ON THE STRUCTURE OF DENSE GRAPHS WITH BOUNDED CLIQUE NUMBER

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Abstract. We study structural properties of graphs with bounded clique number and high minimum degree. In particular, we show that there exists a function $L = L(r, \varepsilon)$, such that every K_r-free graph G on n vertices with minimum degree at least $(\frac{2r-5}{2r-3} + \varepsilon)n$ is homomorphic to a K_r-free graph on at most L vertices. It is known that the required minimum degree condition is approximately best possible for this result.

For $r = 3$ this result was obtained by Łuczak [*On the structure of triangle-free graphs of large minimum degree*, Combinatorica **26** (2006), no. 4, 489–493] and, more recently, Goddard and Lyle [*Dense graphs with small clique number*, J. Graph Theory **66** (2011), no. 4, 319-331] deduced the general case from Łuczak's result. Łuczak's proof was based on an application of Szemerédi's regularity lemma and, as a consequence, it only gave rise to a tower-type bound on $L(3, \varepsilon)$. The proof presented here replaces the application of the regularity lemma by a probabilistic argument, which yields a bound for $L(r, \varepsilon)$ that is doubly exponential in poly(ε).

§1. Introduction

1.1. **Chromatic thresholds of graphs.** The graphs we consider here are finite, undirected, simple, and have no loops and for a graph $G = (V, E)$ we denote by $V = V(G)$ its *vertex set* and by $E = E(G) \subseteq V^{(2)} = \{X \subseteq V : |X| = 2\}$ its *edge set*. We are interested in structural properties of large graphs G with large minimum degree that do not contain a fixed graph F as (not necessarily induced) subgraph, i.e., G is F*-free*. Let

$$\mathrm{Forb}(F) = \{G : F \nsubseteq G\}$$

be the class of F-free graphs and for $n \in \mathbb{N}$ we set

$$\mathrm{Forb}_n(F) = \{G \in \mathrm{Forb}(F) : |V(G)| = n\} .$$

For example, if $F = K_r$ is a clique with r vertices and $\delta(G) \geqslant (r-2)\lfloor\frac{|V(G)|}{r-1}\rfloor$, then Turán's theorem [17] implies that G is $(r-1)$-partite and, in particular, the chromatic number of graphs G is bounded by a constant independent of $|V(G)|$. More generally, Andrásfai,

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Erdős, and Sós [2] raised the following question: For a given graph F and an integer k, what is the smallest minimum degree condition such any (large) graph $G \in \mathrm{Forb}(F)$ satisfying this minimum degree condition has chromatic number at most k? Here we are interested in the case when the minimum degree condition yields an upper bound on $\chi(G)$ independent from the graph G itself. This leads to the so called *chromatic threshold* for a given graph F

$$\delta_\chi(F) = \inf\big\{\alpha \in [0,1] \colon \text{there is some } k \in \mathbb{N} \text{ such that} \\ \text{every } G \in \mathrm{Forb}(F) \text{ with } \delta(G) \geqslant \alpha|V(G)| \text{ satisfies } \chi(G) \leqslant k\big\}\,.$$

If $F' \subseteq F$, then $\mathrm{Forb}(F') \subseteq \mathrm{Forb}(F)$, so obviously $\delta_\chi(F') \leqslant \delta_\chi(F)$. Moreover, it follows from the Erdős–Stone theorem [6] that $\delta_\chi(F) \leqslant \frac{\chi(F)-2}{\chi(F)-1}$ for every graph F with at least one edge. For $F = K_3$ it was shown in [5] that $\delta_\chi(K_3) \geqslant 1/3$. In the other direction, Thomassen [15] obtained a matching upper bound and, therefore, $\delta_\chi(K_3) = 1/3$. In fact, Erdős and Simonovits [5] asked whether all triangle-free graphs G with $\delta(G) \geqslant (1/3 + o(1))|V(G)|$ are 3-colorable. This was answered negatively by Häggkvist [8], but recently Brandt and Thomassé [3] showed that the chromatic number of such graphs is bounded by four. Goddard and Lyle [7] and Nikiforov [13] extended these results from the triangle to arbitrary cliques. They showed

$$\delta_\chi(K_r) = \frac{2r-5}{2r-3} \tag{1.1}$$

for every $r \geqslant 3$ that and $\chi(G) \leqslant r+1$ for every K_r-free graph G with $\delta(G) > \frac{2r-5}{2r-3}|V(G)|$.

In the case when $F = C_{2r+1}$ is an odd cycle of length at least five it was shown by Thomassen [16] that the chromatic threshold is zero and Łuczak and Thomassé [12] proved that

$$\delta_\chi(F) \notin (0, 1/3)$$

for all graphs F and that $\delta_\chi(F) = 0$ if F is nearly bipartite (a graph is nearly bipartite if it is triangle-free and it admits a vertex partition into two parts such that one part is independent and the other part induces a graph with maximum degree one). Recently, Allen et al. [1] extended the work of Łuczak and Thomassé and determined the chromatic threshold for every graph F.

1.2. **Homomorphism thresholds of graphs.** Viewing $\chi(G) \leqslant r$ as the property that G admits a (graph) homomorphism into K_r, which we denote by $G \xrightarrow{\text{hom}} K_r$, one may ask for a graph $G \in \mathrm{Forb}(F)$, whether K_k can be replaced by a graph H of bounded size (independent of G), that is F-free itself. More precisely, in [15] Thomassen posed the following question: For which constant α, does there exist a finite family of triangle-free graphs such that every triangle-free graph on n vertices with minimum degree greater than αn is homomorphic to some graph of this family? To formalise this question we define the *homomorphism threshold* of a graph F

$$\delta_{\hom}(F) = \inf\big\{\alpha \in [0,1] \colon \text{there is some } k \in \mathbb{N} \text{ and a graph } H \in \mathrm{Forb}_k(F) \text{ such that every } G \in \mathrm{Forb}(F) \text{ with } \delta(G) \geqslant \alpha|V(G)| \text{ satisfies } G \xrightarrow{\hom} H\big\}\,.$$

Thomassen then asked to determine $\delta_{\hom}(K_3)$. Since $G \xrightarrow{\hom} H$ implies $\chi(G) \leqslant |V(H)|$, we clearly have

$$\delta_{\hom}(F) \geqslant \delta_{\chi}(F).$$

Łuczak [10] proved $\delta_{\hom}(K_3) = 1/3$ and, hence, for the triangle K_3 the homomorphism threshold and the chromatic threshold are equal. The work of Goddard and Lyle [7] generalised Łuczak's result and showed

$$\delta_{\hom}(K_r) = \delta_{\chi}(K_r) = \frac{2r-5}{2r-3} \tag{1.2}$$

for every $r \geqslant 3$. Łuczak's proof in [10] and the generalisation of Goddard and Lyle [7], which uses Łuczak's result as a base case in an inductive proof, utilise Szemerédi's regularity lemma [14] and lead to a tower-type bound on the size of the K_r-free homomorphic image H in terms of r and the given minimum degree density $\delta(G)/|V(G)|$. We give a different proof of the upper bound of (1.2) based on a simple probabilistic argument, which avoids the regularity lemma and yields a doubly exponential dependency.

Theorem 1.1. *For every integer $r \geqslant 3$ and every $\varepsilon > 0$ there exits some $L = 2^{2^{\mathrm{poly}(r,1/\varepsilon)}}$ such that for every K_r-free graph G with*

$$\delta(G) \geqslant \Big(\frac{2r-5}{2r-3} + \varepsilon\Big)|V(G)|$$

there exists a K_r-free graph H on at most L vertices with $G \xrightarrow{\hom} H$.

It seems an interesting open question to determine the homomorphism threshold for other graphs than cliques. In particular, the case of odd cycles of length at least five seems to be a first interesting open case. Very recently, in [4, 11] an upper bound of the form

$$\delta_{\hom}(C_{2k+1}) \leqslant \frac{1}{2k+1}$$

was obtained and it would be interesting to establish a matching lower bound.

Question 1. Is $\delta_{\hom}(C_{2k+1}) = \frac{1}{2k+1}$ for $k \geqslant 2$?

A somewhat related question concerns the homomorphism threshold for forbidden families of graphs. Note that the definitions of $\mathrm{Forb}(F)$ and $\delta_{\hom}(F)$ straightforwardly extend from one forbidden graph F to forbidden families $\mathcal{F}$ of graphs. In view of Question 1 it is natural to consider the family $\mathcal{C}_{2k+1} = \{C_3, \dots, C_{2k+1}\}$ of odd cycles of length at most $2k+1$. We remark that for these families the homomorphism threshold was obtained and in [4,11] it was shown that $\delta_{\hom}(\mathcal{C}_{2k+1}) = \frac{1}{2k+1}$.

Organisation. In the next section we review a few useful facts for the proof of the main theorem. The proof of Theorem 1.1 is presented in Section 3.

§2. Simple observations

For an integer $r \geqslant 3$ and $\varepsilon > 0$ the following subclass of $\mathrm{Forb}(K_r)$ will play a prominent rôle

$$\mathcal{F}(r,\varepsilon) = \left\{G \in \mathrm{Forb}(K_r)\colon \delta(G) \geqslant \left(\tfrac{2r-5}{2r-3} + \varepsilon\right)|V(G)|\right\},$$

and Theorem 1.1 asserts that there exists some function $L = L(r,\varepsilon)$ and $H \in \mathrm{Forb}_L(K_r)$ such that for every $G \in \mathcal{F}(r,\varepsilon)$ we have $G \xrightarrow{\hom} H$. For a subset $U \subseteq V$ we define the *common (or joint) neighbourhood* of U as

$$N(U) = \bigcap_{u \in U} N(u)\,.$$

For later reference we note that the size of $N(U)$ can be easily bounded from below in terms of the minimum degree of $G = (V,E)$ by

$$|N(U)| \geqslant |U| \cdot \delta(G) - (|U| - 1) \cdot |V|\,. \tag{2.1}$$

We begin with a few observations concerning common neighbourhoods in *maximal K_r-free graphs*, i.e., K_r-free graphs $G = (V,E)$ with the property that $(V, E \cup \{xy\})$ contains a copy of K_r for every $xy \in V^{(2)} \smallsetminus E$.

Proposition 2.1. *For $r \geqslant 3$ let $G = (V,E)$ be a maximal K_r-free graph. If two distinct vertices $u, v \in V$ are non-adjacent, then $|N(u) \cap N(v)| \geqslant r\delta(G) - (r-2)|V|$.*

Proof. Since $G = (V,E)$ is maximal K_r-free and $uv \notin E$, the common neighbourhood $N(u) \cap N(v)$ induces a K_{r-2}. Applying (2.1) to the $r-2$ vertices $w_1, \dots, w_{r-2}$ that span K_{r-2} in the joint neighbourhood of u and v yields $N(\{w_1, \dots, w_{r-2}\}) \geqslant (r-2)\delta(G) - (r-3)|V|$. Moreover, since $N(\{w_1, \dots, w_{r-2}\})$ must be disjoint from $N(u) \cup N(v)$, we obtain

$$\begin{aligned}|V| &\geqslant (r-2)\delta(G) - (r-3)|V| + |N(u) \cup N(v)| \\ &= (r-2)\delta(G) - (r-3)|V| + |N(u)| + |N(v)| - |N(u) \cap N(v)|\end{aligned}$$

and the proposition follows. $\square$

In the proof of the last proposition we used the observation that the neighbourhood of any two non-adjacent vertices in a maximal K_r-free graph induces a K_{r-2}. Next we note that for maximal K_r-free graphs in $\mathcal{F}(r,\varepsilon)$, we can strengthen this observation and ensure that the clique K_{r-2} is disjoint from an arbitrary given small set of vertices.

Proposition 2.2. *For $r \geqslant 3$ and $\varepsilon > 0$, let $G = (V, E)$ be a maximal K_r-free graph from $\mathcal{F}(r,\varepsilon)$. If two distinct vertices u, $v \in V$ are non-adjacent in G and $U \subseteq V$ satisfies $|U| < \varepsilon|V|$, then $K_{r-2} \subseteq G[(N(u) \cap N(v)) \setminus U]$.*

Proof. Given u, v and U as stated, we first consider any set of $r-3$ vertices $w_1, \dots, w_{r-3} \in V$ and owing to (2.1) we have

$$N(\{w_1, \dots, w_{r-3}\}) \geqslant (r-3)\delta(G) - (r-4)|V|\,.$$

Moreover, since u and v are non-adjacent Proposition 2.1 tells us that

$$|N(u) \cap N(v)| \geqslant r\delta(G) - (r-2)|V|\,.$$

Consequently, the joint neighbourhood of u, v and $w_1, \dots, w_{r-3}$ satisfies

$$\begin{aligned}|N(\{u, v, w_1, \dots, w_{r-3})| &\geqslant N(\{w_1, \dots, w_{r-3}\}) - (|V| - |N(u) \cap N(v)|)\\ &\geqslant (2r-3)\delta(G) - (2r-5)|V|\end{aligned}$$

and the minimum degree condition from $G \in \mathcal{F}(r,\varepsilon)$ implies that

$$|N(\{u, v, w_1, \dots, w_{r-3})| \geqslant (2r-3)\varepsilon|V| \geqslant 3\varepsilon|V| > |U|\,.$$

Summarising, we have shown that any collection of $r-3$ vertices together with u and v have a joint neighbour outside of U. Selecting w_1 from $(N(u) \cap N(v)) \setminus U$ and inductively w_{i+1} from $N(\{u, v, w_1, \dots, w_i) \setminus U$ for $i = 1, \dots, r-3$ yields the desired clique on $w_1, \dots, w_{r-2}$. $\square$

Our final observation asserts that any sufficiently large subset of vertices induces a K_{r-2} in a graph G from $\mathcal{F}(r,\varepsilon)$.

Proposition 2.3. *For $r \geqslant 3$ and $\varepsilon > 0$ let $G = (V, E)$ be a graph from $\mathcal{F}(r,\varepsilon)$. If $Z \subseteq V$ satisfies $|Z| \geqslant (\frac{2r-6}{2r-3} + \varepsilon)|V|$, then $K_{r-2} \subseteq G[Z]$.*

Proof. Similarly as in the proof of Proposition 2.2 we consider an arbitrary set of $(r-3)$ vertices $w_1, \dots, w_{r-3} \in V$ and from (2.1) we infer

$$\begin{aligned}|N(\{w_1, \dots, w_{r-3}\}) \cap Z| &\geqslant (r-3)\delta(G) - (r-4)|V| - (|V| - |Z|)\\ &\geqslant \Big((r-3)\frac{2r-5}{2r-3} + \frac{2r-6}{2r-3} - (r-3)\Big)|V| + (r-2)\varepsilon|V|\\ &= (r-2)\varepsilon|V| > 0\,.\end{aligned}$$

Consequently, any set of $k-3$ vertices has a joint neighbour in Z. Hence, selecting w_1 in Z and inductively w_{i+1} from $N(\{w_1,\dots,w_i\})\cap Z$ for $i=1,\dots,r-3$ yields the desired clique on $w_1,\dots,w_{r-2}$. □

§3. Proof of the main result

In the proof of Theorem 1.1 we partition the vertex set of a maximal K_r-free graph $G\in\mathcal{F}(r,\varepsilon)$ into a bounded number of stable sets, which are the preimages of the desired graph homomorphism. Moreover, we show that any two such independent sets are spanning only complete or empty bipartite graphs between them. Consequently, G is a blow-up of a K_r-free graph of bounded size, which is equivalent to the property that G has a K_r-free homomorphic image of bounded size.

We obtain the independent sets in two steps: Roughly speaking, in the first step we consider a random subset $X\subset V(G)$ of bounded size and partition the vertices of $V(G)$ according to their neighbourhood in X and as it will turn out most (in fact all but one) of these sets will be independent. However, since X has only bounded size, a small (but linear sized) set of vertices may have no or only a few neighbours in X and we deal with those vertices in the second step, by considering the neighbourhood into the independent sets from the first step.

Proof of Theorem 1.1. Let $r\geqslant 3$. Owing to (1.1) we have $\delta_\chi(K_r)=\frac{2r-5}{2r-3}$ and since by definition $\delta_\chi(K_r)\leqslant\delta_{\hom}(K_r)$, we have to prove the matching upper bound on $\delta_{\hom}(K_r)$. Let $\varepsilon>0$ and set

$$m=\max\{r,\,\lceil 4\ln(8/\varepsilon)/\varepsilon^2\rceil+1\}\,,\quad T=2^m\,,\quad\text{and}\quad L=2^T+T\,. \tag{3.1}$$

We will show that for any $n>L$ and for every maximal K_r-free graph $G=(V,E)$ from $\mathcal{F}(r,\varepsilon)$ there exists some $H\in\mathrm{Forb}_L(K_r)$ such that $G\xrightarrow{\hom}H$, which clearly suffices to prove the theorem.

In the first part we consider a random subset $X\subseteq V$ of size m chosen uniformly at random from all m-element subsets of V and we consider the random set

$$U_X=\Big\{v\in V\colon|N(v)\cap X|<\big(\tfrac{2r-5}{2r-3}+\tfrac{\varepsilon}{2}\big)m\Big\}$$

of vertices with "small" degree in X. We show that with positive probability $|U_X|\leqslant\varepsilon n/4$ and $|X\cap U_X|\leqslant\varepsilon m/4$.

It follows from Chernoff's inequality for the hypergeometric distribution (see, e.g., [9, eq. (2.6) and Theorem 2.10]) that for a given vertex $v\in V$ we have

$$\mathbb{P}(v\in U_X)\leqslant\exp(-\varepsilon^2m/4)\,. \tag{3.2}$$

Consequently,

$$\mathbb{E}[|U_X|] \leqslant \exp(-\varepsilon^2 m/4) \cdot n \overset{(3.1)}{<} \varepsilon n/8$$

and by Markov's inequality we have

$$\mathbb{P}(|U_X| \leqslant \varepsilon n/4) > 1/2\,. \tag{3.3}$$

In other words, with probability more than $1/2$ all but at most $\varepsilon n/4$ vertices inherit approximately the minimum degree condition on the randomly chosen set X.

Next we show that with probability at least $1/2$ the intersection of X with U_X is small. This follows from a standard double counting argument. In fact, the same argument giving (3.2) shows that for every $v \in V$ there are at most $\exp(-\varepsilon^2(m-1)/4)\binom{n-1}{m-1}$ different $(m-1)$-element subsets Y of V for which

$$|N(v) \cap Y| \leqslant \left(\tfrac{2r-5}{2r-3} + \tfrac{\varepsilon}{2}\right) \cdot (m-1)\,. \tag{3.4}$$

Hence, there are at most $n\exp(-\varepsilon^2(m-1)/4)\binom{n-1}{m-1}$ pairs (v, Y) such that (3.4) holds. Therefore, there are at most

$$\frac{n \cdot \exp(-\varepsilon^2(m-1)/4)\binom{n-1}{m-1}}{\varepsilon m/4} \overset{(3.1)}{\leqslant} \frac{1}{2}\binom{n}{m}$$

m-element subsets $X \subseteq V$ that contain at least $\varepsilon m/4$ vertices v such that v and $Y = X \smallsetminus \{v\}$ satisfy (3.4). Combining this with (3.3) shows that there exists an m-element set $X \subseteq V$ with the promised properties

$$|U_X| \leqslant \frac{\varepsilon}{4} n \quad \text{and} \quad |X \cap U_X| < \frac{\varepsilon}{4} m\,.$$

Finally, we set

$$Y = X \smallsetminus U_X \quad \text{and} \quad U_Y = \Big\{v \in V : |N(v) \cap Y| < \left(\tfrac{2r-5}{2r-3} + \tfrac{\varepsilon}{4}\right)|Y|\Big\}$$

and we note that the induced subgraph on Y satisfies

$$G[Y] \in \mathcal{F}(r, \varepsilon/4)$$

and since $U_Y \subseteq U_X$ we also have

$$|U_Y| \leqslant |U_X| \leqslant \varepsilon n/4\,.$$

Next we define a vertex partition of $V \smallsetminus U_Y$ given by the neighbourhoods in Y. For that we say two vertices v, $w \in V \smallsetminus U_Y$ are equivalent w.r.t. Y, if they have the same neighbours in Y, i.e., $N(v) \cap Y = N(w) \cap Y$. Let $V_1 \cup \ldots \cup V_t = V \smallsetminus U_Y$ be the corresponding partition given by the equivalence classes and let Y_i be the neighbourhood of the vertices from V_i in Y, i.e., for any $v_i \in V_i$ we have

$$N(v_i) \cap Y = Y_i\,.$$

Clearly, $t \leqslant 2^{|Y|} \leqslant 2^{|X|} = 2^m = T$ and by definition of U_Y we have $|Y_i| \geqslant \big(\frac{2r-5}{2r-3} + \frac{\varepsilon}{4}\big)|Y|$ for every $i \in [t]$.

We observe that the vertex classes $V_1, \dots, V_t$ are independent sets in G, i.e., for every $i = 1, \dots, t$ we have

$$E_G(V_i) = \varnothing\,. \tag{3.5}$$

In fact, since every vertex $v \in V \smallsetminus U_Y$ has at least $(\frac{2r-5}{2r-3} + \varepsilon/4)|Y|$ neighbours in Y and since the induced subgraph $G[Y] \in \mathcal{F}(r, \varepsilon/4)$ it follows from Proposition 2.3 applied to $G[Y]$ and $Z = Y_i$ that Y_i induces a K_{r-2}. Consequently, the K_r-freeness of G implies that no two vertices v_i, $w_i \in V_i$ can be adjacent in G and (3.5) follows.

Next we observe that the induced bipartite graphs given by the partition of equivalence classes contain no or all edges, i.e., for every $1 \leqslant i < j \leqslant t$ we have

$$e_G(V_i, V_j) = 0 \quad \text{or} \quad e_G(V_i, V_j) = |V_i||V_j|\,. \tag{3.6}$$

Suppose for a contradiction that there are (not necessarily distinct) vertices v_i, $w_i \in V_i$ and v_j, $w_j \in V_j$ such that $v_iv_j \in E(V_i, V_j)$ and $w_iw_j \notin E(V_i, V_j)$. Due to the edge v_iv_j the intersection $Y_i \cap Y_j$ must be K_{r-2}-free and, hence, in view of Proposition 2.3 applied to $G[Y]$ and $Z = Y_i \cap Y_j$ we have

$$|Y_i \cap Y_j| < \Big(\frac{2r-6}{2r-3} + \frac{\varepsilon}{4}\Big)|Y|$$

and, therefore,

$$|Y_i \cup Y_j| = |Y_i| + |Y_j| - |Y_i \cap Y_j| > \Big(2\frac{2r-5}{2r-3} - \frac{2r-6}{2r-3} + \frac{\varepsilon}{4}\Big)|Y| = \Big(\frac{2r-4}{2r-3} + \frac{\varepsilon}{4}\Big)|Y|\,. \tag{3.7}$$

Next we use that $w_i \in V_i$ and $w_j \in V_j$ are non-adjacent. Owing to the maximality of G we can apply Proposition 2.2 to G and U_Y and obtain a clique K_{r-2} outside U_Y in the joint neighbourhood of w_i and w_j. Let R be the vertex set of this K_{r-2}. Since $R \subseteq V \smallsetminus U_Y$ and since the sets V_k are independent for every $k = 1, \dots, t$ the set R intersects $r-2$ classes $V_{k_1}, \dots, V_{k_{r-2}}$ different from V_i and V_j. We consider the joint neighbourhood of R in Y

$$N(R) \cap Y = Y_{k_1} \cap \dots \cap Y_{k_{r-2}}$$

and note that

$$|N(R) \cap Y| \overset{(2.1)}{\geqslant} (r-2)\Big(\frac{2r-5}{2r-3} + \frac{\varepsilon}{4}\Big)|Y| - (r-3)|Y| = \Big(\frac{1}{2r-3} + \frac{\varepsilon}{4}\Big)|Y|\,.$$

However, combined with (3.7) this implies that either $Y_i \cap N(R) \neq \varnothing$ or $Y_j \cap N(R) \neq \varnothing$. In either case this gives rise to a K_r in G, which yields the desired contradiction and (3.6) follows.

Note that (3.5) shows that $G[V \smallsetminus U_Y]$ is homomorphic to a graph H' on $t \leqslant T$ and it follows from (3.6) that $G[V \smallsetminus U_Y]$ is a blow-up of H'. So in particular H' is K_r-free.

It remains to deal with the vertices in U_Y. For that we first observe that for every vertex $u \in U_Y$ and $i = 1, \dots, t$ we have

$$N(u) \cap V_i = \varnothing \quad \text{or} \quad N(u) \cap V_i = V_i\,. \tag{3.8}$$

In fact, suppose for a contradiction, that for some v_i, $w_i \in V_i$ we have $uv_i \in E$ while u and w_i are not adjacent. Again the maximality of G and Proposition 2.2 show that $N(u) \cap N(w_i)$ contains a K_{r-2} avoiding U_Y. However, since by (3.5) and (3.6) the vertices v_i and w_i have the same neighbourhood in $V \smallsetminus U_Y$ the same K_{r-2} is also in the neighbourhood of v_i, which together with v_i and u yields a K_r in G. This contradicts $K_r \nsubseteq G$ and (3.8) follows.

Next we partition U_Y according to the neighbourhoods of its vertices in $V \smallsetminus U_Y$. For every $S \subseteq [t] = \{1, \dots, t\}$ we set

$$V_S = \left\{ u \in U_Y \colon N(u) \smallsetminus U_Y = \bigcup_{s \in S} V_s \right\},$$

which yields a partition of U_Y into at most $2^t \leqslant 2^T$ classes. Similar as in (3.6) and (3.8) we next observe that for any S, $S' \subseteq [t]$ with $S \neq S'$ we have

$$e_G(V_S, V_{S'}) = 0 \quad \text{or} \quad e_G(V_S, V_{S'}) = |V_S||V_{S'}|\,. \tag{3.9}$$

The proof is very similar to the proof of (3.8). Suppose for a contradiction without loss of generality there exist vertices v_S, $w_S \in V_S$ and $u \in V_{S'}$ such that $uv_S \in E$ while u and w_S are not adjacent. Then by the maximality of G Proposition 2.2 yields a K_{r-2} in $N(u) \cap N(w_S)$ avoiding U_Y. Owing to (3.8) the vertices v_S and w_S have the same neighbourhood in $V \smallsetminus U_Y$ and, hence, the same K_{r-2} is also in the neighbourhood of v_S, which together with v_S and u yields a K_r in G. This contradicts $K_r \nsubseteq G$ and (3.9) follows.

The last thing we have to show is that V_S is independent in G, i.e., for every $S \subseteq [t]$ we have

$$E_G(V_S) = \varnothing\,. \tag{3.10}$$

This is a direct consequence of (3.8) and Proposition 2.3. In fact, it follows from (3.8) that any two vertices u, $v \in V_S$ have the same neighbourhood in $V \smallsetminus U_Y$. Hence, their joint neighbourhood has size at least $(\frac{2r-5}{2r-3} + \frac{3\varepsilon}{4})n$ and Proposition 2.3 yields a K_{r-2} in the joint neighbourhood of u and v. Therefore, u and v cannot be adjacent in G and (3.10) follows.

Summarising, we have shown that there exists a vertex partition

$$\dot{\bigcup_{i=1}^{t}} V_i \cup \dot{\bigcup_{S \subseteq [t]}} V_S = V$$

of V into independent sets (see (3.5) and (3.10)) such that all naturally induced bipartite graphs are either complete or empty (see (3.6), (3.8), and (3.9)). Hence, there exists a graph H on $2^T + T \leqslant L$ vertices such that G is a blow-up of H and, therefore, $G \xrightarrow{\text{hom}} H$ and H itself must be K_r-free. This concludes the proof of Theorem 1.1. □

We close by noting that the same approach used in the proof of Theorem 1.1 can be used to show Thomassen's result from [16] that the chromatic threshold of odd cycles of at least five is 0. Moreover, an adaptation of this proof also led to an upper bound for the homomorphism threshold for odd cycles in [4].

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