidth at most βn and maximum degree at most Δ, then G contains a copy of H.*

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1 Introduction and results

One of the fundamental results in extremal graph theory is the theorem by Erdős and Stone [6] which implies that any *fixed* graph H of chromatic number r is forced to appear as a subgraph in any sufficiently large graph G if the average degree of G is at least $(\frac{r-2}{r-1}+\gamma)n$, for an arbitrarily small positive constant γ.

In this extended abstract we prove a similar result for *spanning* subgraphs H of small bandwidth that was conjectured by Bollobás and Komlós. It is obvious that for a spanning graph H, it no longer suffices to guarantee that G has a large *average* degree, since we need (to be able to control) every single vertex of G, and thus we shift our emphasis to a large *minimum* degree instead. Also, it is clear that in this regime the lower bound has to be raised at least to $\delta(G) \geq \frac{r-1}{r}n$: simply consider the example where G is the complete r-partite graph with partition classes almost, but not exactly, of the same size (thus G has minimum degree almost $\frac{r-1}{r}n$) and let H be the spanning union of vertex disjoint r-cliques.

There are a number of results where a minimum degree of $\frac{r-1}{r}n$ is indeed sufficient to guarantee the existence of a certain spanning subgraph H. A well known example is Dirac's theorem [5]. It asserts that any graph G on n vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle. Another classical result of that type by Corrádi and Hajnal [4] states that every graph G with n vertices and $\delta(G) \geq 2n/3$ contains $\lfloor n/3 \rfloor$ vertex disjoint triangles. This was generalised by Hajnal and Szemerédi [7], who proved that every graph G with $\delta(G) \geq \frac{r-1}{r}n$ must contain a family of $\lfloor n/r \rfloor$ vertex disjoint cliques, each of size r.

Pósa and Seymour [14] suggested a further extension of this theorem. They conjectured that, at the same threshold $\delta(G) \geq \frac{r-1}{r}n$, such a graph G must in fact contain a copy of the $(r-1)$-st power of a Hamiltonian cycle (where the $(r-1)$-st power of an arbitrary graph is obtained by inserting an edge between every two vertices of distance at most $r-1$ in the original graph). This was proven in 1998 by Komlós, Sárközy, and Szemerédi [12] for sufficiently large n.

Recently, several other results of a similar flavour have been obtained which deal with a variety of spanning subgraphs H, such as, e.g., trees, F-factors, and planar graphs (see the survey [13] and the references therein). In an attempt to move away from results that concern only graphs H with a special, rigid structure, Bollobás and Komlós [9, Conjecture 16] conjectured that every r-chromatic graph on n vertices of bounded degree and bandwidth at most $o(n)$, can be embedded into any graph G on n vertices with $\delta(G) \geq (\frac{r-1}{r}+\gamma)n$.

(A graph is said to have bandwidth at most b, if there exists a labelling of the vertices by numbers $1, \ldots, n$, such that for every edge $\{i, j\}$ of the graph we have $|i - j| \leq b$.) In this extended abstract we present a proof of this conjecture.

Theorem 1.1 *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds.*

If H is an r-chromatic graph on n vertices with $\Delta(H) \leq \Delta$ and bandwidth at most βn and if G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.

The analogue of Theorem 1.1 for bipartite graphs H was announced by Abbasi [1] in 1998, and a proof based on our methods can be found in [8]. In [3] we proved the 3-chromatic case of this theorem. One central ingredient to the proof was the existence of the square of a Hamiltonian cycle in graphs of high minimum degree as asserted by the affirmative solution of the conjecture of Pósa mentioned above. However, it turned out that the $(r-1)$-st power of a Hamiltonian cycle is not well connected enough to carry over these methods to the r-chromatic case.

The following simple example shows that the statement of Theorem 1.1 becomes false when the bandwidth condition on H is dropped. Let H be a random bipartite graph with bounded maximum degree and partition classes of size $n/2$ each, and let G be the graph formed by two cliques of size $(1/2+\gamma)n$ each, which share exactly $2\gamma n$ vertices. It is then easy to see that G cannot contain a copy of H, since in H every set of vertices of size $(1/2 - \gamma)n$ has more than $2\gamma n$ external neighbours.

Also, the γ term in the minimum degree condition on G is necessary in the following sense: Abbasi [2] showed that if $\gamma \to 0$ and $\Delta \to \infty$ then β must tend to 0 in Theorem 1.1. However, the bound on β coming from our proof is rather poor, having a tower-type dependence on $1/\gamma$.

Let us finally address the rôle of the chromatic number of H in Theorem 1.1. In the same way that the Hamiltonian cycle on an odd number of vertices is forced as a spanning subgraph in any graph of minimum degree $n/2$ (although it is 3- and not 2-chromatic), other $(r+1)$-chromatic graphs are forced already when $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$. As already observed by Komlós in [10], it seems that the crucial question here is whether all $r + 1$ colours are needed by *many* vertices.

The following extension of Theorem 1.1 tries to go into a somewhat similar direction. Assume that the vertices of H are labelled $1, \ldots, n$. For two positive integers x, y, a proper $(r+1)$-colouring $\sigma : V(H) \to \{0, \ldots, r\}$ of H is said to be (x, y)-*zero free* with respect to such a labelling, if for each $t \in [n]$ there exists a t' with $t \leq t' \leq t + x$ such that $\sigma(u) \neq 0$ for all $u \in [t', t' + y]$.

Theorem 1.2 *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds.*

Let H be a graph with $\Delta(H) \leq \Delta$ whose vertices are labelled $1, \dots, n$ such that, with respect to this labelling, H has bandwidth at most βn, an $(r+1)$-colouring that is $(8r\beta n, 4r\beta n)$-zero free, and uses colour 0 for at most βn vertices in total.

If G is a graph on n vertices with minimum degree $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.

We conclude with the remark that our proof is constructive and yields a polynomial time algorithm, which finds an embedding of H in G if H is given along with a valid r-colouring (respectively, $(r+1)$-colouring) and a labelling of the vertices respecting the bandwidth bound βn.

2 Outline of the proof

Roughly speaking, the proof of Theorem 1.2 is split into two main lemmas. While they deal exclusively with the graph G and the graph H respectively, they are linked to each other in the following way: the lemma for G suggests a partition of G and communicates the structure of this partition (but not the graph G) to the lemma for H . The lemma for H then tries to find a partition of H with a very similar structure, and returns the sizes of the partition classes to the lemma for G. The latter then adjusts its partition classes by shifting a few vertices of G, until they fit exactly the class sizes of H.

The initial partition constructed by the lemma for G is obtained using the regularity lemma of Szemerédi [15]. This lemma guarantees that the vertex set of every graph G can be partitioned in such a way that most of its edges belong to sufficiently "random–like" induced bipartite graphs (so–called ε-regular pairs).

Once compatible partitions of G and H have been found via the lemma for G and the lemma for H, respectively, we find an embedding of H in G with the help of the blow-up lemma of Komlós, Sárközy, and Szemerédi [11]. This lemma asserts that r-chromatic spanning graphs of bounded degree can be embedded into the union of r classes that form $\binom{r}{2}$ ε-regular pairs with minimum degree dn for some small constant d.

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