COUNTING RESULTS FOR SPARSE PSEUDORANDOM HYPERGRAPHS I

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Abstract. We establish a so-called counting lemma that allows embeddings of certain linear uniform hypergraphs into sparse pseudorandom hypergraphs, generalizing a result for graphs [Embedding graphs with bounded degree in sparse pseudorandom graphs, Israel J. Math. 139 (2004), 93–137]. Applications of our result are presented in the companion paper [Counting results for sparse pseudorandom hypergraphs II].

§1. Introduction

Many problems in extremal combinatorics concern embeddings of graphs and hypergraphs of fixed isomorphism type into a large host graph/hypergraph. The systematic study of pseudorandom graphs was initiated by Thomason [15, 16] and since then many embedding results have been developed for host pseudorandom graphs. For example, a well-known consequence of the Chung–Graham–Wilson theorem [2] asserts that dense pseudorandom graphs G contain the "right" number of copies of any fixed graph, where "right" means approximately the same number of copies as expected in a random graph with the same density as G. In view of this result, the question arises to which extent it can be generalized to sparse pseudorandom graphs, and results in this direction can be found in [3–5, 12]. We continue this line of research for embedding properties of sparse pseudorandom hypergraphs. Counting lemmas for pseudorandom hypergraphs were also investigated by Conlon, Fox and Zhao [6].

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Let $G = (V, E)$ be a k-uniform hypergraph. For every $1 \leqslant i \leqslant k-1$ and every i-element set $\{x_1, \ldots, x_i\} \in \binom{V}{i}$, let

$$N_G(x_1, \ldots, x_i) = \left\{ \{x_{i+1}, \ldots, x_k\} \in \binom{V}{k-i} : \{x_1, \ldots, x_k\} \in E \right\},$$

i.e., $N_G(x_1, \ldots, x_i)$ is the set of elements of $\binom{V}{k-i}$ that form an edge of G together with $\{x_1, \ldots, x_i\}$. In what follows, our hypergraphs will usually be k-uniform and will have n vertices. The parameters n and k will often be omitted if there is no danger of confusion.

Property 1.1 (Boundedness Property)**.** *Let $k \geqslant 2$. We define* $\mathrm{BDD}(d, C, p)$ *as the family of n-vertex k-uniform hypergraphs $G = (V, E)$ such that, for all $1 \leqslant r \leqslant d$ and all families of distinct sets $S_1, \ldots, S_r \in \binom{V}{k-1}$, we have*

$$|N_G(S_1) \cap \ldots \cap N_G(S_r)| \leqslant Cnp^r. \tag{1}$$

Property 1.2 (Tuple Property)**.** *Let $k \geqslant 2$. We define* $\mathrm{TUPLE}(d, \delta, p)$ *as the family of n-vertex k-uniform hypergraphs $G = (V, E)$ such that, for all $1 \leqslant r \leqslant d$, the following holds.*

$$\big| |N_G(S_1) \cap \ldots \cap N_G(S_r)| - np^r \big| < \delta np^r \tag{2}$$

for all but at most $\delta\binom{\binom{n}{k-1}}{r}$ families $\{S_1, \ldots, S_r\}$ of r distinct sets of $\binom{V}{k-1}$.

The notion of pseudorandomness considered in this paper is given in Definition 1.3 below.

Definition 1.3. *A k-uniform hypergraph $G = (V, E)$ is (d_1, C, d_2, δ, p)-*pseudorandom *if $|E| = p\binom{n}{k}$ and G satisfies* $\mathrm{BDD}(d_1, C, p)$ *and* $\mathrm{TUPLE}(d_2, \delta, p)$*.*

Note that property TUPLE implies edge-density close to p, but we put the condition $|E| = p\binom{n}{k}$ in the definition of pseudorandomness for convenience. We remark that similar notions of pseudorandomness in hypergraphs were considered in [8, 9].

Our main result, Theorem 1.4 below, estimates the number of copies of some linear k-uniform hypergraphs in sparse pseudorandom hypergraphs. An *embedding* of a hypergraph H into a hypergraph G is an injective mapping $\phi \colon V(H) \to V(G)$ such that $\{\phi(v_1), \ldots, \phi(v_k)\} \in E(G)$ whenever $\{v_1, \ldots, v_k\} \in E(H)$. An edge e of a linear k-uniform hypergraph $E(H)$ is called *connector* if there exist $v \in V(H) \smallsetminus e$ and k edges $e_1, \ldots, e_k$ containing v such that $|e \cap e_i| = 1$ for $1 \leqslant i \leqslant k$. Note that, for $k = 2$, a connector is an edge that is contained in a triangle. Moreover, since H is linear, $e \cap e_i \neq e \cap e_j$ for all $1 \leqslant i < j \leqslant k$. Given a k-uniform hypergraph H, let

$$d_H = \max\{\delta(J) \colon J \subset H\} \ \text{ and } \ D_H = \min\{kd_H, \Delta(H)\},$$

where $\delta(J)$ and $\Delta(J)$ stand, respectively, for the minimum and maximum degree of a vertex in $V(J)$. Note that $d_H \leqslant D_H$.

Kohayakawa, Rödl and Sissokho [12] proved the following counting lemma: *given a fixed triangle-free graph H and $p = p(n) \gg n^{-1/D_H}$ with $p = o(1)$, for all $\varepsilon > 0$ and $C > 1$, there exists $\delta > 0$ such that, if G is an n-vertex $(D_H, C, 2, \delta, p)$-pseudorandom graph and n is sufficiently large, then*

$$\big||\mathcal{E}(H,G)| - n^{v(H)}p^{e(H)}\big| < \varepsilon n^{v(H)}p^{e(H)},$$

where $\mathcal{E}(H,G)$ stands for the set of all embeddings from H into G. Our main theorem generalizes this result for k-uniform hypergraphs.

Theorem 1.4. *Let $k \geqslant 2$ and $m \geqslant 4$ be integers and let $\varepsilon > 0$ and $C > 1$ be fixed. Let H be a linear k-uniform connector-free hypergraph on m vertices. Then there exists $\delta > 0$ for which the following holds for any $p = p(n)$ with $p \gg n^{-1/D_H}$ and $p = o(1)$ and for any sufficiently large n.*

If G is an n-vertex k-uniform hypergraph that is $(D_H, C, 2, \delta, p)$-pseudorandom, then

$$\big||\mathcal{E}(H,G)| - n^{m}p^{e(H)}\big| < \varepsilon n^{m}p^{e(H)}.$$

This paper is organized as follows. In Section 2 we state an important result, Lemma 2.3, and we give some results that are needed for the proof of Lemma 2.3. In Section 3 we state the so-called "Extension Lemma", an important step in the proof of Theorem 1.4. In Section 4, we prove Lemma 2.3 and Theorem 1.4. We finish with some concluding remarks in Section 5.

§2. Auxiliary results

We begin by generalizing the definitions of BDD and TUPLE to deal not only with sets of $k-1$ vertices, but with sets of i vertices, for any $1 \leqslant i \leqslant k-1$.

Property 2.1 (General Boundedness Property)**.** *Let $k \geqslant 2$ and $1 \leqslant i \leqslant k-1$. We define $\mathrm{BDD}_i(d,C,p)$ as the family of n-vertex k-uniform hypergraphs $G = (V,E)$ such that, for all $1 \leqslant r \leqslant d$ and all families of distinct sets $S_1, \ldots, S_r \in \binom{V}{i}$, we have*

$$|N_G(S_1) \cap \ldots \cap N_G(S_r)| \leqslant Cn^{k-i}p^r. \tag{3}$$

Note that $\mathrm{BDD}_{k-1}(d,C,p)$ is the same as $\mathrm{BDD}(d,C,p)$.

Property 2.2 (General Tuple Property)**.** *Let $k \geqslant 2$ and $1 \leqslant i \leqslant k-1$. We define $\mathrm{TUPLE}_i(d,\delta,p)$ as the family of n-vertex k-uniform hypergraphs $G = (V,E)$ such that, for* *all* $1 \leqslant r \leqslant d$*, the following holds.*

$$\left| |N_G(S_1) \cap \ldots \cap N_G(S_r)| - \binom{n}{k-i} p^r \right| < \delta \binom{n}{k-i} p^r \tag{4}$$

for all but at most $\delta\binom{\binom{n}{i}}{r}$ *families* $\{S_1, \ldots, S_r\}$ *of* r *distinct sets of* $\binom{V}{i}$*. We note that* $\mathrm{TUPLE}_{k-1}(d, \delta, p)$ *is the same as* $\mathrm{TUPLE}(d, \delta, p)$*.*

Let $d \geqslant 2$ be an integer and let $\delta > 0$. Roughly speaking, the next result (Lemma 2.3) states that if G is a $(2, C, 2, \delta', p)$-pseudorandom k-uniform hypergraph on n vertices and $p = p(n) \gg n^{-1/d}$, then G is in fact $(2, C, d, \delta, p)$-pseudorandom for all sufficiently large n as long as δ' is sufficiently small.

Lemma 2.3. *For all* $\delta > 0$*,* $C > 1$ *and integers* k*,* $d \geqslant 2$*, there exists* $\delta' > 0$ *such that the following holds when* $p = p(n) \gg n^{-1/d}$ *and* n *is sufficiently large: if* G *is a* $(2, C, 2, \delta', p)$*-pseudorandom* k*-uniform hypergraph, then* G *is* $(2, C, d, \delta, p)$*-pseudorandom.*

Since we have $n^{-1/D_H} \geqslant n^{-1/d_H}$ for any k-graph H, Lemma 2.3 tells us that it suffices to consider (D_H, C, d_H, δ, p)-pseudorandom hypergraphs G in the proof of Theorem 1.4.

In the remainder of this section we prove some results that are important in the proof of Lemma 2.3. We start with some simple combinatorial facts and in Section 2.1 we, roughly speaking, show how to obtain properties BDD_i for every $1 \leqslant i \leqslant k-1$ and TUPLE_1 from our pseudorandomness assumption. The proof of the following well-known lemma can be seen in [12].

Fact 2.4. *For every* $\delta > 0$*, there exists* $\gamma > 0$ *such that, if a family of real numbers* $a_i \geqslant 0$*, for* $1 \leqslant i \leqslant N$*, satisfies the following inequalities:*

(*i*) $\sum_{i=1}^N a_i \geqslant (1-\gamma)Na$,
(*ii*) $\sum_{i=1}^N {a_i}^2 \leqslant (1+\gamma)Na^2$,

then

$$\big|\{i \colon |a_i - a| < \delta a\}\big| > (1-\delta)N.$$

Let $r \geqslant 1$ and $a > 0$. Since $\binom{na}{r}/a^r\binom{n}{r} \to 1$ when $n \to \infty$, we obtain the following lemma, which provides a combinatorial inequality that will be used often.

Fact 2.5. *Let* $\sigma > 0$*,* $r \geqslant 1$ *and* $a > 0$*. Then, the following holds for a sufficiently large* n*.*

$$\left|\binom{na}{r} - a^r\binom{n}{r}\right| \leqslant \sigma a^r \binom{n}{r}.$$

2.1. **Extending properties** BDD **and** TUPLE**.** In this section we prove two results, Lemmas 2.6 and 2.7, that give conditions for a hypergraph G to satisfy properties BDD_i for every $1 \leqslant i \leqslant k-1$, and TUPLE_1.

Lemma 2.6. *Let $C > 1$ be an integer, let G be an n vertex k-uniform hypergraph and consider $0 < p = p(n) \leqslant 1$. If G satisfies $\mathrm{BDD}(2, C, p)$, then G satisfies $\mathrm{BDD}_i(2, C, p)$ for all $1 \leqslant i \leqslant k-1$.*

Proof. The proof follows by induction on $i = k-1, \ldots, 1$ and a simple averaging argument. □

The next result gives necessary conditions for a k-uniform hypergraph to satisfy property $\mathrm{TUPLE}_1(2, \delta, p)$.

Lemma 2.7. *For all $C > 1$, $\delta > 0$ and an integer $k \geqslant 2$, there exists $\sigma > 0$ such that the following holds for $p \gg n^{-1/2}$ and sufficiently large n.*

If G is a $(2, C, 2, \sigma, p)$-pseudorandom n-vertex k-uniform hypergraph, then G satisfies $\mathrm{TUPLE}_1(2, \delta, p)$.

Proof. We must prove that (4) holds for $1 \leqslant r \leqslant 2$. Since the proofs of the cases $r = 1$ and $r = 2$ are similar, we present only the proof for the case $r = 2$. We will show that the two inequalities required to apply Fact 2.4 hold.

Fix $C > 1$, $\delta > 0$ and an integer $k \geqslant 2$. Let $\gamma > 0$ be obtained by an application of Fact 2.4 with parameter $\delta > 0$ and let $\sigma = \sigma(C, \gamma)$ be a sufficiently small constant. Now let $p \gg n^{-1/2}$ and consider a sufficiently large n. Suppose that $G = (V, E)$ is a $(2, C, 2, \sigma, p)$-pseudorandom n-vertex k-uniform hypergraph. Thus,

$$
\begin{aligned}
\sum_{\{u,v\} \in \binom{V}{2}} |N(u) \cap N(v)| &= \sum_{S \in \binom{V}{k-1}} \binom{|N(S)|}{2} \\
&\geqslant \sum_{S \in \binom{V}{k-1} :\, |N(S)| \geqslant (1-\sigma)np} \binom{|N(S)|}{2} \\
&\geqslant (1-\sigma)\binom{n}{k-1}\binom{(1-\sigma)pn}{2} \\
&\geqslant (1-\sigma)^4 \binom{n}{2}\binom{n}{k-1} p^2 \\
&\geqslant (1-\gamma)\binom{n}{2}\binom{n}{k-1} p^2, \qquad (5)
\end{aligned}
$$

where the first inequality is trivial and the second follows from $\mathrm{TUPLE}(2, \sigma, p)$ and in the third inequality we apply Fact 2.5.

Heading for an application of Fact 2.4 we consider the following sum.

$$\begin{aligned}\sum_{\{u,v\}\in\binom{V}{2}} |N(u)\cap N(v)|^2 &= \sum_{(S_1,S_2)\in\binom{V}{k-1}^2} \binom{|N(S_1)\cap N(S_2)|}{2} \\ &= \sum_{\substack{(S_1,S_2)\in\binom{V}{k-1}^2\\ S_1\neq S_2}} \binom{|N(S_1)\cap N(S_2)|}{2} + \sum_{S_1\in\binom{V}{k-1}} \binom{|N(S_1)|}{2}. \qquad (6)\end{aligned}$$

We will bound the two sums in (6). By $\mathrm{BDD}(2,C,p)$, the choice of p and an application of Fact 2.5, we have

$$\begin{aligned}\sum_{S_1\in\binom{V}{k-1}} \binom{|N(S_1)|}{2} &\leqslant \binom{n}{k-1}\binom{Cnp}{2} \\ &\leqslant (1+\sigma)C^2\binom{n}{2}\binom{n}{k-1}p^2.\end{aligned}$$

Since $p \gg n^{-1/2}$, we obtain

$$\sum_{S_1\in\binom{V}{k-1}} \binom{|N(S_1)|}{2} \leqslant \sigma\binom{n}{2}\left(\binom{n}{k-1}p^2\right)^2. \qquad (7)$$

To bound the remaining sum, define A and B as the families of pairs $\{S_1,S_2\}$ with $S_1,S_2\in\binom{V}{k-1}$ and $S_1\neq S_2$ such that $|N(S_1)\cap N(S_2)|\leqslant(1+\sigma)np^2$ for all pairs in A, and $|N(S_1)\cap N(S_2)|>(1+\sigma)np^2$ for all pairs in B. By Fact 2.5 we obtain

$$\begin{aligned}\sum_{\{S_1,S_2\}\in A} \binom{|N(S_1)\cap N(S_2)|}{2} &\leqslant \frac{1}{2}\binom{n}{k-1}^2\binom{(1+\sigma)p^2n}{2} \\ &\leqslant \frac{(1+\sigma)^3}{2}\binom{n}{2}\left(\binom{n}{k-1}p^2\right)^2. \qquad (8)\end{aligned}$$

By $\mathrm{BDD}(2,C,p)$, $\mathrm{TUPLE}(2,\sigma,p)$ and Fact 2.5 applied with σ, $r=2$ and $a=Cp^2$, we have

$$\begin{aligned}\sum_{\{S_1,S_2\}\in B} \binom{|N(S_1)\cap N(S_2)|}{2} &\leqslant \frac{\sigma}{2}\binom{n}{k-1}^2\binom{Cp^2n}{2} \\ &\leqslant \frac{\sigma(1+\sigma)C^2}{2}\binom{n}{2}\left(\binom{n}{k-1}p^2\right)^2. \qquad (9)\end{aligned}$$

Replacing (7), (8) and (9) in (6), we have

$$\begin{aligned}\sum_{\{u,v\}\in\binom{V}{2}} |N(u)\cap N(v)|^2 &\leqslant \big(\sigma+(1+\sigma)^3+\sigma(1+\sigma)C^2\big)\binom{n}{2}\left(\binom{n}{k-1}p^2\right)^2 \\ &\leqslant (1+\gamma)\binom{n}{2}\left(\binom{n}{k-1}p^2\right)^2. \qquad (10)\end{aligned}$$

Equations (5) and (10) can be seen as inequalities *(i)* and *(ii)* in Fact 2.4. Therefore, we conclude that, for at least $(1-\delta)\binom{n}{2}$ pairs of vertices $\{u,v\} \in \binom{V}{2}$, we have

$$\left| |N(u) \cap N(v)| - \binom{n}{k-1}p^2 \right| < \delta \binom{n}{k-1}p^2.$$

□

§3. Extension Lemma and corollaries

In this section we prove a result called *Extension Lemma* (Lemma 3.1) from where we derive Corollaries 3.3 and 3.4, which are used in the proof of Theorem 1.4.

3.1. Extension Lemma.

Before starting the discussion concerning the Extension Lemma we shall define some concepts. Consider k-uniform hypergraphs G and H. Given sequences $W = w_1, \ldots, w_\ell \in V(H)^\ell$ and $X = x_1, \ldots, x_\ell \in V(G)^\ell$, define $\mathcal{E}(H, G, W, X)$ as the set of embeddings $f \in \mathcal{E}(H, G)$ such that $f(w_i) = x_i$ for all $1 \leqslant i \leqslant \ell$. Furthermore, for a sequence Y define the set of its elements by $Y^{\mathrm{set}} = \{y_1, \ldots, y_\ell\}$. We say that a subset of vertices $V' \subset V(H)$ is *stable* if $E(H[V']) = \varnothing$, i.e., if there is no edge of H contained in V'.

Let H be a hypergraph with m vertices. We say that H is *d-degenerate* if there exists an ordering $v_1, \ldots, v_m$ of $V(H)$ such that $d_{H_i}(v_i) \leqslant d$ for all $1 \leqslant i \leqslant m$, where $H_i = H[\{v_1, \ldots, v_i\}]$. In this case, we say that $v_1, \ldots, v_m$ is a *d-degenerate ordering* of the vertices of H. Given a sequence $W \in V(H)^\ell$, we define $\omega(H, W) = |E(H)| - |E(H[W^{\mathrm{set}}])|$, i.e., $\omega(H, W)$ is the number of edges of H that are not contained in W^{set}.

Lemma 3.1 (Extension Lemma)**.** *Let $C \geqslant 1$, $m \geqslant 1$ and $k \geqslant 2$. Let G and H be k-uniform hypergraphs such that H is linear, $|V(H)| = m$, $|V(G)| = n$ and $p = p(n) = e(G)/\binom{n}{k}$. Suppose that $0 \leqslant \ell \leqslant \max\{k, d_H\}$, and let $W \in V(H)^\ell$ and $X \in V(G)^\ell$ be fixed. If $G \in \mathrm{BDD}(D_H, C, p)$, then*

$$|\mathcal{E}(H, G, W, X)| \leqslant C^{m-\ell} n^{m-\ell} p^{\omega(H,W)}.$$

In particular, if $W^{\mathrm{set}} \subset V(H)$ is stable, then $|\mathcal{E}(H, G, W, X)| \leqslant C^{m-\ell} n^{m-\ell} p^{e(H)}$.

For a k-uniform hypergraph $H = (V, E)$ with $|V| = m$ vertices and for a positive integer $\ell \leqslant \max\{k, d_H\}$, Proposition 3.2 allows us to obtain a D_H-degenerate ordering $v_1, \ldots, v_m$ of V such that $W = v_1, \ldots, v_\ell$ from a d_H-degenerate ordering of V. Consider a sequence L of the vertices of H. Given a subsequence W of V, we write $L \smallsetminus W$ for the sequence of $L \smallsetminus W$ obtained from L by deleting the vertices of W. Given a sequence of vertices Y in V^ℓ, we write $L' = (Y, L \smallsetminus Y)$ to denote the sequence L' of V obtained by removing Y from L and placing it before the elements of L.

Proposition 3.2. *Let $H = (V, E)$ be a linear k-uniform hypergraph and let ℓ be an integer with $0 \leqslant \ell \leqslant \max\{k, d_H\}$. If $W \in V^\ell$, then there exists a D_H-degenerate ordering $w_1, \ldots, w_{|V|}$ of V such that $W = w_1, \ldots, w_\ell$.*

Proof. Fix $k \geqslant 2$ and let H, ℓ and W as in the statement of the proposition. Note that the result is trivial whenever W is empty, and if $D_H = \Delta(H)$, then any ordering of the vertices of H is D_H-degenerate. Therefore, assume $D_H = k \cdot d_H$ and $1 \leqslant |W| = \ell \leqslant \max\{k, d_H\}$.

Let L be a d_H-degenerate ordering of V and put $L' = (W, L \smallsetminus W)$. Given a vertex v of H, define the *left degree* of v in L' as the number of edges e such that v is the rightmost element of e considering the ordering L'. Since L is d_H-degenerate and, by the linearity of H, any vertex v belongs to at most $|W|$ edges containing vertices of W, the left degree of v in L' is at most $|W| + d_H$. We divide the proof into three cases.

Case 1: $d_H > k$. In this case, $|W| \leqslant d_H$. Then, the left degree of any vertex of H in L' is at most $2d_H \leqslant k \cdot d_H = D_H$. Therefore, L' is a D_H-degenerate ordering of V.

Case 2: $2 \leqslant d_H \leqslant k$. Here we have $|W| \leqslant k$. Therefore, since $k \geqslant 2$, the left degree of each vertex of H in L' is at most $k + d_H \leqslant k \cdot d_H = D_H$. Therefore, L' is a D_H-degenerate ordering of V.

Case 3: $d_H = 1$. Here we have $|W| \leqslant k$. Note that the only possibility for a vertex v to have left degree larger than $k \cdot d_H = k$ in L' is if the following holds: $|W| = k$ and, for every $w \in W$, the vertex v belongs to an edge e_w containing w and w is the rightmost element of e_w in L. But note that, since $d_H = 1$, there exists at most one vertex v with this property, otherwise L would not be a d_H-degenerate ordering. Let W' be the ordering $w_1, \ldots, w_\ell, v$. Now consider the ordering $L'' = (W', L \smallsetminus W')$. It is clear that all the vertices of H have left degree at most $2 \leqslant k \cdot d_H = D_H$ in L''. Therefore, L'' is a D_H-degenerate ordering of V. □

Now we prove the Extension Lemma.

Proof of Lemma 3.1. Fix $C \geqslant 1$, $m \geqslant 1$ and $k \geqslant 2$. Let G and H be k-uniform hypergraphs such that H is linear with $|V(H)| = m$, $|V(G)| = n$ and $p = p(n) = e(G)/\binom{n}{k}$. Let ℓ be an integer with $0 \leqslant \ell \leqslant \max\{k, d_H\}$, and let $W \in V(H)^\ell$ and $X \in V(G)^\ell$. Suppose that $G \in \mathrm{BDD}(D_H, C, p)$. By Proposition 3.2, we know that there exists a D_H-degenerate ordering $v_1, \ldots, v_m$ of $V(H)$ such that W is its initial segment. We will prove by induction on h that, for all $\ell \leqslant h \leqslant m$,

$$|\mathcal{E}(H_h, G, W, X)| \leqslant C^{h-\ell} n^{h-\ell} p^{\omega(H_h, W)}, \tag{11}$$

where $H_h = H[\{v_1, \ldots, v_h\}]$.

If $h = \ell$, the statement is trivial. Suppose that $\ell < h \leqslant m$ and

$$|\mathcal{E}(H_{h-1}, G, W, X)| \leqslant C^{h-1-\ell} n^{h-1-\ell} p^{\omega(H_{h-1}, W)}.$$

Since $v_1, \ldots, v_m$ is D_H-degenerate we have $d_{H_h}(v_h) \leqslant D_H$. By $G \in \mathrm{BDD}(D_H, C, p)$, we know that any embedding from H_{h-1} to G can be extended to an embedding from H_h to G in at most $Cnp^{d_{H_h}(v_h)}$ different ways. Since $\omega(H_h, W) = \omega(H_{h-1}, W) + d_{H_h}(v_h)$, applying the induction hypothesis, we conclude that

$$\begin{aligned}|\mathcal{E}(H_h, G, W, X)| &\leqslant Cnp^{d_{H_h}(v_h)} |\mathcal{E}(H_{h-1}, G, W, X)| \\ &\leqslant Cnp^{d_{H_h}(v_h)} C^{h-1-\ell} n^{h-1-\ell} p^{\omega(H_{h-1}, W)} \\ &= C^{h-\ell} n^{h-\ell} p^{\omega(H_h, W)}.\end{aligned}$$

□

3.2. **Corollaries of the Extension Lemma.** Given k-uniform hypergraphs G and H, we write $\mathcal{E}^{\neg\mathrm{ind}}(H, G)$ and $\mathcal{E}^{\mathrm{ind}}(H, G)$ for the set of non-induced and induced embeddings from H into G, respectively. The following corollary bounds from above the number of embeddings in $\mathcal{E}^{\neg\mathrm{ind}}(H, G)$ for some hypergraphs G whenever H is linear.

Corollary 3.3. *Let $C \geqslant 1$, $m, k, \eta > 0$ and $p = p(n) = o(1)$ with $m \geqslant k \geqslant 2$. Then, for all k-uniform hypergraphs G and H, where $|V(G)| = n$ and H is linear with $|V(H)| = m$ the following holds. If $G \in \mathrm{BDD}(D_H, C, p)$ and n is sufficiently large, then*

$$\left|\mathcal{E}^{\neg\mathrm{ind}}(H, G)\right| < \eta n^m p^{e(H)}.$$

Proof. Fix $C \geqslant 1$, $m \geqslant k \geqslant 2$, $\eta > 0$ and let $p = p(n) = o(1)$. Let G and H be as in the statement and let n be sufficiently large.

Fix an edge $\{x_1, \ldots, x_k\} \in E(G)$ and a non-edge $\{w_1, \ldots, w_k\}$ of H. Applying Lemma 3.1 with $W = (w_1, \ldots, w_k)$ and $X = (x_1, \ldots, x_k)$, we conclude that the number of embeddings f from $V(H)$ into $V(G)$ such that $f(w_i) = x_i$ for $1 \leqslant i \leqslant k$ is bounded from above by $C^{m-k} n^{m-k} p^{E(H)}$. Since $G \in \mathrm{BDD}(D_H, C, p)$, we have $|E(G)| \leqslant Cn^k p$, from where we conclude that there exist at most $Cn^k p$ choices for $\{x_1, \ldots, x_k\}$ in $E(G)$. Note that there exist at most $\binom{m}{k}$ choices for $\{w_1, \ldots, w_k\}$ in $\binom{V(H)}{k}$. Then, we can choose $(x_1, \ldots, x_k)$ and $(w_1, \ldots, w_k)$, respectively, in $Ck!n^k p$ and $k!\binom{m}{k}$ ways. Therefore, $|\mathcal{E}^{\neg\mathrm{ind}}(H, G)| \leqslant K n^m p^{e(H)+1}$ for some constant $K = K(C, k, m)$. Since $p = o(1)$ the lemma follows for any $\eta > 0$ and any sufficiently large n. □

Let G and H be k-uniform hypergraphs with $|V(G)| = n$ and consider a set $X \subset \binom{V(H)}{k-1}$. If f is an embedding from H into G, we denote by $f_{k-1}(X)$ the family $\{f(x_1), \ldots, f(x_{k-1})\}$, for all $\{x_1, \ldots, x_{k-1}\} \in X$.

Given $\delta > 0$, define $B_G(\delta, r)$ as the families $\{X_1, \ldots, X_r\}$ of r distinct sets of $\binom{V(G)}{k-1}$ such that

$$\big||N_G(X_1) \cap \ldots \cap N_G(X_r)| - np^r\big| \geqslant \delta np^r.$$

Consider the following definition.

$$B_G^{\mathrm{stb}}(\delta, r) = \left\{ \{X_1, \ldots, X_r\} \in B_G(\delta, r) \colon \bigcup_{i=1}^{r} X_i \text{ is stable in } G \right\}.$$

Given r distinct sets $X_1, \ldots, X_r$ of $\binom{V(G)}{k-1}$, we say that $X = \{X_1, \ldots, X_r\}$ is δ-*bad* if $X \in B_G^{\mathrm{stb}}(\delta, r)$. Let H be a k-uniform hypergraph with m vertices and let $v_1, \ldots, v_m$ be a d_H-degenerate ordering of $V(H)$. Define $H_i = H[v_1, \ldots, v_i]$. We say that an embedding $f \colon V(H_{h-1}) \to V(G)$ is δ-*clean* if $f_{k-1}(N_{H_h}(v_h)) \notin B_G^{\mathrm{stb}}(\delta, d_{H_h}(v_h))$. Moreover, if $f \colon V(H_{h-1}) \to V(G)$ is not δ-clean, then we say that f is δ-*polluted*. We denote the set of embeddings $f \in \mathcal{E}(H_{h-1}, G)$ such that f is δ-polluted by $\mathcal{E}_{\delta\text{-poll}}(H_{h-1}, G)$. Similarly, we denote by $\mathcal{E}_{\delta\text{-clean}}(H_{h-1}, G)$ the set of embeddings $f \in \mathcal{E}(H_{h-1}, G)$ such that f is δ-clean. The next corollary shows that if H is linear and connector-free then most of the embeddings from H_{h-1} into a sufficiently pseudorandom hypergraph G are clean, for $1 \leqslant h \leqslant m$.

Corollary 3.4. *Let $\delta > 0$, $C > 1$, $m \geqslant 4$ and $k \geqslant 2$ be fixed constants. Let H be an m-vertex linear k-uniform hypergraph that is connector-free and let $v_1, \ldots, v_m$ be a d_H-degenerate ordering of $V(H)$. Suppose that $1 < h \leqslant m$ and put $r = d_{H_h}(v_h)$. If G is (D_H, C, d_H, δ, p)-pseudorandom, then*

$$|\mathcal{E}_{\delta\text{-poll}}(H_{h-1}, G)| \leqslant \delta \left(r!((k-1)!)^r C^{h-1-r(k-1)} \right) n^{h-1} p^{e(H_{h-1})}.$$

Proof. Fix constants $\delta > 0$, $C > 1$, $m \geqslant 4$ and $k \geqslant 2$. Let H be an m-vertex linear k-uniform hypergraph that is connector-free. Consider a d_H-degenerate ordering $v_1, \ldots, v_m$ of $V(H)$. Let $1 < h \leqslant m$ and put $r = d_{H_h}(v_h)$. Suppose that G is (D_H, C, d_H, δ, p)-pseudorandom.

By definition, an embedding $f \colon V(H_{h-1}) \to V(G)$ is δ-polluted if $f_{k-1}(N_{H_h}(v_h)) \in B_G^{\mathrm{stb}}(\delta, r)$. Let $N_{H_h}(v_h) = \{W_1, \ldots, W_r\}$ where $W_i = \{w_{i,1}, \ldots, w_{i,k-1}\}$ for all $1 \leqslant i \leqslant r$ (Note that since H is linear, the sets $W_1, \ldots, W_r$ are pairwise disjoint). Let

$$W_{\mathrm{ord}} = (w_{1,1}, \ldots, w_{1,k-1}, w_{2,1}, \ldots, w_{2,k-1}, \ldots, w_{r,1} \ldots, w_{r,k-1})$$

be an ordering of $W_1 \cup \ldots \cup W_r$. Therefore,

$$\mathcal{E}_{\delta\text{-poll}}(H_{h-1}, G) = \bigcup_{X} \left(\bigcup_{X_{\mathrm{ord}}} \mathcal{E}(H_{h-1}, G, W_{\mathrm{ord}}, X_{\mathrm{ord}}) \right),$$

where the first union is over all families $X = \{S_1, \ldots, S_r\} \in B_G^{\mathrm{stb}}(\delta, r)$ and the second union is over all $((k-1)!)^r$ possible orderings of S_i for $1 \leqslant i \leqslant r$, and all $r!$ orderings of X. Therefore,

$$|\mathcal{E}_{\delta\text{-poll}}(H_{h-1}, G)| \leqslant \sum_X \sum_{X_{\mathrm{ord}}} |\mathcal{E}(H_{h-1}, G, W_{\mathrm{ord}}, X_{\mathrm{ord}})|.$$

Note that, since H_h is linear and connector-free, $\bigcup N_{H_h}(v_h)$ is stable in H_h. Since $G \in \mathrm{BDD}(D_H, C, p)$ and $|W_{\mathrm{ord}}| = r(k-1)$, we know from the conclusion of Lemma 3.1 that

$$|\mathcal{E}(H_{h-1}, G, W_{\mathrm{ord}}, X_{\mathrm{ord}})| \leqslant C^{h-1-r(k-1)} n^{h-1-r(k-1)} p^{e(H_{h-1})}.$$

Since $r = d_{H_h}(v_h) \leqslant d_H$ and G satisfies $\mathrm{TUPLE}(d_H, \delta, p)$, we have $\left|B_G^{\mathrm{stb}}(\delta, r)\right| \leqslant \delta n^{r(k-1)}$. Then, the first of the sums contains at most $\delta n^{r(k-1)}$ terms. Since the second sum is over $r!((k-1)!)^r$ terms, we obtain

$$|\mathcal{E}_{\delta\text{-poll}}(H_{h-1}, G)| \leqslant \delta \left(r!((k-1)!)^r C^{h-1-r(k-1)} \right) n^{h-1} p^{e(H_{h-1})}.$$

$\square$

§4. Proof of the main result

Before proving Theorem 1.4 we prove Lemma 2.3. The proof of Lemma 2.3 is simple and rely on Facts 2.4 and 2.5, and Lemma 2.6. For simplicity, we will not explicit the constants used in its proof.

Proof of Lemma 2.3. Fix $\delta > 0$, $C > 1$ and integers $k, d \geqslant 2$, and let $2 \leqslant r \leqslant d$. Let $\gamma > 0$ be obtained by an application of Fact 2.4 with parameters δ. Now let $\sigma = \sigma(k, r, \gamma)$ be a sufficiently small constant. Let $\delta_{2.7}$ be obtained by an application of Lemma 2.7 with parameter C, σ and k and put $\delta' = \min\{\delta, \delta_{2.7}\}$. Consider $p \gg n^{-1/d}$ and let n be sufficiently large.

Suppose $G = (V, E)$ is an n-vertex k-uniform $(2, C, 2, \delta', p)$-pseudorandom hypergraph. By Lemma 2.7, the following two inequalities hold, respectively, for more than $(1-\sigma)n$ vertices $u \in V$ and for more than $(1-\sigma)\binom{n}{2}$ pairs $\{u, v\} \in \binom{V}{2}$.

$$\left| |N(u)| - \binom{n}{k-1} p \right| < \sigma \binom{n}{k-1} p, \tag{12}$$

$$\left| |N(u) \cap N(v)| - \binom{n}{k-1} p^2 \right| < \sigma \binom{n}{k-1} p^2. \tag{13}$$

We must check that the inequalities *(i)* and *(ii)* of Fact 2.4 hold. For inequality *(i)*, consider the following sum over distinct sets $S_1, \dots, S_r \in \binom{V}{k-1}$.

$$\sum_{S_1,\dots,S_r \in \binom{V}{k-1}} |N(S_1) \cap \dots \cap N(S_r)| = \sum_{u \in V} \binom{|N(u)|}{r} \geqslant (1-\sigma) n \binom{(1-\sigma)\binom{n}{k-1}p}{r} \geqslant (1-\gamma) \binom{\binom{n}{k-1}}{r} np^r, \tag{14}$$

where the first inequality follows from (12) and the last one follows from Fact 2.5. It remains to prove that inequality *(ii)* of Fact 2.4 holds. Consider the following sum over distinct sets $S_1, \dots, S_r \in \binom{V}{k-1}$.

$$\sum_{S_1,\dots,S_r \in \binom{V}{k-1}} \left| \bigcap_{i=1}^{r} N(S_i) \right|^2 = \sum_{(u,v) \in V^2} \binom{|N(u) \cap N(v)|}{r} = \sum_{\substack{(u,v) \in V^2 \\ u \neq v}} \binom{|N(u) \cap N(v)|}{r} + \sum_{u \in V} \binom{|N(u)|}{r}. \tag{15}$$

Let us estimate the sums in (15). In view of Lemma 2.6 applied for $i = 1$ we can apply the boundedness property to bound $|N(u)|$ for every $u \in V$, obtaining

$$\sum_{u \in V} \binom{|N(u)|}{r} \leqslant n \binom{Cn^{k-1}p}{r} \leqslant C' \binom{\binom{n}{k-1}}{r} np^r \tag{16}$$

for some $C' = C'(k, r, \sigma)$.

Now we estimate the remaining sum. Define A and B as the families of pairs $\{u, v\} \in \binom{V}{2}$ such that $|N(u) \cap N(v)| \leqslant (1+\sigma)\binom{n}{k-1}p^2$ and $|N(u) \cap N(v)| > (1+\sigma)\binom{n}{k-1}p^2$, respectively. Since $p^2 n^{k-1} \gg 1$, Fact 2.5 implies

$$\sum_{\{u,v\} \in A} \binom{|N(u) \cap N(v)|}{r} \leqslant \frac{n^2}{2} \binom{(1+\sigma)p^2\binom{n}{k-1}}{r} \leqslant \frac{(1+\sigma')}{2} \binom{\binom{n}{k-1}}{r} (np^r)^2, \tag{17}$$

where $\sigma' = \sigma'(r, \sigma)$ is a sufficiently small constant. Similarly, using the boundedness of G and (13) we obtain

$$\sum_{\{u,v\} \in B} \binom{|N(u) \cap N(v)|}{r} \leqslant \frac{\sigma n^2}{2} \binom{C(k-1)^{k-1}p^2\binom{n}{k-1}}{r} \leqslant \sigma' \binom{\binom{n}{k-1}}{r} (np^r)^2. \tag{18}$$

Replacing (16), (17) and (18) in (15), we have

$$\sum_{S_1,\dots,S_r \in \binom{V}{k-1}} \left| \bigcap_{i=1}^{r} N(S_i) \right|^2 \leqslant (1+\gamma) \binom{\binom{n}{k-1}}{r} (np^r)^2, \tag{19}$$

where the above sum is over distinct sets $S_1, \ldots, S_r$.

Inequalities (14) and (19) can be seen as inequalities *(i)* and *(ii)* in Fact 2.4. Therefore, we conclude that, for more than $(1-\delta)\binom{\binom{n}{k-1}}{r}$ families of distinct sets $S_1, \ldots, S_r \in \binom{V}{k-1}$, the following holds for all $1 \leqslant r \leqslant d$.

$$\big||N(S_1) \cap \ldots \cap N(S_r)| - np^r\big| < \delta np^r.$$

To finish the proof, note that, since $\delta' \leqslant \delta$ and $G \in \mathrm{TUPLE}(2, \delta', p)$, the following holds for more than $(1-\delta)\binom{n}{k-1}$ sets $S_1 \in \binom{V}{k-1}$.

$$\big||N(S_1)| - np\big| < \delta np.$$

□

Proof of Theorem 1.4. Let $k \geqslant 2$ and $m \geqslant 4$ be integers and fix $C > 1$. Let H be a linear k-uniform connector-free hypergraph on m vertices. Fix a d_H-degenerate ordering $v_1, \ldots, v_m$ of $V(H)$ and put $H_h = H[\{v_1, \ldots, v_h\}]$.

We will use induction on h to prove that for every $1 \leqslant h \leqslant m$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds when $p \gg n^{-1/D_H}$ and n is sufficiently large: if G is an n-vertex k-uniform (D_H, C, d_H, δ, p)-pseudorandom hypergraph (recall that Lemma 2.3 allows us to consider this stronger pseudorandomness condition on G), then

$$\big||\mathcal{E}(H_h, G)| - n^h p^{e(H_h)}\big| < \varepsilon n^h p^{e(H_h)}. \tag{20}$$

For every $\varepsilon > 0$ and $h = 1$ the result is trivial. Thus, assume $1 < h \leqslant m$ and suppose the result holds for $h-1$ and for all $\varepsilon > 0$.

Let $\varepsilon > 0$ be given, let $\varepsilon' = \min\{\varepsilon/4, \varepsilon/6C\}$ and consider $\delta' = \delta'(\varepsilon')$ given by the induction hypothesis such that for $p \gg n^{-1/D_H}$ with $p = o(1)$ the following holds for sufficiently large n.

$$\big||\mathcal{E}(H_{h-1}, G)| - n^{h-1} p^{e(H_{h-1})}\big| < \varepsilon' n^{h-1} p^{e(H_{h-1})}. \tag{21}$$

Fix $\eta = \varepsilon'/2$ and define $r = d_{H_v}(v_h) \leqslant d_H$. Let δ be a sufficiently small constant and suppose $p \gg n^{-1/D_H}$ with $p = o(1)$ and n is sufficiently large.

Suppose G is an n-vertex k-uniform (D_H, C, d_H, δ, p)-pseudorandom hypergraph. An application of Corollary 3.3 with parameters C, $h-1$, k, η and p for the graphs H_{h-1} and G provides the following upper bound on the number of non-induced embeddings.

$$\big|\mathcal{E}^{-\mathrm{ind}}(H_{h-1}, G)\big| \leqslant \eta n^{h-1} p^{e(H_{h-1})}. \tag{22}$$

By Corollary 3.4 applied with δ, C, m and k for the graphs H_{h-1} and G, we have

$$|\mathcal{E}_{\delta\text{-poll}}(H_{h-1}, G)| \leqslant \eta n^{h-1} p^{e(H_{h-1})}. \tag{23}$$

By (22) and (23),

$$\left|\mathcal{E}^{\neg\mathrm{ind}}(H_{h-1},G)\cup\mathcal{E}_{\delta\text{-poll}}(H_{h-1},G)\right|\leqslant 2\eta n^{h-1}p^{e(H_{h-1})}=\varepsilon' n^{h-1}p^{e(H_h)-r}.$$

Then, (21) implies

$$(1-2\varepsilon')n^{h-1}p^{e(H_h)-r}<\left|\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)\right|<(1+\varepsilon')n^{h-1}p^{e(H_h)-r}. \tag{24}$$

The next step is to bound from below the number of ways we can extend an embedding $f'\in\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)$ to an embedding $f\in\mathcal{E}(H_h,G)$. Let f' be such an embedding. Since f' is clean, $f'_{k-1}(N_{H_h}(v_h))\notin B^{\mathrm{stb}}_G(\delta,r)$, i.e., either $f'(\bigcup N_{H_h}(v_h))$ is not stable in G or $\left|N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)-np^r\right|<\delta np^r$. Since H is linear and connector-free, it is easy to see that $\bigcup N_{H_h}(v_h)$ is stable in H_h. But since f' is an induced embedding, $f'(\bigcup N_{H_h}(v_h))$ is stable in G. Therefore,

$$\left|N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)-np^r\right|<\delta np^r. \tag{25}$$

To obtain an extension $f\in\mathcal{E}(H_h,G)$ from $f'\in\mathcal{E}(H_{h-1},G)$ we must choose $f(v_h)$ in the set $N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)\smallsetminus f'\big(V(H_{h-1})\big)$. Therefore, the number of such extensions is

$$\left|N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)\smallsetminus f'\big(V(H_{h-1})\big)\right|\geqslant(1-\delta)np^r-(h-1)\geqslant(1-2\delta)np^r, \tag{26}$$

where the first inequality is due to (25) and the last one follows from the choice of p. By (24) and (26), we have

$$\begin{aligned}|\mathcal{E}(H_h,G)|&\geqslant|\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_h,G)|\\&\geqslant\left|\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)\right|\left|N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)\smallsetminus f'\big(V(H_{h-1})\big)\right|\\&>(1-2\varepsilon')(1-2\delta)n^{h-1}p^{e(H_h)-r}np^r\\&\geqslant(1-\varepsilon)n^hp^{e(H_h)}.\end{aligned}$$

To finish the proof we must show that $|\mathcal{E}(H_h,G)|<(1+\varepsilon)n^hp^{e(H_h)}$. Fix an embedding $f'\in\mathcal{E}(H_{h-1},G)$. Consider the case $f'\in\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)$. Note that the number of extensions of f' to embeddings from H_h into G is at most $\left|N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)\right|$. Therefore, by (24) and (25), the number of such embeddings is at most

$$\begin{aligned}\left|\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)\right|\left|N_G\big(f'_{k-1}(N_{H_h}(v_h))\big)\right|&\leqslant(1+\varepsilon')n^{h-1}p^{e(H_h)-r}(1+\delta)np^r\\&\leqslant(1+\varepsilon/2)n^hp^{e(H_h)}.\end{aligned} \tag{27}$$

Now suppose $f'\in\big\{\mathcal{E}(H_{h-1},G)\smallsetminus\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)\big\}$. By (21) and (24), we have

$$\left|\mathcal{E}(H_{h-1},G)\smallsetminus\mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1},G)\right|\leqslant 3\varepsilon' n^{h-1}p^{e(H_h)-r}. \tag{28}$$

But since $r=d_{H_h}(v_h)\leqslant d_H\leqslant D_H$ and $G\in\mathrm{BDD}(D_H,C,p)$, every embedding f' from H_{h-1} into G can be extended to at most Cnp^r embeddings $f\in\mathcal{E}(H_h,G)$. In fact, to see this, apply property $\mathrm{BDD}(D_H, C, p)$ to the family $\{f'(S_1), \dots, f'(S_{|N_{H_h}(v_h)|})\}$, where $\{S_1, S_2, \dots, S_{|N_{H_h}(v_h)|}\}$ is the neighbourhood of v_h in H_h. This fact together with (28) implies that the number of extensions of an embedding in $\mathcal{E}(H_{h-1}, G) \setminus \mathcal{E}^{\mathrm{ind}}_{\delta\text{-clean}}(H_{h-1}, G)$ to embeddings from H_h into G is at most $(3\varepsilon' C)n^h p^{e(H_h)} \leqslant (\varepsilon/2)n^h p^{e(H_h)}$. Therefore, using (27) we conclude that $|\mathcal{E}(H_h, G)| < (1+\varepsilon)n^h p^{e(H_h)}$. □

§5. Concluding remarks

We say that a graph $G = (V, E)$ satisfies property $\mathcal{Q}(\eta, \delta, \alpha)$ if, for every subgraph $G[S]$ induced by $S \subset V$ with $|S| \geqslant \eta|V|$, we have $(\alpha - \delta)\binom{|S|}{2} < |E(G[S])| < (\alpha + \delta)\binom{|S|}{2}$. In [11, 14], answering affirmatively a question posed by Erdős (see, e.g.,[7] and [1, p. 363]; see also [13]), Rödl proved that for every positive integer m and for every positive $\alpha, \eta < 1$ there exist $\delta > 0$ and an integer n_0 such that, if $n \geqslant n_0$, then every n-vertex graph G satisfying $\mathcal{Q}(\eta, \delta, \alpha)$ contains all graphs with m vertices as induced subgraphs. In [10], we apply Theorem 1.4 to obtain a variant of this result, which allows one to count the number of copies (not necessarily induced) of some fixed 3-uniform hypergraph in hypergraphs satisfying a property similar to $Q(\eta, \delta, \alpha)$, as long as they are subhypergraphs of sufficiently "jumbled" 3-uniform sparse hypergraphs.

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