COMPLETE PARTITE SUBGRAPHS IN DENSE HYPERGRAPHS

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Abstract. For a given r-uniform hypergraph $\mathcal{F}$ we study the largest blow-up of $\mathcal{F}$ which can be guaranteed in every large r-uniform hypergraph with many copies of $\mathcal{F}$. For graphs this problem was addressed by Nikiforov, who proved that every n-vertex graph that contains $\Omega(n^\ell)$ copies of the complete graph K_ℓ must contain a complete ℓ-partite graph with $\Omega(\log n)$ vertices in each class. We give another proof of Nikiforov's result, make very small progress towards that problem for hypergraphs, and consider a Ramsey-type problem related to a conjecture of Erdős and Hajnal.

1. Introduction

We say a subhypergraph $\mathcal{B}$ of an r-uniform hypergraph $\mathcal{H}$ is a blow-up of an r-uniform hypergraph $\mathcal{F}$ if it is isomorphic to a hypergraph which is obtained from $\mathcal{F}$ by replacing every vertex v of $\mathcal{F}$ by an independent set U_v and for every edge $e = \{v_1, \dots, v_r\}$ of $\mathcal{F}$ we add a complete r-partite r-uniform hypergraph spanned by $U_{v_1}, \dots, U_{v_r}$. We say a blow-up of $\mathcal{F}$ is a t-blow-up if $|U_v| \geq t$ for every $v \in V(\mathcal{F})$. This note concerns the size of the largest blow-up of a given hypergraph $\mathcal{F}$ in a large hypergraph $\mathcal{H}$ which contains many copies of $\mathcal{F}$.

One of the earliest results concerning blow-ups in dense graphs is due to Kövari, Sós, and Turán [12]. In fact, it follows from their result that for sufficiently large n every n-vertex graph with cn^2 edges contains a t-blow-up of K_2 (the graph consisting of a single edge) where $t = \gamma \log n$ and $\gamma > 0$ only depends on c (we write log for the logarithm with base 2 here). Indeed such a result appeared implicitly already in the work of Erdős and Stone [8]. This was generalized by Erdős [4] to r-uniform hypergraphs $r \geq 3$. Erdős showed that for sufficiently large n every r-uniform hypergraph on n vertices with cn^r edges contains a t-blow-up of $K_r^{(r)}$ (the r-uniform hypergraph consisting of a single edge) where

$$t \geq \gamma (\log n)^{\frac{1}{r-1}} \tag{1}$$

and $\gamma > 0$ only depends on c.

Note that the result of Erdős implies the following: if an n-vertex r-uniform hypergraph $\mathcal{H}$ contains at least cn^ℓ copies of $K_\ell^{(r)}$, the complete r-uniform hypergraph

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on ℓ vertices, then $\mathcal{H}$ contains a t-blow-up of $K_\ell^{(r)}$ for some

$$t \geq \gamma(\log n)^{\frac{1}{\ell-1}}\,.$$

For that we consider an auxiliary ℓ-uniform hypergraph on the same vertex set, where every edge corresponds to a copy of $K_\ell^{(r)}$. Applying the result of Erdős to to this auxiliary hypergraph yields a t-blow-up of $K_\ell^{(\ell)}$, which corresponds to a t-blow-up of $K_\ell^{(r)}$ in $\mathcal{H}$.

However, applying (1) here seems to be wasteful. In fact, for graphs Nikiforov [15] obtained a much better bound by an elegant application of the Kövari-Sós-Turán theorem.

Theorem 1 (Nikiforov [15])**.** *For every $\ell \geq 3$ and $c > 0$ there exist $\gamma > 0$ and n_0 such that every graph $G = (V, E)$ on $n \geq n_0$ vertices which contains cn^ℓ copies of K_ℓ contains a t-blow-up of K_ℓ for some $t \geq \gamma \log n$.*

We obtained a different proof of Theorem 1, which we present in Section 2. A simple probabilistic argument shows that up to the constant γ Theorem 1 is optimal. In view (1) and Theorem 1 the following problem for hypergraphs seems plausible.

Problem 2. *For all integers $\ell > r > 2$, every r-uniform hypergraph $\mathcal{F}$ with $V(\mathcal{F}) = [\ell]$, and every $c > 0$ does there exist $\gamma > 0$ and n_0 such that the following holds? If an r-uniform hypergraph $\mathcal{H} = (V, E)$ on $n = |V| \geq n_0$ vertices contains at least cn^ℓ copies of $\mathcal{F}$, then $\mathcal{H}$ contains a t-blow-up of $\mathcal{F}$ for some*

$$t \geq \gamma(\log n)^{\frac{1}{r-1}}\,.$$

Clearly, it would suffice to solve Problem 2 for $\mathcal{F} = K_\ell^{(r)}$ and the results of Kövari, Sós, and Turán [12], Erdős [4], and Nikiforov [15] answer the problem in an affirmative way for $r = 2$ and for $\ell = r > 2$. A Ramsey-type version of Problem 2 for $r = 3$ was solved by Conlon, Fox, and Sudakov [3]. Here, we make very little progress on this problem and answer it affirmatively for some special hypergraphs (see Section 4).

Perhaps the first open case of Problem 2 is when $\mathcal{F} = K_{4-}^{(3)}$, i.e., the 3-uniform hypergraph on four vertices with three edges. For that case one straightforward approach is the following: First one applies an averaging argument to $\mathcal{H}$ to locate a set T of $\Omega(\sqrt{\log n})$ vertices such that their joint link graph G_T (i.e., the graph consisting of all those pairs of vertices that form a hyperedge in $\mathcal{H}$ with every vertex in T) contains $n^3/\exp(O(\sqrt{\log n}))$ triangles. Then one would have to show that G_T contains an $\Omega(\sqrt{\log n})$-blow-up of the triangle K_3. It seems plausible that such a statement might be true in general and we put forward the following problem.

Problem 3. *For every $C > 0$ does there exist $\gamma > 0$ and n_0 such that the following holds? If G is a graph on $n \geq n_0$ vertices which contains at least $n^3/\exp(C\sqrt{\log n})$ copies of K_3, then G contains a t-blow-up of K_3 for some $t \geq \gamma\sqrt{\log n}$.*

If one would like to extend the strategy discussed above to move from $K_{4-}^{(3)}$ to the $K_4^{(3)}$, then one would consider those hyperedges e of $\mathcal{H}$ which are supported by a triangle in G_T. In particular, every such edge e forms a $K_4^{(3)}$ with every vertex in T and if one could show that those edges still span a t-blow-up of an hyperedge for $t = \Omega(\sqrt{\log n})$, then one would solve Problem 2 for the $K_4^{(3)}$.

Such an approach leads to the study of *relatively dense* 3-uniform hypergraphs with respect to an underlying graph. For a graph G and a 3-uniform hypergraph $\mathcal{H}$ on the same vertex set V we say G *underlies* $\mathcal{H}$ if the three vertices of every edge in $\mathcal{H}$ span a triangle in G. By $d(\mathcal{H}|G)$ we denote the relative density of $\mathcal{H}$ with respect to G defined to be the number of edges of $\mathcal{H}$ divided by the number of triangles in G (we only consider graphs containing at least one triangle here). Our next contribution is the following result.

Theorem 4. *For every $K > 0$ there exists some $\beta > 0$ such that for every $c > 0$ there exists n_0 such that the following holds.*

If an n-vertex graph G with at least cn^3 triangles underlies a 3-uniform hypergraph $\mathcal{H}$ and $d(\mathcal{H}|G) \geq (1-\beta)$, then $\mathcal{H}$ contains a $K\sqrt{\log n}$-blow-up of one triplet, i.e., $\mathcal{H}$ contains a complete 3-partite, 3-uniform hypergraph with vertex classes of size at least $K\sqrt{\log n}$.

Note that Theorem 4 is an extension of the result of Erdős [4] for 3-uniform hypergraphs. In fact, for G being the complete graph it was proved by Erdős. However, Theorem 4 shows that $K \to \infty$ as $\beta \to 0$ independent of c. Moreover, letting $\mathcal{H}$ be a random selection of $(1-\beta)cn^3$ triangles from an arbitrary graph with cn^3 triangles shows that Theorem 4 is best possible.

Theorem 4 has some interesting consequences related to a well known problem of Erdős and Hajnal from [6] (see also [5, p. 95]). Those authors conjectured that for every graph F there exists an $\varepsilon > 0$ such that for sufficiently large n every n-vertex graph G that does not contain an induced copy of F, must have the property that either G or its complement must contain a clique K_t for $t = n^{\varepsilon}$.

Despite a lot of effort this conjecture is still wide open. On the other hand, the variant of the conjecture when the clique K_t is replaces by a bipartite graph $K_{t,t}$ can be shown fairly easily (see, e.g. [5, Proposition 1]). A standard application of the regularity method for hypergraphs together with Theorem 4 yields a first step towards the partite version of the Erdős-Hajnal problem for hypergraphs.

Corollary 5. *For every 3-uniform hypergraph $\mathcal{F}$ and every constant $K > 0$ there exists n_0 such that the following holds. If a 3-uniform hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices contains no induced copy of $\mathcal{F}$, then either $\mathcal{H}$ or its complement contains a t-blow-up of a single edge $K_3^{(3)}$ for some $t \geq K\sqrt{\log n}$.*

We remark that Conlon, Fox, and Sudakov [2] recently showed that the constant K in Corollary 5 can be replaced by $(\log n)^{\delta}$ for some sufficiently small constant $\delta > 0$ depending on $\mathcal{F}$.

1.1. **Organization.** We begin in the next section with an alternative proof of Nikiforov's theorem, Theorem 1. In Section 3 we prove Theorem 4 and we sketch the proof of Corollary 5. We close with a few positive examples concerning Problem 2 in Section 4.

2. Complete partite subgraphs in graphs with many cliques

Nikiforov's elegant proof of Theorem 1 appears to be tailored for graphs. The following proof is somewhat different and we hoped that it may pave the way for extensions of Theorem 1. Similarly, as in the original proof we will make use of the Kövari-Sós-Turán theorem.

Theorem 6 (Kövari, Sós & Turán 1954)**.** *If $G = (A\dot\cup B, E)$ is a bipartite graph and for some integer s and t we have*

$$(t-1)\binom{|A|}{s} < |B|\binom{|E|/|B|}{s} \tag{2}$$

then G contains a complete bipartite graph $K_{s,t}$ with the vertex class of size s contained in A and the vertex class of size t contained in B.

In particular, for every $c > 0$ there exists $\gamma_{\mathrm{KST}} > 0$ and n_0 such that every bipartite graph with $n \geq n_0$ vertices in each partition class and with at least cn^2 edges contains a copy of $K_{t,t}$ for some $t \geq \gamma_{\mathrm{KST}} \log n$. □

We will use the following notation. Let $G = (V, E)$ be a graph. For two disjoint sets U and $W \subseteq V$ we write $E_G(U, W)$ for the set of edges in G with one vertex in U and one vertex in W and set $e_G(U, W) = |E_G(U, W)|$. For a vertex $v \in V$, we denote by $N_G(v)$ the *set of neighbours* of v in G, i.e., $N_G(v) = \{u \in V\colon\ uv \in E\}$. For two vertices u, $v \in V$ we denote by $N_G(u, v)$ the *joint neighbourhood* $N_G(u) \cap N_G(v)$. More generally, for a set $U \subseteq V$ we write $N_G(U)$ for $\bigcap_{u\in U} N_G(u)$.

We continue with an alternative proof of Nikiforov's theorem.

Proof of Theorem 1. First we will only consider the case $\ell = 3$. We comment on the general case at the end of the proof. A simple averaging argument shows that we may assume without loss of generality that G is a balanced tripartite graph with vertex set $X\dot\cup Y\dot\cup Z$ of size $|X| = |Y| = |Z| = n$.

For given $c > 0$ let $\gamma_{\mathrm{KST}} > 0$ be the constant guaranteed by Theorem 6 such that every bipartite graph with density $c/4$ and vertex classes of size at least m (for sufficiently large m) contains a $K_{t',t'}$ for $t' \geq \gamma_{\mathrm{KST}} \log m$. We set

$$\gamma = \min\left\{\frac{c}{20\log(8/c)}, \frac{\gamma_{\mathrm{KST}}}{2}\right\}. \tag{3}$$

For sufficiently large n let $G = (X\dot\cup Y\dot\cup Z, E_G)$ be a tripartite graph with vertex sets of size $|X| = |Y| = |Z| = n$ which contains cn^3 triangles. We fix auxiliary constants

$$s = \left\lfloor \frac{\log(n)}{2\log(4/c)} \right\rfloor \quad \text{and} \quad r = \left\lfloor \frac{c\log(n)}{16\log(8/c)} \right\rfloor < \frac{cs}{8}. \tag{4}$$

In a first step we remove all edges from $E_G(X, Y)$ which are contained in only a few triangles. More precisely, we remove all edges xy with

$$N_G(x, y) < cn/2.$$

This way we destroy at most $cn^3/2$ triangles in G and, hence, at least $cn^3/2$ triangles are left after this deletion in the resulting graph. Let G_1 be the resulting graph. Consequently, at least $cn^2/2$ edges between X and Y remain in G_1. We select a subset $X' \subseteq X$ with at least s elements such that for

$$Y' = Y \cap N_{G_1}(X')$$

we have

$$|Y'| \geq \left(\frac{c}{4}\right)^s n \overset{(4)}{>} \sqrt{n}.$$

In fact, such a set X' exists since in view of the fact that $e_{G_1}(X, Y) \geq cn^2/2$ we have

$$\sum_{y\in Y}\binom{|N_{G_1}(y)\cap X|}{s} \geq n\binom{cn/2}{s} \geq \left(\frac{c}{4}\right)^s n\binom{n}{s}.$$

We denote the induced subgraph of G_1 on $X'\dot\cup Y'\dot\cup Z$ by G'. Note that by the definition of Y' the bipartite subgraph $G'[X',Y']$ is complete and $G'[X',Y'] \subseteq G_1[X',Y']$. Consequently, every edge $xy \in E_{G'}(X',Y')$ satisfies

$$|N_{G'}(x,y)| = |N_{G_1}(x,y)| = |N_G(x,y)| \geq cn/2$$

and, hence, the graph G' contains at least $c|X'||Y'|n/2$ triangles.

Next we repeat the same argument for edges between X' and Z. For that we remove all edges $xz \in E_{G'}(X',Z)$ for which $N_{G'}(x,z) < c|Y'|/4$. Similarly to the argument above, at least $c|X'||Y'|n/4$ triangles remain and, hence, $c|X'|n/4$ edges will be left between X' and Z after this deletion. Let $G_2 \subseteq G'$ be the graph consisting of the remaining edges. As a result we have

$$\sum_{z\in Z}\binom{|N_{G_2}(z)\cap X'|}{r} \geq n\binom{c|X'|/4}{r} \geq \left(\frac{c}{8}\right)^r n\binom{|X'|}{r}$$

and, hence, there exists a subset $X'' \subseteq X'$ of size r such that the joint neighbourhood $Z' = Z \cap \bigcap_{x\in X''} N_{G'}(x)$ satisfies

$$|Z'| \geq \left(\frac{c}{8}\right)^r n \overset{(4)}{>} \sqrt{n}\,.$$

Moreover, since every edge xz between X'' and Z' is contained in at least $c|Y'|/4$ triangles the induced subgraph $G'' = G[X'',Y',Z']$ contains at least $c|X''||Y'||Z'|/4$ triangles and $e_{G''}(Y',Z') \geq c|Y'||Z'|/4$. Owing to Theorem 6 we have $G''[Y',Z']$ contains a $K_{t',t'}$ for

$$t' \geq \gamma_{\mathrm{KST}}\log(\sqrt{n})\,.$$

Consequently, G contains a t-blow-up of K_3 for

$$t \overset{(3)}{\geq} \min\{r, \gamma_{\mathrm{KST}}\log(n)/2\} \geq \gamma\log(n)\,,$$

which concludes the proof of Theorem 1 for $\ell = 3$.

For general ℓ we proceed by induction and consider an ℓ-partite graph G with vertex classes $X_1,\dots,X_\ell$ of size n and cn^ℓ copies of K_ℓ. Similarly, as above we first locate a subset $X_1' \subseteq X_1$ of size $\Omega(\log n)$ such that the joint neighbourhood X_2' of X_1' in X_2 is of size at least $\sqrt{n}$ and the subgraph induced on $X_1', X_2', X_3, \dots, X_\ell$ contains at least $(c/2)|X_1^2||X_2'|n^{\ell-2}$ copies of K_ℓ. Note that this time in the n-element set X_1 we only located a subset of size $\Omega(\log n)$.

However, in the same way as we located the subsets X'' in the set X' in proof for $\ell = 3$, we can find a subset $X_1'' \subseteq X_1'$ still of size $\Omega(\log n)$ such that the joint neighbourhood X_3' of X_1'' in X_3 has size $\sqrt{n}$ and the subgraph induced on $X_1'', X_2', X_3', X_4, \dots, X_\ell$ contains at least $(c/4)|X_1''||X_2'||X_3'|n^{\ell-3}$ copies of K_ℓ.

We then repeat this argument $\ell-3$ more times and end up with sets $X_1^{(\ell-1)}$ and $X_2',\dots,X_\ell'$ such that

- $|X_1^{(\ell-1)}| = \Omega(\log n)$ and $|X_2'|,\dots,|X_\ell'| > \sqrt{n}$,
- $X_i' \subseteq \bigcap_{x\in X_1^{(\ell-1)}} N_G(x)$, and
- the number of copies of K_ℓ in the induced subgraph $G[X_1^{(\ell-1)}, X_2', \dots, X_\ell']$ are at least $(c/2^{\ell-1})|X_1^{(\ell-1)}||X_2'|\cdots|X_\ell'|$.

In particular, G induced on $X_2',\dots,X_\ell'$ contains at least $(c/2^{\ell-1})|X_2'|\cdots|X_\ell'|$ copies of $K_{\ell-1}$, which allows us to apply the induction assumption. Consequently, G

induced on $X'_2, \dots, X'_\ell$ contains a t'-blow-up of $K_{\ell-1}$ for $t' = \Omega(\log(\sqrt{n}))$, which together with $X_1^{(\ell-1)}$ yields an $\Omega(\log n)$-blow-up of K_ℓ in G. □

3. Large 3-partite 3-uniform hypergraphs in relatively dense hypergraphs

In this section we verify Theorem 4 and discuss the proof of Corollary 5.

Proof of Theorem 4. First we note that without loss of generality we may assume that G (and hence $\mathcal{H}$) is a balanced tripartite (hyper)graph. In fact, for an n-vertex graph $G = (V, E)$ consider the tripartite graph $G' = (V', E') = G \times K_3$ defined by $V' = V \times [3]$ and $\{(x, i), (y, j)\}$ is an edge in E' if and only if $xy \in E$ and $i \neq j$. The graph G' is tripartite and every triangle in G appears with all six different labelings in G'. Similarly we define $\mathcal{H}' = \mathcal{H} \times K_3^{(3)}$ on the vertex set V' with $\{(x, 1), (y, 2), (z, 3)\}$ being an edge if and only if $xyz \in E(\mathcal{H})$. Every edge of $\mathcal{H}$ appears six times in $\mathcal{H}'$.

As a result we constructed a tripartite hypergraph $\mathcal{H}'$ and a tripartite graph G' with $d(\mathcal{H}'|G') = d(\mathcal{H}|G)$. Now a tripartite version of Theorem 4 yields a copy of $K^{(3)}_{t,t,t}$ in $\mathcal{H}'$ with $t \geq K\sqrt{\log n}$. This copy of $K^{(3)}_{t,t,t}$ may contain copies of the same vertex in more than one of the three partition classes. However, in each class we can select a set of at least $t/3$ vertices such that these collisions can be avoided. Consequently, there is a copy of $K^{(3)}_{s,s,s} \subseteq \mathcal{H}$ with $s \geq K\sqrt{\log n}/3$, which suffices for this reduction and below we may assume that G and $\mathcal{H}$ are balanced tripartite.

Let $K > 0$ be given we fix $\beta > 0$ sufficiently small so that

$$-4K^2 \log(1 - 4\beta) < 1\,. \tag{5}$$

Let $c > 0$ be given. Without loss of generality we may assume that

$$0 < c \leq \beta < 1/6\,. \tag{6}$$

Finally, we let n_0 be sufficiently large such that for every $n \geq n_0$ we have

$$K\sqrt{\log n} + 1 \leq n\frac{\beta^2 c}{8}\left((1-4\beta)\left(\beta\frac{c}{4}\right)^{25/\beta^3 c}\right)^{K\sqrt{\log n}+1} \tag{7}$$

and

$$K\sqrt{\log n} + 1 < n\left(\left(\beta^2\frac{c}{8}\right)^{4/\beta}\right)^{K\sqrt{\log n}+1}(1-4\beta)^{(K\sqrt{\log n}+1)^2}\,. \tag{8}$$

Bounding $K\sqrt{\log n} + 1$ by $2K\sqrt{\log n}$, we observe that in fact condition (8) holds for sufficiently large n due to the choice of β in (5).

Let $G = (X\dot\cup Y\dot\cup Z, E_G)$ be a tripartite graph with $|X| = |Y| = |Z| = n$ and with at least cn^3 triangles. Let $V = V(G) = X\dot\cup Y\dot\cup Z$ and denote by $\mathcal{K}_3(G)$ the set of triangles in G, i.e.,

$$\mathcal{K}_3(G) = \big\{\{x, y, z\} \subseteq V\colon\ xy,\ xz,\ \text{and } yz \in E_G\big\}\,.$$

Let $\mathcal{H} \subseteq \mathcal{K}_3(G)$ be a 3-uniform hypergraph satisfying $d(\mathcal{H}|G) \geq 1 - \beta$. We set

$$t = \left\lceil K\sqrt{\log n}\right\rceil \tag{9}$$

and will show that $K^{(3)}_{t,t,t} \subseteq \mathcal{H}$.

For the proof we also fix the following two auxiliary constants $r > s > t$

$$s = \left\lceil \frac{t}{\beta(1-3\beta)} \right\rceil \leq \frac{3t}{\beta} \qquad \text{and} \qquad r = \left\lceil \frac{8s}{\beta^2 c} \right\rceil \leq \frac{25t}{\beta^3 c}\,. \tag{10}$$

For later reference we observe that the conditions on n_0 in (7) and (8) yield

$$t \leq \frac{1}{2}(1-4\beta)^t \beta^2 \frac{c}{4} \left(\beta \frac{c}{4}\right)^r n \tag{11}$$

and

$$t < \left(\beta^2 \frac{c}{8}\right)^s \left((1-4\beta)^t \beta^2 \frac{c}{8}\right)^t n\,. \tag{12}$$

The proof consists of three steps, each being rendered in one of the claims (Claims 1-3) below. We first state these claims and conclude the proof of Theorem 4 from it, before we verify the claims.

Applying these claims succesively, we will be choosing sets X', X'', and X''' with $X \supseteq X' \supseteq X'' \supseteq X'''$ and $|X'| = r$, $|X''| = s$, and $|X'''| = t$ so that a more and more refined structure can be deduced for the hypergraph restricted to those subsets and their neighbourhoods.

In the first step we select some special set of vertices $X' \subseteq X$ with $|X'| = r$ and with "large" joint neighbourhood in Y, so that the induced graph and hypergraph on X' it neighbourhood in Y and Z still enjoy similar properties, as G and $\mathcal{H}$ at the beginning (see *(iv)* and *(iii)* in the following claim).

Claim 1. *There exist some sets $X' \subseteq X$ and $Y' \subseteq Y$ such that for the induced subgraph $G' = G[X', Y', Z]$ and for $\mathcal{H}' = \mathcal{H} \cap \mathcal{K}_3(G')$ we have*

- *(i)* $|X'| = r$ *and* $|Y'| \geq (\beta c/4)^r n$,
- *(ii)* $Y' \subseteq N_{G'}(X')$,
- *(iii)* $d(\mathcal{H}'|G') \geq 1 - 2\beta$, *and*
- *(iv)* $|\mathcal{K}_3(G')| \geq \beta c|X'||Y'|n/2$.

While Claim 1 gives us some control over pairs $(x, y) \in X' \times Y'$ the next claim achieves the same for pairs from $X'' \times Z'$ for suitably chosen subsets $X'' \subseteq X'$ and $Z' \subseteq Z$.

Claim 2. *There exist some sets $X'' \subseteq X'$ and $Z' \subseteq Z$ such that for the induced subgraph $G'' = G[X'', Y', Z']$ and for $\mathcal{H}'' = \mathcal{H} \cap \mathcal{K}_3(G'')$ we have*

- (I) $|X''| = s$ *and* $|Z'| \geq (\beta^2 c/8)^s n$,
- (II) $Z' \subseteq N_{G''}(X'')$,
- (III) $d(\mathcal{H}''|G'') \geq 1 - 3\beta$, *and*
- (IV) $|\mathcal{K}_3(G'')| \geq \beta^2 c|X''||Y'||Z'|/4$.

The next step concerns the pairs between Y' and Z'. We will select some subset $X''' \subseteq X''$ and a set of edges $F \subseteq E_{G''}(Y', Z')$ such that $xyz \in E(\mathcal{H}'')$ for every $yz \in F$ and $x \in X'''$.

Claim 3. *There exist $X''' \subseteq X''$ and a set of edges $F \subseteq E_{G''}(Y', Z')$ such that*

- *(a)* $|X'''| = t$ *and* $|F| \geq (1-4\beta)^t \beta^2 c|Y'||Z'|/4$ *and*
- *(b)* $xyz \in E(\mathcal{H})$ *for every* $x \in X'''$ *and* $yz \in F$.

One can check that the bounds for the sizes of Y', Z' and F (see (i) of Claim 1, (I) of Claim 2, and (a) of Claim 3) combined with (11) and (12) imply

$$(t-1)\binom{|Y'|}{t} < |Z'|\binom{|F|/|Z'|}{t}.$$

Consequently, Theorem 6 yields a copy of $K_{t,t}$ in the bipartite graph $(Y'\dot\cup Z', F)$. This copy of $K_{t,t}$ together with X''' spans a copy of $K^{(3)}_{t,t,t}$ in $\mathcal{H}$. This concludes the proof of Theorem 4. ☐

Proof of Claim 1. In the proof the following obvious identities will be used. If uv is an edge in G, then $|N_G(u,v)|$ is the number of triangles containing this edge and

$$\begin{aligned}|\mathcal{K}_3(G)| &= \sum_{xy\in E_G(X,Y)} |N_G(x,y)| \\ &= \sum_{xz\in E_G(X,Z)} |N_G(x,z)| = \sum_{yz\in E_G(Y,Z)} |N_G(y,z)|\,.\end{aligned} \tag{13}$$

Moreover, for an edge $uv\in E_G$ we denote by $N_{\mathcal{H}}(u,v)$ the set of those vertices $w\in V$ for which uvw is a hyperedge of $\mathcal{H}$. Clearly,

$$|\mathcal{H}| = \sum_{xy\in E_G(X,Y)} |N_{\mathcal{H}}(x,y)| = \sum_{xz\in E_G(X,Z)} |N_{\mathcal{H}}(x,z)| = \sum_{yz\in E_G(Y,Z)} |N_{\mathcal{H}}(y,z)| \tag{14}$$

and since $\mathcal{H}\subseteq\mathcal{K}_3(G)$ we have

$$N_{\mathcal{H}}(u,v)\subseteq N_G(u,v)\,.$$

First we remove every edge xy from $E_G(X,Y)$ for which either

(a) $|N_{\mathcal{H}}(x,y)| < (1-2\beta)|N_G(x,y)|$ or
(b) $|N_G(x,y)| < \beta cn/2$.

Let G_a be the tripartite subgraph of G, which we obtain after removal of all edges from $E_G(X,Y)$ due to reason (a) and set $\mathcal{H}_a = \mathcal{H}\cap\mathcal{K}_3(G_a)$.

Owing to the identities (13) and (14) we have

$$\begin{aligned}|E(\mathcal{H})\setminus E(\mathcal{H}_a)| &= \sum_{xy\notin E_{G_a}(X,Y)} |N_{\mathcal{H}}(x,y)| \overset{(a)}{\le} \sum_{xy\notin E_{G_a}(X,Y)} (1-2\beta)|N_G(x,y)| \\ &\le (1-2\beta)\sum_{xy\in E_G(X,Y)} |N_G(x,y)| = (1-2\beta)|\mathcal{K}_3(G)|\end{aligned}$$

Moreover, since $|E(\mathcal{H})|\ge(1-\beta)|\mathcal{K}_3(G)|$ by assumption of Theorem 4 we have

$$|\mathcal{K}_3(G_a)| \ge |E(\mathcal{H}_a)| = |E(\mathcal{H})| - |E(\mathcal{H})\setminus E(\mathcal{H}_a)| \ge \beta|\mathcal{K}_3(G)| \ge \beta cn^3\,.$$

Next we remove the edges from $E_{G_a}(X,Y)$ which fail to satisfy condition (b) and let G' be the resulting graph and set $\mathcal{H}' = \mathcal{H}\cap\mathcal{K}_3(G')$. Since $e_{G_a}(X,Y)\le n^2$ we destroy at most $\beta cn^3/2$ triangles this way. Consequently,

$$|\mathcal{K}_3(G')| \ge |\mathcal{K}_3(G_a)| - \beta cn^3/2 \ge \beta cn^3 - \beta cn^3/2 = \beta cn^3/2$$

and, therefore, $e_{G'}(X,Y)\ge\beta cn^2/2$. Note that every edge xy in $E_{G'}(X,Y)$, then satisfies conditions (a) and (b).

A simple double counting argument as was carried out by Kövari, Sós, and Turán in the proof of Theorem 6 will yield a complete bipartite subgraph in $G'[X,Y]$ with the desired properties. In fact,

$$\sum_{y\in Y}\binom{|N_{G'}(y)\cap X|}{r}\geq n\binom{e_{G'}(X,Y)/n}{r}\geq n\binom{\beta cn/2}{r}\geq n\left(\frac{\beta c}{4}\right)^r\binom{n}{r},$$

where we used $\beta cn/4\geq r$ (see (10) and (11)) for the last inequality.

Hence, there exists an r-element subset $X'\subseteq X$ with $|N_{G'}(X')|\geq(\beta c/4)^r n$ and we set $Y'=N_{G'}(X')$. By definition X' and Y' satisfy properties *(i)* and *(ii)* of Claim 1. Moreover, properties *(iii)* and *(iv)* follow from properties *(a)* and *(b)* of the graph G'. □

The proof of Claim 2 is almost identical to the proof of Claim 1.

Proof of Claim 2. Let G' and $\mathcal{H}'$ be given by Claim 1. We remove every edge xz from $E_{G'}(X',Z)$ for which

(*c*) $|N_{\mathcal{H}'}(x,z)|\geq(1-3\beta)|N_{G'}(x,z)|$ or
(*d*) $|N_{G'}(x,z)|\geq\beta^2c|Y'|/4$.

Let G_c be the tripartite subgraph of G', which we obtain after we remove all edges from $E_{G'}(X',Z)$ which violate condition *(c)* and let $\mathcal{H}_c=\mathcal{H}\cap\mathcal{K}_3(G_c)$. Owing to *(iii)* of Claim 1 we have $|\mathcal{H}'|\geq(1-2\beta)|\mathcal{K}_3(G')|$. Similarly as in the proof of Claim 1 one can verify that $|\mathcal{H}'\setminus\mathcal{H}_c|\leq(1-3\beta)|\mathcal{K}_3(G')|$ and, hence, we obtain

$$|\mathcal{K}_3(G_c)|\geq|\mathcal{H}_c|\geq\beta|\mathcal{K}_3(G')|\geq\beta^2c|X'||Y'|n/2\,,$$

where the last inequality follows from *(iv)* of Claim 1. Now delete those edges from $E_{G_c}(X',Z)$ which fail to satisfy *(d)*. Let G'' be the resulting graph and set $\mathcal{H}''=\mathcal{H}\cap\mathcal{K}_3(G'')$. Since $e_{G_c}(X',Z)\leq|X'|n$ we destroy at most $\beta^2c|X'||Y'|n/4$ triangles this way. Consequently,

$$|\mathcal{K}_3(G'')|\geq|\mathcal{K}_3(G_c)|-\beta^2c|X'||Y'|n/4\geq\beta^2c|X'||Y'|n/4$$

and, therefore, $e_{G''}(X',Z)\geq\beta^2c|X'|n/4$. By definition of G'' every edge xz in $E_{G''}(X',Z)$ satisfies conditions *(c)* and *(d)*.

A similar double counting argument as in the proof of Claim 1 yields

$$\sum_{z\in Z}\binom{|N_{G''}(z)\cap X'|}{s}\geq n\binom{e_{G''}(X',Z)/n}{s}\geq n\binom{\beta^2c|X'|/4}{s}\geq n\left(\frac{\beta^2c}{8}\right)^s\binom{|X'|}{s},$$

where we used $\beta^2c|X'|/8=\beta^2cr/8\geq s$ (see (10)) for the last inequality. Therefore, there exists an s-element set $X''\subseteq X'$ with $|N_{G''}(X'')|\geq(\beta^2c/8)^s n$. We set $Z'=N_{G''}(X'')$. By definition X'' and Z' satisfy properties (I) and (II) of Claim 2. Moreover, properties (III) and (IV) follow from properties *(c)* and *(d)* of the graph G''. □

Proof of Claim 3. Owing to part *(ii)* of Claim 1 and to part (II) of Claim 2 the two bipartite graphs $G''[X'',Y']\subseteq G'[X',Y']$ and $G''[X'',Z']$ are complete. Consequently, every edge $yz\in E_{G''}(Y,Z)$ belongs to exactly $|X''|$ triangles in G''. Hence property (III) of Claim 2 implies that on average an edge $yz\in E_{G''}(Y,Z)$ is contained in $(1-3\beta)|X''|$ hyperedges of $\mathcal{H}''$. Consequently,

$$\sum_{yz\in E_{G''}(Y,Z)} \binom{N_{\mathcal{H}''}(y,z)}{t} \geq e_{G''}(Y,Z)\binom{(1-3\beta)|X''|}{t} \geq (1-4\beta)^t e_{G''}(Y,Z)\binom{|X''|}{t},$$

where we used $(1-3\beta)|X''|-t \geq (1-4\beta)|X''|$, which follows since $\beta|X''| = \beta s \geq t$ by (10). Hence, there exists a t-set $X''' \subseteq X''$ such that at least $(1-4\beta)^t e_{G''}(Y,Z)$ of the edges of $E_{G''}(Y,Z)$ form a hyperedge with every $x \in X'''$. It follows from (IV) of Claim 2 that $e_{G''}(Y,Z) \geq \beta^2 c|Y'||Z'|/4$. Consequently, there exists a subset F of $E_{G''}(Y,Z)$ of size at least

$$(1-4\beta)^t e_{G''}(Y,Z) \geq (1-4\beta)^t \beta^2 c|Y'||Z'|/4\,,$$

which satisfies property *(b)* of Claim 3. □

Below we sketch a proof of Corollary 5 based on Theorem 4 and the regularity method for hypergraphs.

Proof of Corollary 5 (sketch). Let a 3-uniform hypergraph $\mathcal{F}$ and a constant $K>0$ be given. Consider an n-vertex, 3-uniform hypergraph $\mathcal{H}$ for sufficiently large n. We assume that neither $\mathcal{H}$ nor its complement contain an s-blow-up $K^{(3)}_{s,s,s}$ of a single edge $K^{(3)}_3$ for any $s \geq K\sqrt{\log n}$. By Theorem 4 this means that there exists a constant $\beta > 0$ such that for any $c > 0$ and sufficiently large n the following holds: If G is a graph on the same vertex set as $\mathcal{H}$ and G contains at least cn^3 triangles, then

$$\beta < d(\mathcal{H}|G) < 1-\beta\,. \tag{15}$$

We will use the hypergraph regularity lemma from [9], which yields some auxiliary constants t and ℓ which can be bounded from above in terms of δ (independent of n and $\mathcal{H}$). In particular, we can ensure the following hierarchy of the constants involved in this proof

$$\frac{1}{K}\,, \frac{1}{|V(\mathcal{F})|} \gg \beta \gg \delta \gg \frac{1}{\ell}, \frac{1}{t} \gg c \gg \frac{1}{n}\,. \tag{16}$$

Below we will use (15) to infer that $\mathcal{H}$ contains an induced copy of $\mathcal{F}$.

To this end we apply the regularity lemma from [9]. We will not describe the concept of regularity of that lemma here, but we will discuss its consequences necessary for the proof. This regularity lemma decomposes the vertex set of $\mathcal{H}$ into t vertex classes $V_1,\dots,V_t$ of size $\lfloor n/t\rfloor$ or $\lceil n/t\rceil$ each and it partitions each of the $\binom{t}{2}$ complete bipartite graphs $K(V_i,V_j)$ into ℓ bipartite graphs P^α_{ij} such that for all but at most $\delta\binom{t}{3}\ell^3$ of the tripartite graphs (so-called *triads*)

$$P^{\alpha_{ij}\alpha_{ik}\alpha_{jk}}_{ijk} = (V_i\dot\cup V_j\dot\cup V_k, P^{\alpha_{ij}}_{ij}\dot\cup P^{\alpha_{ik}}_{ik}\dot\cup P^{\alpha_{jk}}_{jk})$$

for $1 \leq i < j < k \leq t$ and $1 \leq \alpha_{ij}, \alpha_{ij}, \alpha_{jk} \leq \ell$ we have

(a) $P^{\alpha_{ij}\alpha_{ik}\alpha_{jk}}_{ijk}$ contains at least $\frac{1}{2}(\frac{n}{\ell t})^3$ triangles and

(b) $\mathcal{H}$ is δ-regular with respect to $P^{\alpha\beta\gamma}_{ijk}$.

Since almost all triads satisfy properties *(a)* and *(b)* a standard averaging argument (see, e.g., [13]) yields a collection of indicies $1 \leq i_1 \leq \dots \leq i_f \leq t$ where $f = |V(\mathcal{F})|$

and $\alpha_{i_i,i_j} \in [\ell]$ for all $1 \leq i < j \leq f$ such that properties *(a)* and *(b)* hold for all triads given by this selection, i.e., for all $\binom{f}{3}$ triads of the form

$$P^{\alpha_{i_i i_j}\alpha_{i_i i_k}\alpha_{i_j i_k}}_{i_i i_j i_k}.$$

It follows from (15), (16), and property *(a)* that for all those selected triads P satisfy $\beta < d(\mathcal{H}|P) < 1-\beta$. Morover, the regularity lemma also asserts that the involved bipartite graph P^{α}_{ij} are sufficiently regular themselves. This allows us to apply the counting lemma from [14] (see also [11]) in this environment, which then yields an induced copy of $\mathcal{F}$. $\square$

4. A few results towards Problem 2

In view of Erdős' result (see (1)) Problem 2 is true for all r-partite, r-uniform hypergraphs $\mathcal{F}$. In fact, if $\mathcal{F}$ has ℓ vertices and an n-vertex r-uniform hypergraph $\mathcal{H}$ contains $\Omega(n^{\ell})$ copies of $\mathcal{F}$, then it must contain at least $\Omega(n^r)$ hyperedges. Hence, it follows from (1) that $\mathcal{H}$ contains a t-blow-up of $K^{(r)}_r$ for some $t = \Omega((\log n)^{\frac{1}{r-1}})$ and such a t-blow-up contains trivially a (t/ℓ)-blow-up of $\mathcal{F}$.

4.1. A non-partite example.

We solve Problem 2 for the following non-partite r-uniform hypergraph $\mathcal{S}_r$. The hypergraph $\mathcal{S}_r$ has vertex set $\{x_1,\dots,x_{r-1},y_1,\dots,y_r\}$ and it contains the edge $\{y_1,\dots,y_r\}$ and the r edges of the form $\{x_1,\dots,x_{r-1},y_j\}$ with $j\in[r]$.

Proposition 7. *For every integer $r\geq 3$ and every $c>0$ there exist $\gamma>0$ and n_0 such that every r-uniform hypergraph $\mathcal{H}$ on $n\geq n_0$ vertices which contains at least cn^{2r-1} copies of $\mathcal{S}_r$ must contain a t-blow-up of $\mathcal{S}_r$ for some $t\geq\gamma(\log n)^{\frac{1}{r-1}}$.*

The proof of Proposition 7 relies on Lemma 8, which is variant of Erdős' result (see also [16, Theorem 2]). For an r-uniform hypergraph $\mathcal{H}$ and an $(r-1)$-tuple $\vec{v}=(v_1,\dots,v_{r-1})$ of its vertices we denote by $d_{\mathcal{H}}(\vec{v})$ its degree in $\mathcal{H}$, i.e., the number of edges in $\mathcal{H}$ which contain $v_1,\dots,v_{r-1}$.

Lemma 8. *For every integer $r\geq 3$ and every c_1, $c_2>0$, and $C>0$ there exist $\gamma>0$, $D>0$, and n_0 such that for every $n\geq n_0$ the following holds. Let $\mathcal{H}$ be an r-partite, r-uniform hypergraph with vertex classes $V_1,\dots,V_r$ such that $|V_1|=\dots=|V_{r-1}|=n$ and $|V_r|=m\geq c_1(\log n)^{\frac{1}{r-1}}$. Let*

$$F\subseteq\{\vec{v}\in V_1\times\dots\times V_{r-1}\colon\ d_{\mathcal{H}}(\vec{v})\geq c_2 m\}.$$

If

$$|F|\geq\frac{n^{r-1}}{\exp(C(\log n)^{\frac{1}{r-1}})},$$

then there exist subsets $F'\subseteq F$ and $W\subseteq V_r$ such that

- *(i)* $|W|=\gamma(\log n)^{\frac{1}{r-1}}$ *and* $|F'|\geq n^{r-1}\exp(-D(\log n)^{\frac{1}{r-1}})$ *and*
- *(ii)* *for every $(v_1,\dots,v_{r-1})\in F'$ and every $w\in W$ we have $\{v_1,\dots,v_{r-1},w\}$ is an hyperedge in $\mathcal{H}$.*

Proof. Since $d_{\mathcal{H}}(\vec{v})\geq c_2 m$ for every $\vec{v}\in F$ we have

$$\sum_{\vec{v}\in F}\binom{d_{\mathcal{H}}(\vec{v})}{t}\geq|F|\binom{c_2 m}{t}\geq\left(\frac{c_2}{2}\right)^t|F|\binom{m}{t}$$

for every integer $t \le c_2 m/2$. Consequently, there exist a subset $W \subseteq V_r$ with $|W| = t$ and a subset $F' \subseteq F$ of size at least $(c_2/2)^t|F|$ for which *(ii)* holds. Setting

$$t = \left\lfloor \frac{c_1 c_2}{2} (\log n)^{\frac{1}{r-1}} \right\rfloor \le \frac{c_2}{2} m$$

and in view of

$$|F'| \ge \left(\frac{c_2}{2}\right)^t |F| \ge \frac{n^{r-1}}{\exp\left((C + \ln(2/c_2)c_1c_2/2)(\log n)^{\frac{1}{r-1}}\right)},$$

and

$$|W| = t \ge \frac{c_1 c_2}{3} (\log n)^{\frac{1}{r-1}}$$

we infer the lemma for $\gamma = c_1c_2/3$, $D = (C + \ln(2/c_2)c_1c_2/2)$, and sufficiently large n. □

We obtain Proposition 7 from Lemma 8.

Proof of Proposition 7. Similarly as in the proofs of Theorems 1 and 4, we note that it suffices to prove a partite version of Proposition 7. More precisely, we will assume that $\mathcal{H}$ is contained in an n-blow-up of $\mathcal{S}_r$. Let $X_1, \dots, X_{r-1}, Y_1, \dots, Y_r$ be the vertex classes of $\mathcal{H}$. The proof splits into three parts.

In the first step we consider only those hyperedges E_0 of $\mathcal{H}[Y_1, \dots, Y_r]$ with the property, that each of them is contained in at least $cn^{r-1}/2$ copies of $\mathcal{S}_r$. Since this way we ignore at most $cn^{r-1}/2$ copies of $\mathcal{S}_r$ in $\mathcal{H}$, there are at least $cn^{r-1}/2$ copies left. In particular, $|E_0| \ge cn^r/2$ and by an application of Erdős result (see (1)) we locate in E_0 a t_0-blow-up of $K_r^{(r)}$ for $t_0 = \gamma_0(\log n)^{\frac{1}{r-1}}$ for some $\gamma_0 = \gamma_0(c)$. Let $Y_1', \dots, Y_r'$ be the vertex classes of this t_0-blow-up.

In the second step we consider the induced r-uniform subhypergraph $\mathcal{H}_1 = \mathcal{H}[X_1, \dots, X_{r-1}, Y_1', \dots, Y_r']$. Owing to the definition of E_0 the hypergraph $\mathcal{H}_1$ contains at least $(c/2)n^{r-1}|Y_1'| \dots |Y_r'|$ copies of $\mathcal{S}_r$.

Consequently, there exists a set $F_0 \subseteq X_1 \times \dots \times X_{r-1}$ of size at least $(c/4)n^{r-1}$ such that for every $i \in [r]$ every $(x_1, \dots, x_{r-1}) \in F_0$ there exist at least $(c/4)|Y_i'|$ vertices $y \in Y_i'$ such that $\{x_1, \dots, x_{r-1}, y\}$ is an edge in $\mathcal{H}_1$.

The second step consists of r applications of Lemma 8. We will successively select sets $F_1 \subseteq F_0$ and $Y_1'' \subseteq Y_1'$, then $F_2 \subseteq F_1$ and $Y_2'' \subseteq Y_2'$ and so on such that for every $j = 0, 1, \dots, r$ the set F_j is still "large" and, moreover, for every $(r-1)$-tuple $(x_1, \dots, x_{r-1}) \in F_j$ and every $y \in Y_1'' \cup \dots \cup Y_j''$ the r-tuple $\{x_1, \dots, x_{r-1}, y\}$ forms an edge in $\mathcal{H}_1$.

For the first application we apply it to the hypergraph $\mathcal{H}_1[X_1, \dots, X_{r-1}, Y_1']$ and the set F_0. Lemma 8 yields subsets $Y_1'' \subseteq Y_1'$ and $F_1 \subseteq F_0$ satisfying for some constants $\gamma_1 = \gamma_1(c, \gamma_0) = \gamma_1(c) > 0$ and $C_1 = C_1(c, \gamma_1)$

- *(i)* $|Y_1''| = \gamma_1(\log n)^{\frac{1}{r-1}}$ and $|F_1| \ge n^{r-1}\exp(-C_1(\log n)^{\frac{1}{r-1}})$ and
- *(ii)* for every $(x_1, \dots, x_{r-1}) \in F_1$ and every $y \in Y_1''$ we have $\{x_1, \dots, x_{r-1}, y\}$ is an hyperedge in $\mathcal{H}_1$.

We continue with the second application of Lemma 8. For that we note that since by definition of $F_0 \supseteq F_1$ every $\vec{x} \in F_1$ extends to $(c/4)|Y_2'|$ edges in $\mathcal{H}$ with the r-th vertex being a member of Y_2'. Consequently, we can apply Lemma 8 again and obtain subsets $Y_2'' \subseteq Y_2'$ and $F_2 \subseteq F_1$ such that

- *(i)* $|Y_2''| = \gamma_2(\log n)^{\frac{1}{r-1}}$ and $|F_2| \ge n^{r-1}\exp(-C_2(\log n)^{\frac{1}{r-1}})$ and (*ii*) for every $(x_1,\dots,x_{r-1})\in F_2$ and every $y\in Y_2''$ we have $\{x_1,\dots,x_{r-1},y\}$ is an hyperedge in $\mathcal{H}_1$

for some constants γ_2 and C_2 only depending on c.

We repeat this procedure $r-2$ more times and obtain subsets Y_j'' for every $j\in[r]$ and $F_0\supseteq F_1\supseteq F_2\supseteq\dots\subseteq F_r$ such that for every $j\in[r]$ we have

(*i*) $|Y_j''|=\gamma_j(\log n)^{\frac{1}{r-1}}$ and $|F_j|\ge n^{r-1}\exp(-C_j(\log n)^{\frac{1}{r-1}})$ and

(*ii*) for every $(x_1,\dots,x_{r-1})\in F_j$ and every $y\in Y_j''$ we have $\{x_1,\dots,x_{r-1},y\}$ is an hyperedge in $\mathcal{H}_1$

for some constants γ_j and C_j only depending on c.

In the third and last step we consider the set F_r, which we view as an $(r-1)$-partite, $(r-1)$-uniform hypergraph with vertex classes $X_1,\dots,X_{r-1}$. Since

$$|F_r|\ge\frac{n^{r-1}}{\exp(C_r(\log n)^{\frac{1}{r-1}})}\ge\frac{n^{r-1}}{n^{1/s^{r-2}}}$$

for some integer $s\ge\gamma(\log n)^{\frac{1}{r-1}}$ with $\gamma>0$ only depending on C_r, which only depends on the given c. Hence, it follows from Erdős' result [4, Theorem 1] that F_r contains a copy of a s-blow-up of $K_{r-1}^{(r-1)}$. Owing to (*ii*) for every $j\in[r]$ this s-blow-up together with the sets $Y_1'',\dots,Y_r''$ spans a t-blow-up of $\mathcal{S}_r$ for the integer $t=\min\{t_1,\dots,t_r,s\}$. □

4.2. **Partite blow-ups.** As discussed in Section 4.1 for r-partite r-uniform hypergraphs $\mathcal{F}$ a positive answer for Problem 2 is a direct consequence of (1). However, for partite blow-ups (defined below) the problem is different. In this section we address such a version of the problem for tight 3-uniform paths.

A tight path $P_\ell^{(3)}$ on vertex set $[\ell]=\{1,\dots,\ell\}$ consists of all "consecutive" triplets of vertices, i.e., $e\in E(P_\ell^{(3)})$ if and only if $e=\{i,i+1,i+2\}$ for some $i=1,\dots,\ell-2$. Moreover, we say an ℓ-partite hypergraph $\mathcal{H}$ with vertex classes $V_1,\dots,V_\ell$ contains a *partite copy* of an ℓ-vertex hypergraph $\mathcal{F}$ with $V(\mathcal{F})=[\ell]$ if the copy of the vertex $i\in V(\mathcal{F})$ is contained in V_i. We will also say that $\mathcal{H}$ contains a *partite blow-up* of $\mathcal{F}$, if it contains a blow-up with vertex classes $U_1,\dots,U_\ell$ such that $U_i\subseteq V_i$ for every $i\in V(\mathcal{F})$.

Proposition 9. *For all integers $\ell\ge3$ and every $c>0$ there exists $\gamma>0$ and n_0 such that the following holds. If a 3-uniform, ℓ-partite hypergraph $\mathcal{H}$ is contained in an n-blow-up of the tight path $P_\ell^{(3)}$ and it contains at least cn^ℓ partite copies of $P_\ell^{(3)}$, then $\mathcal{H}$ contains a partite t-blow-up of $P_\ell^{(3)}$ with $t\ge\gamma\sqrt{\log n}$.*

The proof of Proposition 9 relies on the following version for 3-uniform hypergraphs of the result of Erdős from [4].

Lemma 10. *For every $c\in(0,1)$ there exist $\gamma>0$ and n_0 such that for every $n\ge n_0$ the following holds. Let $\mathcal{H}=(X\dot\cup Y\dot\cup Z,E)$ be a tripartite 3-uniform hypergraph with $|X|\ge(c/8)\sqrt{\log n}$, $|Y|\ge\sqrt{n}$, and $|Z|=n$. If every pair $(x,y)\in X\times Y$ satisfies $d_{\mathcal{H}}(x,y)\ge cn$, then there exist subsets $X'\subseteq X$, $Y'\subseteq Y$, and $Z'\subseteq Z$ such that*

(*a*) *$|X'|\ge\gamma\sqrt{\log n}$, $|Y'|\ge(c/8)\sqrt{\log n}$, and $|Z'|\ge\sqrt{n}$ and*

(*b*) *$\mathcal{H}[X',Y',Z']$ is a complete, tripartite, 3-uniform hypergraph.*

Proof. The proof relies on two applications of the Kövari-Sós-Turán theorem. We set

$$\gamma=c^2/20\,.$$

Let n be sufficiently large and set

$$r = \lceil \gamma\sqrt{\log n}\rceil < c|X|/2\,, \quad s = \lceil (c/8)\sqrt{\log n}\rceil\,, \quad \text{and} \quad t = \lceil \sqrt{n}\rceil\,. \tag{17}$$

Owing to the minimum degree condition for the pairs in $X \times Y$ we have the lower bound $c|X|$ for the average degree of the pairs in $Y \times Z$. Consequently, we have

$$\sum_{(y,z)\in Y\times Z}\binom{d_{\mathcal{H}}(y,z)}{r} \geq |Y||Z|\binom{c|X|}{r} \geq \left(\frac{c}{2}\right)^r |Y||Z|\binom{|X|}{r}.$$

Hence, in view of $\gamma < 1$ there exist sets $X' \subseteq X$ of size r and $F \subseteq Y \times Z$ of size

$$|F| \geq \left(\frac{c}{2}\right)^r |Y||Z| \geq \left(\frac{c}{2}\right)^{\sqrt{\log n}} |Y||Z| \tag{18}$$

such that for every $(y,z) \in F$ and $x \in X'$ we have that $\{x,y,z\}$ is an edge of $\mathcal{H}$.

Next we apply Theorem 6 to the auxiliary bipartite graph G with vertex partition $Y\dot\cup Z$ and edge set F. Note that due to the assumptions on the sizes of Y and Z and due to (18) it follows from the choice of s and t in (17) that for sufficiently large n we have

$$(t-1)\binom{|Y|}{s} < |Z|\left(\frac{c}{4}\right)^{s\sqrt{\log n}}\binom{|Y|}{s} < |Z|\binom{|F|/|Z|}{s},$$

since $(c/8)\log(4/c) < 1/2$ for every $c \in (0,1)$. Hence, Theorem 6 yields a copy of $K_{s,t}$ in G. Set $Y' \subseteq Y$ be the vertex class of size s and $Z' \subseteq Z$ be the vertex class of size t of this copy of $K_{s,t}$. It then follows that X', Y' and Z' satisfy the conclusion of the lemma. □

Proposition 9 is a consequence of $\ell - 2$ applications of Lemma 10.

Proof of Proposition 9. For given $\ell \geq 3$ and $c > 0$ let $\gamma > 0$ and n_0 be the constants guaranteed by Lemma 10 applied with ℓ and

$$c' = \frac{c}{2(\ell-2)}\,.$$

Let $n \geq n_0$ be sufficiently large and let $\mathcal{H}$ be a 3-uniform hypergraph which is contained in an n-blow-up of the tight path $P_\ell^{(3)}$ with ℓ vertices and which itself contains at least cn^ℓ partite copies of $P_\ell^{(3)}$. Let $V_1, \dots, V_\ell$ be the vertex classes of $\mathcal{H}$.

We first remove all hyperedges from $\mathcal{H}$ which contain a pair which fails to satisfy the following degree condition. For $i = 1, \dots, \ell-2$ and a pair of vertices $(v_i, v_{i+1}) \in V_i \times V_{i+1}$ we denote by $d_{\mathcal{H},i+2}(v_i, v_{i+1})$ the number of neighbours of those two vertices in V_{i+2}, i.e.,

$$d_{\mathcal{H},i+2}(v_i, v_{i+1}) = \big|\{v \in V_{i+2}\colon \{v_i, v_{i+1}, v\} \in E(\mathcal{H})\}\big|\,.$$

Successively, we will keep removing hyperedges $\{v_i, v_{i+1}, v\}$ from $\mathcal{H}$ with $v_i \in V_i$, $v_{i+1} \in V_{i+1}$ and $v \in V_{i-1} \cup V_{i+2}$ for some $i = 1, \dots, \ell-2$ as long as $d_{\mathcal{H},i+2}(v_i, v_{i+1}) < c'n$. Let $\mathcal{H}_1$ be the hypergraph of the remaining edges. By definition $\mathcal{H}_1$ has the property, that if a pair $(v_i, v_{i+1}) \in V_i \times V_{i+1}$ is contained in some edge of $\mathcal{H}_1$, then

$$d_{\mathcal{H},i+2}(v_i, v_{i+1}) \geq c'n\,.$$

Note that since there are $\ell - 2$ choices for i, we removed less than

$$(\ell-2)n^2 \times c'n \times n^{\ell-3} \leq cn^\ell/2$$

copies of $P_\ell^{(3)}$ from $\mathcal{H}$ this way. Hence, there are at least $cn^\ell/2$ partite copies of $P_\ell^{(3)}$ in $\mathcal{H}_1$ left and, consequently, $\mathcal{H}_1[V_1, V_2, V_3]$ contains at least $cn^3/2$ hyperedges. Let F be the set of those pairs in $V_1 \times V_2$ which are contained in at least one (and hence in at least $c'n$) hyperedges in $\mathcal{H}_1[V_1, V_2, V_3]$. Clearly, $|F| \geq cn^2/2$ and, moreover, for sufficiently large n we have

$$\sqrt{n}\binom{|V_1|}{\lceil (c'/8)\sqrt{\log n}\rceil} < n\binom{(1/2)cn^2/|V_2|}{\lceil (c'/8)\sqrt{\log n}\rceil}.$$

Consequently, Theorem 6 yields the existence of a copy of $K_{s,t}$ in F for integers $s \geq (c'/8)\sqrt{\log n}$ and $t \geq \sqrt{n}$. Let V_1' and V_2' be the corresponding vertex sets in V_1 and V_2 with sizes $|V_1'| = s$ and $|V_2'| = t$.

Since every pair of $(v_1, v_2) \in F$ was contained in an hyperedge of $\mathcal{H}_1$, we have $d_{\mathcal{H},3}(v_1, v_2) \geq c'n$. Therefore, we can apply Lemma 10 and obtain vertex sets $W_1 \subseteq V_1'$, $V_2'' \subseteq V_2'$ and $V_3' \subseteq V_3$ of sizes

$$|W_1| \geq \gamma\sqrt{\log n}\,, \quad |V_2''| \geq (c'/8)\sqrt{\log n}\,, \quad \text{and} \quad |V_3'| \geq \sqrt{n}$$

such that $\mathcal{H}_1[W_1, V_2'', V_3']$ is a complete tripartite hypergraph. In particular, every pair in $(v_2, v_3) \in V_2'' \times V_3'$ is contained in an edge of $\mathcal{H}_1$. Hence by definition of $\mathcal{H}_1$, $d_{\mathcal{H},4}(v_2, v_3) \geq c'n$. In other words $\mathcal{H}_1[V_2'', V_3', V_4]$ meets the assumptions of Lemma 10 for $X = V_2''$, $Y = V_3'$, and $Z = V_4$. Therefore, there exist sets $W_2 \subseteq V_2''$, $V_3'' \subseteq V_3'$ and $V_4' \subseteq V_4$ of sizes

$$|W_2| \geq \gamma\sqrt{\log n}\,, \quad |V_3''| \geq (c'/8)\sqrt{\log n}\,, \quad \text{and} \quad |V_4'| \geq \sqrt{n}$$

such that $\mathcal{H}_1[W_2, V_3'', V_4']$ is a complete tripartite hypergraph.

We repeat the same argument for every $i = 3, \ldots, \ell-2$ and obtain sets $W_3, \ldots, W_{\ell-2}$ and $V_{\ell-1}''$ and V_ℓ'. If follows that $\mathcal{H}_1[W_1, \ldots, W_{\ell-2}, V_{\ell-1}'', V_\ell']$ contains a t-blow-up of $P_\ell^{(3)}$ for $t \geq \gamma\sqrt{\log n}$. □

5. Concluding remarks

Much of the work here was motivated by Problem 2. We point out that a somewhat weaker conjecture was formulated by Bollobás, Erdős and Simonovits in [1]. For an r-uniform hypergraph $\mathcal{F}$ let $\pi_{\mathcal{F}} \geq 0$ denote its Turán density defined by

$$\lim_{n\to\infty} \frac{\max\{|E(\mathcal{H})|\colon\ \mathcal{H} \text{ is an } \mathcal{F}\text{-free } n\text{-vertex } r\text{-uniform hypergraph}\}}{\binom{n}{r}}.$$

It is not hard to show that this limit in fact exists (see, e.g., [10]). Moreover, it is known (see, e.g., [7]) that for fixed $\varepsilon > 0$ and sufficiently large n every n-vertex r-uniform hypergraph with at least $(\pi_{\mathcal{F}} + \varepsilon)\binom{n}{r}$ edges must contain cn^ℓ copies of $\mathcal{F}$, where $\ell = |\mathcal{F}|$ and $c = c(\varepsilon) > 0$ only depends on ε. In other words, such hypergraphs $\mathcal{H}$ satisfy the assumptions of Problem 2 and, in fact, it was conjectured in [1] that those satisfy the conclusion of Problem 2. (In [1] such a result was proved for graphs.)

Our contributions here rely mainly on applications of the results of Kövari, Sós, and Turán [12] and Erdős [4] and very similar ideas. We believe those ideas may be used to approach Problem 2 for other particular hypergraphs $\mathcal{F}$. However, for major progress on this problem new ideas seem to be needed.

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