Discrepancy and Eigenvalues of Cayley Graphs (extended abstract)

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Abstract

We consider *quasirandom* properties for *Cayley graphs* of finite abelian groups. In particular, we show that having uniform edge-distribution (i.e., small discrepancy) and having large eigenvalue gap are equivalent properties for Cayley graphs, even if they are *sparse*. This positively answers a question of Chung and Graham ["Sparse quasi-random graphs", Combinatorica **22** (2002), no. 2, 217–244] for the particular case of Cayley graphs, while in general the answer is negative.

1 Introduction

Our aim here is to investigate certain aspects of a well known connection between the eigenvalue gap property and quasirandomness of graphs.

Let an n-vertex graph G be given. Recall that the eigenvalues of G are simply the eigenvalues of the n by n, 0–1 adjacency matrix of G, with 1 indicating edges. As usual, let $\lambda_k = \lambda_k(G)$ be the kth largest eigenvalue of G, in absolute value. Recall that G is said to be "quasirandom" if the edges of G are "uniformly distributed" (we postpone the precise definition, see Definition 1).

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Thanks to the work of Tanner [14], Alon and Milman [3] and Alon [1] (see also Alon and Spencer [4, Chapter 9]) it is well known that a *gap* between the largest and the second largest eigenvalue of a graph G is related to the quasirandomness of G. Here, the concept of "quasirandomness" will be that of Chung, Graham, and Wilson [7].

Recall that [7] presents a "theory of quasirandomness" for *dense* graphs, exhibiting several, quite disparate almost sure properties of random graphs that are, quite surprisingly, equivalent in a deterministic sense. (Earlier work in this direction is due to Thomason [15] (see also [16]), and several other authors [1, 2, 8, 13].) One of the so-called "quasirandom properties" that is presented in [7] is the eigenvalue gap between λ_1 and λ_k ($k \geq 2$).

More recently, Chung and Graham [6] set out to investigate the extension of the results in [7] to *sparse graphs*, that is, graphs with vanishing edge-density. As it turns out, a naïve approach to such a project fails, as the results in [7] *do not* fully generalise to the "sparse case" in the expected manner (for a thorough discussion on this point, see [6] and also to [9, 11]).

On the other hand, some positive results have been established. In particular it was shown in [6] that a large eigenvalue gap implies uniform edge-distribution. Chung and Graham asked whether the converse also holds (see [6, p. 230]). An affirmative answer to this question would fully generalise the relationship between these two concepts to the sparse case.

However, Krivelevich and Sudakov [12] discovered that, unfortunately, the answer to the question posed by Chung and Graham is negative, by constructing a suitable family of counterexamples. Here, our aim is to show that *the answer is positive if one considers Cayley graphs of finite abelian groups, regardless of the density of the graph.* We leave the non-abelian case as an open question. It is worth noting that several explicit constructions of quasirandom graphs are indeed Cayley graphs (see, e.g., [16] and [12, Section 3]).

Before we proceed to state our result precisely, we mention that our proof method also sheds some light on the investigation of quasirandom subsets of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, in the spirit of Chung and Graham [5], in the *sparse case* (and for general abelian groups, as suggested in [5, p. 85]). We shall come back to this topic in the near future.

2 Statement of the main result

We use the following notation. If $G = (V, E)$ is a graph, we write $e(G)$ for the number of edges $|E|$ in G. If $U \subset V$ is a set of vertices of G, then $G[U]$ denotes the subgraph of G induced by U. Furthermore, if $W \subset V$ is disjoint from U, then we write $G[U, W]$ for the bipartite subgraph of G naturally induced by the pair (U, W). We also sometimes write $E(U, W) = E_G(U, W)$ for the edge set of $G[U, W]$. If $\delta > 0$, we write $x \sim_\delta y$ to mean that

$$(1 - \delta)y \leq x \leq (1 + \delta)y.$$

Definition 1 (DISC(δ))**.** *Let $0 < \delta \leq 1$ be given. We say that an n-vertex graph G ($n \geq 2$) satisfies property* DISC(δ) *if the following assertion holds: for all $U \subset V(G)$ with $|U| \geq \delta n$, we have*

$$e(G[U]) \sim_\delta e(G)\binom{|U|}{2} \Big/ \binom{n}{2}.$$

Given a graph G, let $\mathbf{A} = \mathbf{A}(G) = (a_{vv'})_{v,v' \in V(G)}$ be the 0–1 adjacency matrix of G, with 1 denoting edges. The *eigenvalues* of G are simply the eigenvalues of $\mathbf{A}$. Since $\mathbf{A}$ is symmetric, its eigenvalues are real. As usual, we adjust the notation so that these eigenvalues are such that

$$\lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|. \tag{1}$$

Definition 2 (EIG(ε))**.** *Let $0 < \varepsilon \leq 1$ be given. We say that an n-vertex graph G satisfies property* EIG(ε) *if the following holds. Let $\bar{d} = \bar{d}(G) = 2e(G)/n$ be the average degree of G, and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G, with the notation adjusted in such a way that* (1) *holds. Then*

(i) $\lambda_1 \sim_\varepsilon \bar{d}$,

(ii) $|\lambda_i| \leq \varepsilon \bar{d}$ *for all* $1 < i \leq n$.

Finally, we define Cayley graphs.

Definition 3 (Cayley graph $G = G(\Gamma, A)$)**.** *Let Γ be an abelian group, and suppose $A \subset \Gamma \setminus \{0\}$ is symmetric, that is, $A = -A$. The Cayley graph $G = G(\Gamma, A)$ is defined to be the graph on Γ, with two vertices γ and $\gamma' \in \Gamma$ adjacent in G if and only if $\gamma' - \gamma \in A$.*

We only consider finite graphs and finite abelian groups.

The following theorem answers a question of Chung and Graham from [6] in the positive for the interesting class of Cayley graphs.

Theorem 4. *For any $\varepsilon > 0$, there are constants $\delta > 0$ and $n_0 \geq 1$ for which the following holds. If $G = G(\Gamma, A)$ is a Cayley graph for some abelian group Γ and symmetric set $A = -A \subseteq \Gamma \setminus \{0\}$, the number of vertices $n = |\Gamma|$ of G satisfies $n \geq n_0$, and G satisfies property* DISC(δ)*, then G satisfies* EIG(ε).

The somewhat technical proof of Theorem 4 is given in the full version of this paper [10].

3 Remarks on the result

Let us discuss some points concerning Theorem 4 (for the proofs see [10]). We first observe that Theorem 4 together with the results of Chung and Graham [6] imply that properties DISC and EIG are equivalent for Cayley graphs. We say that DISC implies EIG for Cayley graphs if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that, for any sequence of n-vertex d_n-regular Cayley graphs G_n with d_n tending to infinity as $n \to \infty$, the following holds: *if all but finitely many graphs G_n satisfy* DISC(δ)*, then all but finitely many G_n satisfy* EIG(ε). Theorem 4 tells us that DISC implies EIG for sequences of Cayley graphs. In [6] it is proved that EIG implies DISC in the same sense for sequences of arbitrary graphs with average degree tending to infinity.

Secondly, we note that in general it is not true that DISC implies EIG for arbitrary sequences of graphs. This was already pointed out by Krivelevich and Sudakov in [12]. For every $\varepsilon > 0$ and every $\delta > 0$, they constructed an infinite sequence of graphs that satisfy DISC(δ) but fail to satisfy (i) in the definition of EIG(ε) (see Definition 2). A different construction to be presented in the full version of this paper gives additional control over a constant number of the largest eigenvalues.

At last, we wish to compare our main result, Theorem 4, with the earlier work of Chung and Graham [6]. Let us consider the following property.

Definition 5 (CIRCUIT$_t(\xi)$)**.** *Let $0 < \xi \leq 1$ and an integer $t \geq 3$ be given. We say that an n-vertex graph G with average degree $\bar{d}(G)$ satisfies property* CIRCUIT$_t(\xi)$ *if the number of t-circuits C_t^* in G, i.e., closed walks of length t, satisfies*

$$\#\{C_t^* \hookrightarrow G\} \sim_\xi \bar{d}(G)^t \,.$$

Similarly to our Theorem 4, Theorem 6 in [6] establishes the implication DISC $\Rightarrow$ EIG. This implication is proved in [6] for arbitrary graphs under some additional conditions. These additional conditions, combined with DISC, also imply CIRCUIT$_{2\ell}$ for some $\ell > 1$. The following fact shows that Theorem 6 in [6] does not imply our main result, as it says that there are sequences of Cayley graphs that satisfy both DISC and EIG, but fail to meet CIRCUIT$_{2\ell}$ for every $\ell > 1$.

Fact 6. *There is an infinite sequence G_N of N-vertex Cayley graphs ($N \to \infty$) for which the following holds:*

(i) *for every $\delta > 0$ all but finitely many graphs G_N satisfy* DISC(δ)*,*

(ii) *for every $\varepsilon > 0$ all but finitely many graphs G_N satisfy* EIG(ε)*, and*

(iii) *for every integer $\ell > 1$ and every $\xi > 0$ only finitely many graphs G_N satisfy* CIRCUIT$_{2\ell}(\xi)$*.*

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