RAMSEY PROPERTIES OF RANDOM k-PARTITE k-UNIFORM HYPERGRAPHS

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Abstract. We investigate the threshold probability for the property that every r-coloring of the edges of a random binomial k-uniform hypergraph $\mathbb{G}^{(k)}(n,p)$ yields a monochromatic copy of some fixed hypergraph G. In this paper we solve the problem for arbitrary $k \geq 3$ and k-partite, k-uniform hypergraphs G.

1. Introduction

Given two hypergraphs, G and F, we write $F \longrightarrow G$ if every two-coloring of the edges of F results in a monochromatic copy of G. We then say that F has *the Ramsey property with respect to* G. Note that for fixed G, this property, viewed as the family of hypergraphs $\{F : F \longrightarrow G\}$, is increasing, that is, it is closed under taking super-hypergraphs.

In this paper we study Ramsey properties of random k-uniform hypergraphs. Given $k \geq 2$ and $0 \leq p = p(n) \leq 1$, let $\mathbb{G}^{(k)}(n,p)$ be a random hypergraph obtained by declaring each k-element subset of $\{1, 2, \dots, n\} = [n]$ an edge, independently, with probability p.

We say that $\mathbb{G}^{(k)}(n,p)$ possesses a property $\mathcal{P}$ *asymptotically almost surely*, abbreviated to *a.a.s.*, if $\mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \in \mathcal{P}\right) \to 1$ as $n \to \infty$. For an increasing hypergraph property $\mathcal{P}$, the most relevant question in the theory of random hypergraphs is to find a threshold sequence $\widehat{p}(n)$, above which the random hypergraph possesses $\mathcal{P}$ a.a.s., while below which it does not possess $\mathcal{P}$ a.a.s. More precisely, we say that property $\mathcal{P}$ has a threshold $\widehat{p}(n)$ if

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \in \mathcal{P}\right) = \begin{cases} 0 & \text{if} \quad p = o(\widehat{p}) \\ 1 & \text{if} \quad \widehat{p} = o(p)\,. \end{cases}$$

The two parts of the above definition will be referred to as the *0-statement* and the *1-statement*, respectively. It is known that for increasing set properties a threshold always exists (see [3] and [12]).

In [18] and [20] thresholds for Ramsey properties of graphs have been found. To state this and other results we need further notation.

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The numbers of vertices and edges of a hypergraph G will be denoted by v_G and e_G (or $e(G)$), respectively. For a k-uniform hypergraph G with at least one edge, we define the parameters

$$d_G^{(k)} = \begin{cases} \frac{e_G - 1}{v_G - k} & \text{if} \quad e_G > 1 \\ \frac{1}{k} & \text{if} \quad e_G = 1 \end{cases},$$

and

$$m_G^{(k)} = \max\{d_G^{(k)}\colon\ H \subseteq G \text{ and } e_H \geq 1\}\,.$$

Note that for all hypergraphs G with at least one edge

- $m_G^{(k)} > 0$,
- $m_G^{(k)} = 1/k$ if and only if $\Delta(G) = 1$, that is, G consists of isolated edges and vertices,
- $m_G^{(k)} \geq 1/(k-1)$, otherwise.

The parameter $m_G^{(k)}$ is defined in such a way that for $p = \Omega(n^{-1/m_G^{(k)}})$, we have $n^{v_H} p^{e_H} = \Omega(n^k p)$ for each $H \subseteq G$ with $e_H > 0$, that is, the expected number of copies of each subgraph H of G in $\mathbb{G}^{(k)}(n,p)$ is at least of the order of magnitude of the expected number of edges. This seems to be a necessary condition for the property $\mathbb{G}^{(k)}(n,p) \longrightarrow G$ to hold a.a.s. (see [18] for a proof in the graph case).

Below is an abridged version of the threshold theorem for Ramsey properties of random graphs ($k = 2$). For the full version see [12, Theorem 8.1].

Theorem 1 ([18, 20])**.** *Given a graph G, other than a forest, the threshold for the Ramsey property with respect to G is $p(n) = n^{-1/m_G^{(2)}}$. Moreover, there exist constants, c and $C > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(2)}(n,p) \longrightarrow G\right) = \begin{cases} 0 & \textit{if} \quad p \leq cn^{-1/m_G^{(2)}} \\ 1 & \textit{if} \quad p \geq Cn^{-1/m_G^{(2)}}\,. \end{cases} \tag{1}$$

As a by-product of our approach in this paper, we obtain a simple proof of the 1-statement of Theorem 1. This proof will be outlined in Section 4. As opposed to [20], where a stronger version of Theorem 1 with arbitrarily many colors was shown, our current proof does not require any use of the regularity lemma from [23].

Note that (1) is stronger than what the definition of the threshold says. It has been recently shown in [7] that in the case $G = K_3$ the threshold is even sharper.

The only result about Ramsey properties of random hypergraphs was obtained in [21]. There it is shown that $\lim_{n\to\infty} \mathbb{P}(\mathbb{G}^{(3)}(n,p) \longrightarrow K_4^{(3)}) = 1$ if $p \gg n^{-1/3}$, where $K_4^{(3)}$ is the complete 3-uniform hypergraph on four vertices. (Note that $m^{(3)}_{K_4^{(3)}} = 3$.) That proof used a recent regularity lemma for hypergraphs from [6] and the ideas from [19] and [20].

In this paper we extend the 1-statement of Theorem 1 to the class of k-partite, k-uniform hypergraphs for all $k \geq 2$. Recall that a k-uniform hypergraph is *k-partite* if its vertex set can be partitioned into k nonempty sets in such a way that every edge intersects every set of the partition in exactly one vertex.

Theorem 2. *For all $k \geq 2$ and every k-uniform, k-partite hypergraph G with $\Delta(G) \geq 2$ there exists $C > 0$ such that for every sequence $p = p(n) \geq Cn^{-1/m_G^{(k)}}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \longrightarrow G\right) = 1\,.$$

For hypergraphs G with $\Delta(G) = 1$, Theorem 2 is not true (cf. discussion after the statement of Theorem 9 below). For some other special hypergraphs G, like stars, the actual threshold is lower than $n^{-1/m_G^{(k)}}$. More precisely, for integers $k, t \geq 2$ and $s \geq 1$ let $S_{s,t}^{(k)}$ denote the star (or Δ-system or sunflower) with t edges in which every two edges intersect in precisely the same set of s vertices (e.g., $S_{1,t}^{(2)} = K_{1,t}$). Clearly, a hypergraph has the Ramsey property w.r.t. $S_{1,t}^{(2)} = K_{1,t}$ if it contains a copy of $S_{s,2t-1}^{(k)}$ as a sub-hypergraph. Hence, if $p \gg n^{-k+s-s/(2t-1)}$, then

$$\lim_{n\to\infty} \mathbb{P}\big(\mathbb{G}^{(k)} \longrightarrow S_{s,t}^{(k)}\big) = 1\,.$$

On the other hand, for $t \geq 2$ we have for $S = S_{s,t}^{(k)}$

$$n^{-1/m_S^{(k)}} = n^{-(t(k-s)+s-k)/(t-1)} \gg n^{-k+s-s/(2t-1)}\,.$$

However, we believe that the corresponding 0-statement is true with C replaced by a smaller constant c for "most" hypergraphs G. For $k = 2$, a tedious proof was given in [18]. We are convinced that it will be possible to extend it for $k > 2$ and we hope to get back to this in the near future. In fact, in [21] a similar proof of the 0-statement of an analogous threshold result in the vertex-coloring case for hypergraphs was given.

We provide a complete proof of Theorem 2 in Section 3. In this proof, the special structure of k-partite hypergraphs allows for replacing the regularity lemma by an old result of Erdős, Lemma 8 below. As a technical tool, we will be actually proving a stronger theorem, Theorem 9. Being stronger, it will easily imply yet another related result.

We write $F \xrightarrow{\text{ind}} G$ if every two-coloring of the edges of F results in an *induced* monochromatic copy of G. Note that this property is not monotone.

Theorem 3. *For all $\varepsilon > 0$, $k \geq 2$, and every k-uniform, k-partite hypergraph G with $\Delta(G) \geq 2$ there exists $C > 0$ such that for every sequence $p = p(n)$ with $Cn^{-1/m_G^{(k)}} \leq p(n) \leq 1 - \varepsilon$*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \xrightarrow{\text{ind}} G\right) = 1\,.$$

All three results, Theorem 1, Theorem 2, and Theorem 3 have generalizations to an arbitrary number $r \geq 2$ of colors. Interestingly enough, the parameter r does not influence the order of magnitude of the threshold (only the constant C depends on r). In Section 4 we outline the proof of such a generalization of Theorem 2. This is possible, because in Theorem 2 we restrict ourselves to k-partite hypergraphs only. In general, even for graphs the proofs for $r \geq 3$ colors are technically more involved. In particular, we are unable to simplify the proof of the r-color version of Theorem 1 from [20], as we do here for $r = 2$.

2. Preliminaries

If not noted otherwise, throughout the paper all hypergraphs are k-uniform for a fixed integer $k \geq 2$. We will use notation $G - f$ for a hypergraph obtained from G by removing the edge f, and $G + f$ for the hypergraph obtained from G by adding the edge f, where f is a fixed set of k vertices of G.

2.1. **Exponentially small probabilities.** Let Γ be a finite set, $|\Gamma| = N$, let $0 \le p \le 1$ and $0 \le M \le N$, where M is an integer. Then the random subset Γ_p is obtained by including to Γ_p each element of Γ, independently, with probability p. The random subset Γ_M is obtained by selecting uniformly at random one M-element subset of Γ.

By Chernoff's bound (see, e.g., [12], page 26, inequality (2.6)), we have

$$\mathbb{P}(|\Gamma_p| \le Np - t) \le \exp\{-t^2/(2Np)\}. \tag{2}$$

Further, let $\mathcal{S}$ be a family of subsets of Γ, and for $A \in \mathcal{S}$, let I_A be the indicator random variable equal to 1 if $A \subset \Gamma_p$, and equal to 0 otherwise. Finally, let $X = \sum_{A\in\mathcal{S}} I_A$ be the random variable counting all subsets belonging to $\mathcal{S}$ which are present in Γ_p. The following inequality may be thought of as an extension of (2) to sums of dependent indicators.

Lemma 4 (Janson's Inequality [11])**.** *With the above notation, let*

$$\bar{\Delta} = \sum_{A\in\mathcal{S}} \sum_{B\in\mathcal{S}: A\cap B\neq\emptyset} \mathbb{E}(I_A I_B).$$

Then, for all $t \ge 0$

$$\mathbb{P}(X \le \mathbb{E}X - t) \le \exp\left\{-\frac{t^2}{2\bar{\Delta}}\right\}. \tag{3}$$

As a useful illustration, let $\Gamma \subset \binom{[n]}{k}$ be a k-uniform hypergraph, and let $\mathcal{S}$ be the family of the edge sets of all copies of a given hypergraph G, present in Γ. Then Γ_p is a random sub-hypergraph of Γ, and X counts the copies of G in Γ_p. Set

$$\Psi_G = n^{v_G} p^{e_G}$$

and

$$\Phi_G = \min\{\Psi_{G'}\colon\ G' \subseteq G \text{ and } e_{G'} \ge 1\}.$$

Assume that $|\mathcal{S}| \ge cn^{v_G}$ for some $c > 0$. Then $\mathbb{E}X \ge cn^{v_G}p^{e_G} = c\Psi_G$. Note that for every $H \subseteq G$, there are in Γ no more than $n^{2v_G - v_H}$ pairs of copies of G which intersect in a subgraph isomorphic to H. Thus,

$$\bar{\Delta} \le \sum_{H\subseteq G, e_H\ge 1} n^{2v_G - v_H} p^{2e_G - e_H} \le 2^{e_G}\frac{\Psi_G^2}{\Phi_G},$$

and, for every $\varepsilon > 0$, by (3) with $t = \varepsilon\mathbb{E}X$,

$$\mathbb{P}(X \le (1-\varepsilon)\mathbb{E}X) \le \exp\left\{-\frac{1}{2}\varepsilon^2 c^2 2^{-e_G}\Phi_G\right\}. \tag{4}$$

In our main proof we will also need a stronger property to be held by Γ_p. Namely, that with probability very close to one, the number of copies of a given hypergraph remains large even after deleting from Γ_p a fraction of its edges. To this end, for an increasing family $\mathcal{P}$ of subsets of Γ and a nonnegative integer s, define

$$\mathcal{P}_s = \{A \in \mathcal{P}\colon\ \forall B \subseteq A \text{ with } |B| \le s,\ A \setminus B \in \mathcal{P}\}.$$

Note that $\mathcal{P}_s$ is also increasing.

The following lemma has appeared already in different forms in [20] and [12]. We provide the short proof for completeness.

Lemma 5. *Let Γ be a set of size N. For every $0 < p < 1$, every $0 < \delta < 1$ and $b > 0$ such that $\delta(2+\log(1/\delta)) \le b$, every $0 < s \le \delta Np/2$, and for every increasing family $\mathcal{P}$ of subsets of Γ, if $N/\log n \gg p$ and*

$$\mathbb{P}(\Gamma_{(1-\delta)p} \in \neg\mathcal{P}) < \mathrm{e}^{-bNp},$$

then

$$\mathbb{P}(\Gamma_p \in \neg\mathcal{P}_s) < \mathrm{e}^{-0.1\delta^2 Np}.$$

Proof. We will switch to the uniform model Γ_M and utilize the relations between the two probability spaces. Without loss of generality, we may assume that $s = \delta Np/2$. By Chernoff's bound (2),

$$\mathbb{P}(|\Gamma_p| \le Np - s) \le \mathrm{e}^{-\delta^2 Np/8}.$$

Hence, for every increasing property $\mathcal{P}$,

$$\mathbb{P}(\Gamma_p \in \neg\mathcal{P}) \le \mathbb{P}(\Gamma_{Np-s} \in \neg\mathcal{P}) + \mathrm{e}^{-\delta^2 Np/8}.$$

After applying the above inequality to $\mathcal{P}_s$, it remains to estimate $\mathbb{P}(\Gamma_{Np-s} \in \neg\mathcal{P}_s)$. To do this, it is convenient to view Γ_M as a random sequence of M elements, chosen one by one from Γ, uniformly and without replacements. Observe that any subsequence of length $M' \le M$ generates a random copy of $\Gamma_{M'}$ of its own.

If $\Gamma_{Np-s} \in \neg\mathcal{P}_s$, then, by the definition of $\mathcal{P}_s$, there exists a subsequence of length $Np-2s$ such that the set of the elements of this subsequence does not have property $\mathcal{P}$. Thus, by Boole's inequality,

$$\mathbb{P}(\Gamma_{Np-s} \in \neg\mathcal{P}_s) \le \binom{Np-s}{Np-2s}\mathbb{P}(\Gamma_{Np-2s} \in \neg\mathcal{P}).$$

Since $\binom{n}{k} \le (en/k)^k$ for all n and k, the binomial term can be bounded by

$$\binom{Np-s}{Np-2s} = \binom{Np-s}{s} \le (2e/\delta)^s.$$

By Pittel's inequality (see, e.g., [12], page 17),

$$\mathbb{P}(\Gamma_{Np-2s} \in \neg\mathcal{P}) \le 3\sqrt{N}\mathbb{P}(\Gamma_{(1-\delta)p} \in \neg\mathcal{P}).$$

Hence, by our assumption on δ and b,

$$\begin{aligned}\mathbb{P}(\Gamma_p \in \neg\mathcal{P}_s) &\le (2e/\delta)^s 3\sqrt{N}\mathrm{e}^{-bNp} + \mathrm{e}^{-\delta^2 Np/8}\\ &\le 3\sqrt{N}\mathrm{e}^{-bNp/2} + \mathrm{e}^{-\delta^2 Np/8} \le \mathrm{e}^{-0.1\delta^2 Np},\end{aligned}$$

where the last inequality holds for sufficiently large N. □

2.2. **Intersecting copies.** Next we prove an elementary result about the number of small sub-hypergraphs of $\mathbb{G}^{(k)}(n,p)$, with a special structure relevant to our proof of Theorem 2. We will need a simple fact first.

Given a hypergraph H, let X_H be the number of copies of H in $\mathbb{G}^{(k)}(n,p)$. We recall from Section 2.1 that the expectation of X_H can be well upper-bounded by

$$\Psi_H = n^{v_H} p^{e_H}$$

and that

$$\Phi_H = \min\{\Psi_{H'}\colon\ H' \subseteq H \text{ and } e_{H'} \ge 1\}.$$

Claim 6. *If $\Phi_H \to \infty$ then a.a.s. $X_H \le 2\mathbb{E}X_H$.*

Proof. By estimates similar to those in the case of random graphs (see e.g. [12, Lemma 3.5])

$$\operatorname{Var} X_H = O\left(\sum_{H'\subseteq H, e_{H'}>0} \frac{(\mathbb{E}X_H)^2}{\Psi_{H'}}\right),$$

and so, by Chebyshev's inequality,

$$\mathbb{P}(X_H > 2\mathbb{E}X_H) \le \mathbb{P}(|X_H - \mathbb{E}X_H| > \mathbb{E}X_H) \le \frac{\operatorname{Var} X_H}{(\mathbb{E}X_H)^2} = o(1).$$

□

Now we are ready to prove the main result of this subsection.

Lemma 7. *Let G be a k-uniform hypergraph with $\Delta(G) \ge 2$, and let T be a union of two copies G_1 and G_2 of G, intersecting in at least one edge. Furthermore, let $\widetilde{T}$ be obtained from T by removing an edge $f \in G_1 \cap G_2$. If $p = p(n) \ge n^{-1/m_G^{(k)}}$ then a.a.s.*

$$X_{\widetilde{T}} \le 2n^{2v_G - k} p^{2e_G - 2}.$$

Proof. Set $I = G_1 \cap G_2$ and, for every $H \subseteq T$, set $\widetilde{H} = H - f$, regardless of whether $f \in H$ or not. Then

$$\Psi_{\widetilde{T}} = \frac{\Psi_{\widetilde{G}_1}\Psi_{\widetilde{G}_2}}{\Psi_{\widetilde{I}}}. \tag{5}$$

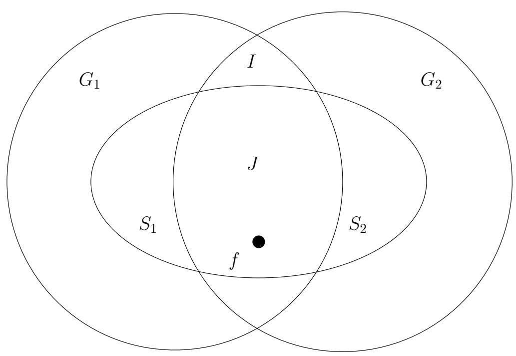

FIGURE 1. Illustration for the proof of Lemma 7

The probability $p = p(n)$ was chosen in such a way that for every $H \subseteq G$, $e_H \ge 1$, we have

$$\Psi_H = n^{v_H} p^{e_H - 1} p \ge n^{v_H} n^{-(e_H-1)/m_H^{(k)}} p \ge n^k p, \tag{6}$$

and, in particular, for $H = I$, we get

$$\Psi_{\widetilde{I}} = \frac{1}{p}\Psi_I \ge n^k$$

and consequently,

$$\mathbb{E}X_{\widetilde{T}} \le \Psi_{\widetilde{T}} \le n^{2v_G - k} p^{2e_G - 2}.$$

Hence, if $\Phi_{\widetilde{T}} \to \infty$ then we are done by Claim 6. On the other hand, by (5), if $n^{-k}\Psi_{\widetilde{I}} \to \infty$ then $\mathbb{E}X_{\widetilde{T}} = o\left(n^{2v_G-k}p^{2e_G-2}\right)$, and, by Markov's inequality,

$$\mathbb{P}(X_{\widetilde{T}} > 2n^{2v_G-k}p^{2e_G-2}) = o(1).$$

It remains to show that either $\Phi_{\widetilde{T}} \to \infty$ or $n^{-k}\Psi_{\widetilde{I}} \to \infty$. Quite arbitrarily, suppose that $\Phi_{\widetilde{T}} \leq \sqrt{n}$. Note that for every $H \subseteq T$ we have

$$\Psi_H = \begin{cases} \Psi_{\widetilde{H}} & \text{if} \quad f \notin H \\ p\Psi_{\widetilde{H}} & \text{if} \quad f \in H \end{cases}$$

and thus, $\Psi_H \leq \Psi_{\widetilde{H}}$ and $\Phi_T \leq \Phi_{\widetilde{T}} \leq \sqrt{n}$.

Let $\Phi_T = \Psi_S$, that is, S is a sub-hypergraph of T which achieves the minimum in Φ_T. Set $S_i = S \cap G_i$, $i = 1, 2$ and $J = S_1 \cap S_2$ (see Figure 1). Note that $S \cap I = J$. Note also that $e(S_i) \geq 1$, $i = 1, 2$, since otherwise S would consist of a subgraph S' of G and, possibly, some isolated vertices. However, then we would have, by (6), $\Psi_S \geq \Psi_{S'} \geq n^k p$ and since $\Delta(G) \geq 2$ we have

$$n^k p \geq n^{k-1/m_G^{(k)}} \geq n^{k-(k-1)} = n \tag{7}$$

a contradiction with the choice of S.

But then, again by (6), we have

$$\Psi_S = \frac{\Psi_{S_1}\Psi_{S_2}}{\Psi_J} \geq \frac{(n^k p)^2}{\Psi_J}$$

which yields that

$$\Psi_J \geq \frac{(n^k p)^2}{\sqrt{n}}.$$

Finally, observe that

$$\Psi_S \leq \Psi_{S\cup I} = \frac{\Psi_S \Psi_I}{\Psi_J},$$

so $\Psi_I \geq \Psi_J$ and consequently,

$$\Psi_{\widetilde{I}} = \frac{1}{p}\Psi_I \geq n^k \frac{n^k p}{\sqrt{n}} \overset{(7)}{\geq} n^k \sqrt{n}.$$

□

2.3. **Erdős' k-partite counting lemma.** For two hypergraphs, F and H, let $N(F, H)$ stand for the number of *labeled copies* of H in F, that is, the number of injective mappings $f : V(H) \to V(F)$ such that if $e \in E(H)$ then $f(e) \in E(F)$. For a fixed labeling on $V(H)$, say, $v_1, \dots, v_{v_H}$, we will identify an labeled copy f of H in F with the sequence $(f(v_1), \dots, f(v_{v_H}))$. We use labeled copies just for convenience, noting that the number of ordinary copies of H in F, that is, the number of sub-hypergraphs of F which are isomorphic to H, equals $N(F, H)/aut(H)$.

Lemma 8. *For every integer $k \geq 2$, every $d > 0$, and every k-uniform k-partite hypergraph H, there exist $c > 0$ and n_0 such that for every k-uniform hypergraph F on $n \geq n_0$ vertices with $e_F \geq dn^k$, we have $N(F, H) \geq cn^{v_H}$.*

A similar statement was first proved by Erdős in [4] (see also [5]). For completeness we give a short proof.

Proof. It suffices to prove Lemma 8 for complete k-uniform k-partite hypergraphs H. Let $k \geq 2$ and $d > 0$ be given and let $H = K(\ell_1, \dots, \ell_k)$ be the complete k-uniform k-partite hypergraph with vertex classes $W_1 \dot\cup \dots \dot\cup W_k = V(H)$ of sizes $W_i = \ell_i$. (For the sake of defining labeled copies of H in F, we impose on $V(H)$ an arbitrary labeling in which each vertex of W_i precedes each vertex of W_{i+1}, $i = 1, \dots, k-1$.)

Let L_H be the set of indices i for which $\ell_i = |W_i| > 1$, i.e.,

$$L_H = \{i \in [k]\colon\ \ell_i \geq 2\}.$$

The proof is by induction on $|L_H|$. The induction base is trivial, as for $|L_H| = 0$ the hypergraph $H = K(1, \dots, 1)$ contains only one edge and we can choose $c = dk!$.

Suppose $|L_H| = \ell > 0$ and Lemma 8 holds for hypergraphs H' with $|L_{H'}| < \ell$. Without loss of generality assume that $k \in L_H$ and consider the sub-hypergraph $H' = K(\ell_1, \dots, \ell_{k-1}, 1)$. Clearly, $|L_{H'}| = \ell - 1$ and from the induction assumption we infer that

$$N(F, H') \geq c' n^{v(H')} = c' n^{v(H) - \ell_k + 1} \tag{8}$$

for some constant $c' = c'(k, d, H')$. Set $\tilde\ell = v(H) - \ell_k$ and consider the set $\mathcal{X}$ of all $\tilde\ell$-element sequences of distinct vertices of F. Note that

$$|\mathcal{X}| = n(n-1)\cdots(n - \tilde\ell + 1) = (n)_{\tilde\ell} < n^{\tilde\ell}. \tag{9}$$

For a sequence $X = (v_1, \dots, v_{\tilde\ell}) \in \mathcal{X}$ we define

$$\deg(X) = \big|\{v \in V(F)\colon\ (v_1, \dots, v_{\tilde\ell}, v) \text{ is an labeled copy of } H' \text{ in } F\}\big|.$$

Therefore,

$$N(F, H') = \sum_{X \in \mathcal{X}} \deg(X). \tag{10}$$

By Jensen's inequality and by (8), (9) and (10) we conclude that

$$N(F, H) = \sum_{X \in \mathcal{X}} (\deg(X))_{\ell_k} \geq |\mathcal{X}| \left(\frac{N(F, H')}{|\mathcal{X}|}\right)_{\ell_k} \geq (n)_{\tilde\ell} \left(\frac{c' n^{\tilde\ell + 1}}{n^{\tilde\ell}}\right)_{\ell_k} \geq c n^{v(H)}$$

for some suitably chosen $c = c(c', H', H) = c(k, d, H)$ and n sufficiently large. □

3. Proof of Theorem 2

3.1. **The idea of the proof.** The underlying idea of our proof comes from classical Ramsey theory, where often to force a monochromatic object, a coloring process is put into a dead-end. A simplest and best known illustration of this strategy is the proof of the "six-person-party theorem," which says that every 2-coloring of the edges of K_6 results in a monochromatic triangle. In that proof, at some point a vertex is known to be connected to three other by edges of the same color, while the edges between the three neighbors are yet uncolored. But then no matter how they are colored, a monochromatic triangle is guaranteed.

To facilitate this idea in the context of random hypergraphs, we employ the two-round exposure technique (see [12, Section 1.1]), where the random hypergraph $\mathbb{G}^{(k)}(n, p)$ is generated in two installments, that is, it is expressed as the union of two independent random hypergraphs $\mathbb{G}_1 = \mathbb{G}^{(k)}(n, p_1)$ and $\mathbb{G}_2 = \mathbb{G}^{(k)}(n, p_2)$ with p_1 and p_2 suitably chosen and such that

$$p_1 + p_2 - p_1 p_2 = p. \tag{11}$$

From now on, by a coloring we will always mean a 2-coloring where the colors are *blue* and *red*. For every instance of $\mathbb{G}_1$ and every coloring χ of its edges, we will consider a hypergraph $\Gamma_\chi = \Gamma_\chi(\mathbb{G}_1)$ consisting of all edges $f \notin \mathbb{G}_1$ such that $\mathbb{G}_1 + f$ contains a copy G_f of G, whose one edge is f and all other edges are of the same color (in fact, $\mathbb{G}_1$ will contain many such copies – see the precise definition later). Depending on the color of $G_f - f$, we may refer to each $f \in \Gamma_\chi$ as "blue-closing" or "red-closing", and thus express Γ as a union

$$\Gamma_\chi = \Gamma_\chi^{\text{blue}} \cup \Gamma_\chi^{\text{red}},$$

with the obvious meaning of the superscripts. Note that $\Gamma_\chi^{\text{blue}}$ and Γ_χ^{red} are not necessarily disjoint, as an edge f may close blue and red copies of $G - f$ at the same time. We think of Γ_χ as the hypergraph of "closing edges" after round one or, alternatively, as the hypergraph of "useful" edges for round two.

The ultimate goal of the first round is to show that a.a.s. for every χ, either $\Gamma_\chi^{\text{blue}}$ or Γ_χ^{red} contains many copies of G. Say, it is the case of Γ_χ^{red}. Then, in the second round, we focus exclusively on the random sub-hypergraph of Γ_χ^{red}, that is, on

$$\left(\Gamma_\chi^{\text{red}}\right)_{p_2} = \Gamma_\chi^{\text{red}} \cap \mathbb{G}_2$$

and argue that, with probability very close to 1, at least one copy G_0 of G in Γ_χ^{red} (in fact, many) will be present in $\mathbb{G}_2$. But if this is the case, then there is no way to extend χ without creating a monochromatic copy of G. Indeed, either every edge of G_0 is blue, or an edge $f \in G_0$ is red, turning G_f into a red copy of G.

There is one important refinement to the above simplified argument. Whatever we claim to hold in round two, must hold for all colorings χ of the outcome of the first round, $\mathbb{G}_1$. Thus, it must hold with probability so close to 1 that the probability of failure, multiplied by the number of colorings, still converges to 0. Since a.a.s. $e(\mathbb{G}_1) = \Theta(n^k p_1)$, the number of colorings of $\mathbb{G}_1$ is $2^{\Theta(n^k p_1)}$, and we need the probability of having a copy of G in $\left(\Gamma_\chi^{\text{red}}\right)_p$ to be $1 - \exp\{-\Theta(n^k p_2)\}$. To achieve this goal we will prove first that a.a.s. the number of copies of G in Γ_χ^{red} is $\Theta(n^{v_G})$ and then apply Janson's inequality.

It remains to explain how we prove that a.a.s. Γ_χ^{red} contains $\Theta(n^{v_G})$ copies of G. Since G is k-partite, by Erdős' k-partite counting lemma, Lemma 8, it is enough to show that a.a.s. $|\Gamma_\chi| = \Theta(n^k)$, and then apply Lemma 8 to the majority color class, Γ_χ^{red} or $\Gamma_\chi^{\text{blue}}$.

To show that $|\Gamma_\chi| = \Theta(n^k)$, we will argue that a.a.s. for every coloring χ of $\mathbb{G}_1$ there are $\Theta(n^{v_G})$ monochromatic copies of $\widetilde{G} := G - f_0$, a hypergraph obtained by removing from G one, fixed edge f_0. This is how we come across the idea of using induction on e_G. But the induction hypothesis must be stronger than the theorem itself, claiming not one but $\Theta(n^{v_G} p^{e_G})$ monochromatic copies of G in every coloring (see Theorem 9 below).

As a consequence of strengthening Theorem 2, our argument has to be modified slightly. First, we should request that $f \in \Gamma_\chi$ if $\mathbb{G}_1 + f$ contains not one but $\Theta(n^{v_G - k} p_1^{e_G - 1})$ copies of G which contain f, and, except from f, are monochromatic. Assume, again, that red is the majority color. Then, after $\mathbb{G}_2$ is exposed, either an extension of coloring χ colors at least $\Theta(n^k p_2)$ edges of $\Gamma_\chi^{\text{red}} \cap \mathbb{G}_2$ red, creating

$$\Theta(n^k p_2 \times n^{v_G - k} p_1^{e_G - 1}) = \Theta(n^{v_G} p^{e_G})$$

red copies of G, or not. In the latter case, though, Janson's inequality combined with Lemma 5 guarantee, with probability $1-\exp\{-\Theta(n^k p_2)\}$, that the remaining, blue part of $\Gamma_\chi^{\rm red}\cap\mathbb{G}_2$ contains $\Theta(n^{v_G}p_2^{e_G})$ copies of G.

Returning to round one, it is a bit tedious to show that having $\Theta(n^{v_G})$ monochromatic copies of $\widetilde{G}$ in $\mathbb{G}_1$ implies $|\Gamma_\chi|=\Theta(n^k)$. The proof involves Jensen's inequality and an upper tail estimate for the number of pairs of copies of $\widetilde{G}$ in $\mathbb{G}_1$ sharing the same non-edge.

As an example, suppose $k=2$ and $G=C_4$, the four-cycle. Then $\widetilde{G}=P_4$, the path on four vertices, and an edge f belongs to Γ_χ if together with some $\Theta(n^2p_1^3)$ monochromatic copies of P_4 in $\mathbb{G}_1$ it forms a copy of C_4. Hence, many monochromatic copies of P_4 in $\mathbb{G}_1$ will give rise to many edges in Γ_χ, provided not too many P_4's will "sit" on the same edge f. One way to forbid this is to bound the number of six-cycles C_6 in $\mathbb{G}_1$, which can be viewed as pairs of copies of P_4 sharing the same "closing non-edge" but otherwise disjoint.

3.2. **The strengthening.** We will be, in fact, proving by induction on e_G the following strengthening of Theorem 2. For a real number $a>0$, we write $F\xrightarrow{a}G$ if every coloring of the edges of F results in at least $aN(F,G)$ monochromatic copies of G. For example, it is well known that $K_6\xrightarrow{0.1}K_3$, since every two-coloring of K_6 yields two monochromatic triangles. Note that for given G and a, property $F\xrightarrow{a}G$ is not a monotone property of F.

Theorem 9. *For all $k\geq 2$ and every k-uniform, k-partite hypergraph G with at least one edge there exist $C\geq 1$ and $a>0$ such that if $p=p(n)>Cn^{-1/m_G^{(k)}}$, $n^kp\to\infty$ but $p\to 0$, then*

$$\lim_{n\to\infty}\mathbb{P}\left(\mathbb{G}^{(k)}(n,p)\xrightarrow{a}G\right)=1\,.$$

By a standard application of the second moment method, it can be easily proved that in the above range of p, $\mathbb{G}^{(k)}(n,p)$ contains at least one copy (in fact, $\Theta(n^{v_G}p^{e_G})$ copies) of G. Hence, Theorem 9 does, indeed, imply Theorem 2. (We may assume that $p\to 0$, since the Ramsey property in Theorem 2 is increasing.) Although Theorem 9 is about a non-monotone property, it is also true for p constant, the fact which we will not need here.

Another consequence of Theorem 9 is Theorem 3 – the induced version of Theorem 2. Indeed, if $p\to 0$, then a.a.s. only $o(n^{v_G}p^{e_G})$ copies of G in $\mathbb{G}^{(k)}(n,p)$ are not induced. Thus, in view of Theorem 9, a.a.s. for every coloring of $\mathbb{G}^{(k)}(n,p)$ there is at least one (in fact, many) induced copy of G which is monochromatic.

To prove Theorem 3 also for $p<1$ constant, we argue as follows. By the result from [1, 17], there exists a hypergraph F such that $F\xrightarrow{\rm ind}G$. For p constant it is easy to show that a.s.s. there is at least one induced copy of F in $\mathbb{G}^{(k)}(n,p)$ (see [2] for the graph case), and thus every coloring of $\mathbb{G}^{(k)}(n,p)$ produces an induced, monochromatic copy of G.

Our proof of Theorem 9 is by induction on e_G, the number of edges in G, and it is convenient to begin with the case $e_G=1$. (This is why, unlike in Theorem 2, we included here the case $\Delta(G)=1$.) But then $m_G^{(k)}=1/k$ and thus, for $p=\Theta(n^{-1/m_G^{(k)}})$, the expected number of edges in $\mathbb{G}^{(k)}(n,p)$ equals $\Theta(n^kp)=\Theta(1)$. This is why we added the assumption that $n^kp\to\infty$. Note that in this case C is irrelevant. As another convenience, in Theorem 9 we require that $C \geq 1$, which is not a restriction at all.

3.3. **The case $\Delta(G) = 1$.** To begin the inductive proof of Theorem 9, let $\Delta(G) = 1$, which includes the case $e_G = 1$. The following two properties are true for all $p = p(n)$ satisfying $n^k p \to \infty$. The random variable $e(\mathbb{G}^k(n,p))$ has the binomial distribution with $\mathbb{E}e(\mathbb{G}^k(n,p)) = \binom{n}{k}p < n^k p$ and $\operatorname{Var} e(\mathbb{G}^k(n,p)) = \binom{n}{k}p(1-p)$. Hence, by Chebyshev's inequality

$$\mathbb{P}\left(|e(\mathbb{G}^k(n,p)) - p\tbinom{n}{k}| > \tfrac{1}{2}p\tbinom{n}{k}\right) \leq \frac{\operatorname{Var} e(\mathbb{G}^k(n,p))}{(\frac{1}{2}p\binom{n}{k})^2} < \frac{4}{p\binom{n}{k}} = o(1). \tag{12}$$

For each $\ell \geq 2$, let X_ℓ be the number of (unordered) ℓ-tuples of distinct edges of $\mathbb{G}^{(k)}(n,p)$, not all of which are pairwise disjoint. We have $\mathbb{E}X_\ell \leq n^{\ell k-1}p^\ell$, and by Markov's inequality,

$$\mathbb{P}(X_\ell > (n^k p)^\ell/\sqrt{n}) \leq 1/\sqrt{n}.$$

Hence, a.a.s., we have $e(\mathbb{G}^k(n,p)) > \frac{1}{2}\binom{n}{k}p$, and, taking $\ell = e_G$, $X_{e_G} \leq (n^k p)^{e_G}/\sqrt{n}$. Consequently, a.a.s., after coloring the edges of $\mathbb{G}^{(k)}(n,p)$, the number of e_G-tuples of edges of $\mathbb{G}^{(k)}(n,p)$ which are pairwise disjoint and monochromatic (in the majority color alone) is, a.a.s., at least

$$\binom{\frac{1}{4}\binom{n}{k}p}{e_G} - \frac{1}{\sqrt{n}}(n^k p)^{e_G} = (1-o(1))\binom{\frac{1}{4}\binom{n}{k}p}{e_G}.$$

Each set of e_G pairwise disjoint edges can be extended to $\binom{n-ke_G}{v_G-ke_G}$ copies of G, by adding $v_G - ke_G$ arbitrary vertices. Therefore, a.a.s., for every coloring there are at least

$$(1-o(1))\binom{\frac{1}{4}\binom{n}{k}p}{e_G} \times \binom{n-ke_G}{v_G-ke_G} > an^{v_G}p^{e_G}$$

monochromatic copies of G, for some constant $a > 0$ independent of n. This proves Theorem 9 for all graphs with $\Delta(G) = 1$.

3.4. **Mainstream proof.** The proof of the induction step requires several constants. We will specify those constants later in the proof instead of defining them all up front. We believe this eases the reading. The dependencies of the main constants are given in Figure 2.

Assume that $e_G \geq 2$ and $p \geq Cn^{-1/m_G^{(k)}}$, where C will be specified later (see (30)). Let p_1 and p_2 be suitably chosen (see (27) and (31)), so that p_2 is sufficiently larger than p_1 and (11) holds. Throughout the proof we will assume that p, p_1, and p_2 all tend to 0 as $n \to \infty$. As before, we will be using abbreviated notation $\mathbb{G} = \mathbb{G}^{(k)}(n,p)$, $\mathbb{G}_1 = \mathbb{G}^{(k)}(n,p_1)$ and $\mathbb{G}_2 = \mathbb{G}^{(k)}(n,p_2)$.

For a suitably selected constant $a > 0$ (see (28) and (29) below), let BAD be the event that there is a coloring of the edges of $\mathbb{G}$ with less than $an^{v_G}p^{e_G}$ monochromatic copies of G. Since by (12), a.a.s. $e(\mathbb{G}) = \Theta(n^k p)$, Theorem 9 is equivalent to the fact that $\mathbb{P}(\text{BAD}) = o(1)$ for some $a > 0$.

Fix an arbitrary edge f_0 of G and let $\widetilde{G}$ be the hypergraph obtained from G by removing the edge f_0. By the induction assumption applied to $\widetilde{G}$, there exist $\widetilde{a}$ and $\widetilde{C}$ such that if $p \geq \widetilde{C}n^{-1/m_{\widetilde{G}}^{(k)}}$, then $\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \xrightarrow{\widetilde{a}} \widetilde{G}\right) = 1\,.$

For a copy $\widetilde{G}'$ of $\widetilde{G}$ in $\mathbb{G}_1$, let $\mathrm{cl}(\widetilde{G}')$ be the set of edges $f \in K_n^{(k)}$ such that $\widetilde{G}' + f$ is isomorphic to G. For a coloring χ of $\mathbb{G}_1$, we define the auxiliary hypergraph

$$\Gamma_\chi^{\mathrm{blue}} = \{f \in K_n^{(k)} \setminus \mathbb{G}_1 \colon\ |\{\widetilde{G}' \subseteq \mathbb{G}_1 \colon\ f \in \mathrm{cl}(\widetilde{G}') \text{ and } \widetilde{G}' \text{ is blue copy of } \widetilde{G}\}| \geq z\}\,,$$

where

$$z = \frac{\widetilde{a}}{2} n^{v_G - k} p_1^{e_G - 1}\,. \tag{13}$$

We set

$$\Gamma_\chi = \Gamma_\chi^{\mathrm{blue}} \cup \Gamma_\chi^{\mathrm{red}}\,,$$

where $\Gamma_\chi^{\mathrm{red}}$ is defined as $\Gamma_\chi^{\mathrm{blue}}$ with the word "blue" replaced by "red". Further, let

$$d = \frac{\widetilde{a}^2}{2^{\binom{2v_G - k}{k} + 8} v_G^k}\,, \tag{14}$$

and let GOOD be the event that

$$e(\mathbb{G}_1) < n^k p_1 \tag{15}$$

and, for every coloring χ of $\mathbb{G}_1$

$$\max\{|\Gamma_\chi^{\mathrm{red}}|, |\Gamma_\chi^{\mathrm{blue}}|\} \geq d n^k\,. \tag{16}$$

Note that GOOD is not the complement of BAD.

Conditioning on $\mathbb{G}_1$ and fixing some coloring χ of $\mathbb{G}_1$, let BAD_χ be the event that there is an extension of χ, $\bar{\chi}\colon\ \mathbb{G} \to \{\text{blue,red}\}$, with less than $a n^{v_G} p^{e_G}$ monochromatic copies of G in both colors. We will later verify the following two facts.

Fact 10. *The event* GOOD *holds a.a.s.*

Fact 11. *For every* $\mathbb{G}_1 \in$ GOOD *and every coloring* χ*, of* $\mathbb{G}_1$

$$\mathbb{P}(\mathrm{BAD}_\chi \mid \mathbb{G}_1) \leq \mathrm{e}^{-n^k p_1}\,.$$

Assuming these two facts, we may easily complete the proof of Theorem 9. Indeed, we have

$$\mathbb{P}(\mathrm{BAD}) \leq \mathbb{P}(\neg\mathrm{GOOD}) + \sum_{\mathbb{G}_1 \in \mathrm{GOOD}} \mathbb{P}(\mathrm{BAD} \mid \mathbb{G}_1)\mathbb{P}(\mathbb{G}_1)\,,$$

and, for every $\mathbb{G}_1 \in$ GOOD,

$$\mathbb{P}(\mathrm{BAD} \mid \mathbb{G}_1) = \sum_\chi \mathbb{P}(\mathrm{BAD}_\chi \mid \mathbb{G}_1) \leq 2^{n^k p_1} \mathbb{P}(\mathrm{BAD}_{\chi_0} \mid \mathbb{G}_1)\,,$$

where the summation is taken over all, at most $2^{n^k p_1}$, colorings χ of the edges of $\mathbb{G}_1$, and χ_0 maximizes the conditional probability. Therefore, by Facts 10 and 11,

$$\mathbb{P}(\mathrm{BAD}) \leq o(1) + (2/e)^{n^k p_1} = o(1).$$

3.5. **Round one – Proof of Fact 10.** Due to the choice of p_1 (c.f. (31)) and the concentration of the number of edges in $\mathbb{G}_1$ as given in (12), $\mathbb{G}_1$ a.a.s. contains at most $n^k p_1$ edges as claimed in (15). For the rest of this subsection our goal will be to prove that a.a.s. (16) also holds.

Since $m^{(k)}_{\widetilde{G}} \leq m^{(k)}_G$, we have by (31) $p_1 \geq \widetilde{C} n^{-1/m^{(k)}_{\widetilde{G}}}$, and we are in position to apply the induction assumption, that is Theorem 9, to $\widetilde{G}$. Consequently, a.a.s., for every coloring χ of the edges of $\mathbb{G}_1$ there is a color (say, red) such that at least

$$\ell := \frac{\widetilde{a}}{2} n^{v_G} p_1^{e_G - 1} \tag{17}$$

copies of $\widetilde{G}$ are colored red in $\mathbb{G}_1$. Note that, by (13), (31), and (14)

$$\ell = zn^k \geq \frac{\widetilde{a}}{2} n^k \geq 8dn^k. \tag{18}$$

For every $f \in K_n^{(k)}$, let x_f be the number of red copies $\widetilde{G}'$ of $\widetilde{G}$ in $\mathbb{G}_1$ for which $f \in \mathrm{cl}(\widetilde{G}')$. Then, a.a.s.

$$\sum_{f \in K_n^{(k)}} x_f \geq \ell. \tag{19}$$

Let $\mathscr{T} = \{T_1, T_2, \ldots, T_t\}$ be the family of all pairwise non-isomorphic hypergraphs which are the unions of two copies of $\widetilde{G}$, say $\widetilde{G}' \cup \widetilde{G}''$, with the property that there is a k-tuple f which makes both $\widetilde{G}' + f$ and $\widetilde{G}'' + f$ isomorphic to G. We will say that f is a *common G-closing non-edge* of $\widetilde{G}'$ and $\widetilde{G}''$. Clearly, $|\mathscr{T}|$ does not exceed the number of all graphs on $2v_G - k$ vertices, that is,

$$t := |\mathscr{T}| \leq 2^{\binom{2v_G - k}{k}}.$$

Then, by Lemma 7, $\mathbb{G}_1$ contains a.a.s. at most

$$2tn^{2v_G - k} p_1^{2e_G - 2} \tag{20}$$

copies of members of $\mathscr{T}$. As $|\mathrm{cl}(\widetilde{G})| \leq \binom{v_G}{k} < v_G^k$, a particular copy of a graph from $\mathscr{T}$ may be obtained as a union of two copies of $\widetilde{G}$ with a common G-closing non-edge in at most v_G^k ways. Thus, a.a.s.

$$\sum_{f \in K_n^{(k)}} \binom{x_f}{2} < 2tv_G^k n^{2v_G - k} p_1^{2e_G - 2}. \tag{21}$$

Let $Z \subseteq K_n^{(k)}$ be the set of edges f such that $x_f \geq z$. We are going to show that

$$|Z| \geq 2dn^k. \tag{22}$$

First, observe that

$$\sum_{f \in K_n^{(k)} \setminus Z} x_f < z \binom{n}{k} \leq \frac{\ell}{2}$$

and consequently, in view of (19),

$$\sum_{f \in Z} x_f \geq \frac{\ell}{2}. \tag{23}$$

If $|Z| \geq \ell/4$, then, by (18), inequality (22) holds. Assuming that $|Z| \leq \ell/4$, we derive by Jensen's inequality and by (23) that

$$\sum_{f\in Z}\binom{x_f}{2} \geq |Z|\binom{\frac{\sum_{f\in Z} x_f}{|Z|}}{2} \geq |Z|\binom{\frac{\ell}{2|Z|}}{2} \geq \frac{\ell^2}{16|Z|} = \frac{\widetilde{a}^2 n^{2v_G} p_1^{2e_G-2}}{64|Z|},$$

which, by (21) and (14), yields (22) again. Thus, by (15) and the fact that $p_1 \to 0$, a.a.s.

$$|\Gamma^{\mathrm{red}}_\chi| \geq 2dn^k - |\mathbb{G}_1| > (2d-p_1)n^k \geq dn^k,$$

and the property GOOD holds.

3.6. **Round two – Proof of Fact 11.** We condition on the event that $\mathbb{G}_1$ satisfies property GOOD. Let a coloring χ of the edges of $\mathbb{G}_1$ be given. According to property (16), let, say, $|\Gamma^{\mathrm{red}}_\chi(\mathbb{G}_1)| \geq dn^k$. Set $\Gamma^{\mathrm{red}} = \Gamma^{\mathrm{red}}_\chi(\mathbb{G}_1)$.

Let $c = c(G,d)$ be given by Lemma 8. Hence $N(\Gamma^{\mathrm{red}}, G) \geq cn^{v_G}$. Later we use Lemma 5. Therefore, we first consider a random sub-hypergraph Γ^{red}_q, with

$$q := (1-\delta)p_2\,, \tag{24}$$

where $\delta > 0$ is so small that

$$\delta(2-\log\delta) < b := \frac{c^2}{400\cdot 2^{e_G}}. \tag{25}$$

We want to apply Janson's inequality (in the form of inequality (4)) with $\varepsilon = 0.1$, $\Gamma = \Gamma^{\mathrm{red}}$, and $X = N(\Gamma^{\mathrm{red}}_q, G)$. Note that $\mathbb{E}[X] \geq cn^{v_G}q^{e_G}$. By (24), (31) and (27), we have $q \geq n^{-1/m_G^{(k)}}$ and, consequently, $n^{v_K}q^{e_K} \geq n^k q$ for every $K \subseteq G$ with $e_K \geq 1$. Hence,

$$\begin{aligned}\mathbb{P}(X \leq 0.9cn^{v_G}q^{e_G}) &\leq \mathbb{P}(X \leq 0.9\mathbb{E}X)\\ &\leq \exp\left(-\frac{c^2n^kq}{200\cdot 2^{e_G}}\right) \leq \mathrm{e}^{-bn^kp_2} \leq \mathrm{e}^{-b|\Gamma^{\mathrm{red}}|p_2}\,,\end{aligned}$$

where we also use the fact that $\delta < 1/2$.

Next we apply Lemma 5 with

$$s := \delta(dn^k)p_2/2 \leq \delta|\Gamma^{\mathrm{red}}|p_2/2\,. \tag{26}$$

We conclude that, for sufficiently large n, with probability at least

$$1 - 3\sqrt{|\Gamma^{\mathrm{red}}|}\mathrm{e}^{-b|\Gamma^{\mathrm{red}}|p_2/2} - \mathrm{e}^{-\delta^2|\Gamma^{\mathrm{red}}|p_2/8} \geq 1 - \mathrm{e}^{-\delta^2cn^kp_2/10} \geq 1 - \mathrm{e}^{-n^kp_1}$$

we have

$$N(\Gamma^{\mathrm{red}}_{p_2}\setminus D, G) \geq 0.9cn^{v_G}q^{e_G} \geq 0.8cn^{v_G}p_2^{e_G}$$

for all $D \subseteq \Gamma^{\mathrm{red}}_{p_2}$ of size $|D| \leq s$. For the last inequality in the above bound on probability, we need the relation

$$p_2 \geq 10p_1/(c\delta^2). \tag{27}$$

We will verify now that, with probability at least $1-\mathrm{e}^{-n^kp_1}$, for every extension $\bar{\chi}$ of the coloring χ, $\Gamma^{\mathrm{red}}_{p_2}$ either contains at least $an^{v_G}p^{e_G}$ blue copies of G or it completes at least $an^{v_G}p^{e_G}$ red copies of G in $\mathbb{G}$.

Let D be the set of edges of $\Gamma_{p_2}^{\text{red}}$ colored red by $\bar{\chi}$. If $|D| < s$ then, by the above property and with a suitably chosen a, there are at least

$$0.8cn^{v_G}p_2^{e_G} \geq an^{v_G}p^{e_G} \tag{28}$$

copies of G in $\Gamma_{p_2}^{\text{red}} \setminus D$, all of them blue.

If, on the other hand, $|D| \geq s$, then, as each edge of Γ^{red} closes at least z red copies of $\widetilde{G}$ in $\mathbb{G}_1$, there are, with a suitably chosen a, at least

$$\frac{s \times z}{v_G^k} \geq \frac{\delta d n^k p_2 \times \widetilde{a} n^{v_G - k} p^{e_G - 1}}{4 v_G^k} \geq an^{v_G}p^{e_G} \tag{29}$$

red copies of G in $\mathbb{G}$.

To complete the proof, we choose

$$C = \widetilde{C}\left(\frac{10}{c\delta^2} + 1\right), \tag{30}$$

where δ is defined by (25) and $c = c(G, d)$ comes from Lemma 8. Then, with

$$p_1 = \widetilde{C} n^{-1/m_G^{(k)}}, \tag{31}$$

(27) is satisfied. We leave the determination of the constant a for an anxious reader.

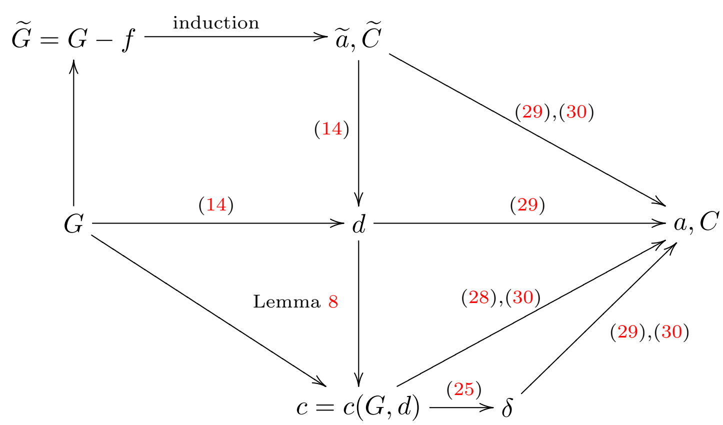

FIGURE 2. Flowchart of constants for the proof of Theorem 9

4. Outlines of other proofs

4.1. **Theorem 1 – two colors.** The proof we present here follows the main strategy of the proof from [20] but avoids the use of regularity lemma. Therefore, rather than outlining the whole proof, we just point out how it differs from the original argument. To this end, we first give a sketch of the original proof in [20], in a simplified version for $r = 2$ colors.

One of the ingredients of the proof of Theorem 1 in [20] was the following simple result which could be viewed as an extension of Lemma 8 to the non-partite case, but limited to graphs only.

For $0 < d \leq 1$ and $0 < \rho \leq 1$ we call an n-vertex graph F (ρ, d)-*dense* if every induced subgraph of F on $v = \lceil \rho n \rceil$ vertices contains at least $d\binom{v}{2}$ edges.

Lemma 12 ([20])**.** *For every $d > 0$ and every graph H there exist $\rho > 0$ and $c > 0$ such that for every n-vertex (ρ, d)-dense graph F we have $N(F, H) \geq cn^{v_H}$.*

Thus, Lemma 8 specifies that for bipartite graphs H, Lemma 12 holds with $\rho = 1$.

The original proof of Theorem 1, similarly to the above presented proof of Theorem 9, was based on induction on e_G and the two-round exposure technique. Applying the induction assumption to all induced subgraphs of $\mathbb{G}(n, p_1)$ on ρn vertices, viewed as random graphs on their own, resulted in showing that a.a.s., for every coloring χ, the graph Γ_χ was (ρ, d)-dense.

Then, by an application of Szemerédi's regularity lemma for graphs, it was shown that either $\Gamma_\chi^{\text{blue}}$ or Γ_χ^{red} contained a (ρ', d')-dense subgraph F with some new parameters. By Lemma 12 with $H := G$, the graph F contained lots of copies of G and from that point on, the proof went along the same lines as the proof of Theorem 9.

Now, we describe how one can avoid the use of regularity lemma. The crucial change is to apply Lemma 12 directly to the graph $F = \Gamma_\chi$ with $H = K_R$, where $R = R(G)$ is the Ramsey number for the graph G. As a result, Γ_χ contains $\Theta(n^R)$ copies of K_R. Consequently, by the definition of $R(G)$, the partition $\Gamma_\chi = \Gamma_\chi^{\text{blue}} \cup \Gamma_\chi^{\text{red}}$, contains $\Theta(n^{v_G})$ copies of G in one class, say Γ_χ^{red}, and the proof can be completed as before.

4.2. **Theorem 2 – more colors.** As we have mentioned in Section 1, Theorems 1, 2, and 3 remain true for $r \geq 3$ colors, but the proofs become more technical. While for $r > 2$, the r-colored version of Theorem 1 seems to be much harder to prove than the 2-colored version, for Theorem 2 the proofs of these two cases do not differ essentially.

Below we outline the proof of the r-colored version of Theorem 2, $r \geq 2$. We write $F \longrightarrow (G, r)$ if every r-coloring of the edges of F results in a monochromatic copy of G.

Theorem 13. *For all $k \geq 2$ and $r \geq 2$ and for every k-uniform, k-partite hypergraph G with $\Delta(G) \geq 2$ there exists $C > 0$ such that for every sequence $p = p(n) \geq Cn^{-1/m_G^{(k)}}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\mathbb{G}^{(k)}(n,p) \longrightarrow (G,r)\right) = 1\,.$$

For two colors we argued that in round two, either $\Gamma_{p_2}^{\text{red}}$ had many edges colored red, or it contained many copies of G colored blue. With more colors we may only claim that either $\Gamma_{p_2}^{\text{red}}$ has many edges colored red, or not so many. Since, in view of Lemma 5, these few red edges can be deleted, this calls for induction on the number of colors r.

To make this idea work, we have to generalize and strengthen the statement of Theorem 13 in three ways. First, note that $\Gamma_{p_2}^{\text{red}}$ is a random sub-hypergraph of an incomplete hypergraph Γ^{red}. Hence, for the sake of induction, we must generalize our statement to random sub-hypergraphs F_p of dense hypergraphs F. But then, not every closing non-edge is in F, and we better restrict our attention to those monochromatic copies of $\widetilde{G}$ in F whose complements are also in F. We call such copies of G *nested*. We write

$$F \xrightarrow[\text{nested}]{a} (G, r)$$

if every r-coloring of the edges of F results in at least $aN(F,G)$ nested, monochromatic copies of G.

Finally, since the second round will now be successful if our statement holds for $r-1$ colors, the probability of the failure must be, as all failures in round two, exponentially small (to beat the number of colorings χ from the first round). All in all, we are to prove the following statement.

Theorem 14. *For all integers $k \geq 2$ and $r \geq 1$, every k-uniform k-partite hypergraph G with at least one edge, and for every real $0 < d \leq 1$, there exist positive numbers a, b, C, and n_0 such that if*

(i) $n > n_0$
(ii) *F is a k-uniform hypergraph with $e_F \geq dn^k$, and*
(iii) $p = p(n) > Cn^{-1/m_G^{(k)}}$,

then

$$\mathbb{P}\left(F_p \xrightarrow[\text{nested}]{a} (G,r)\right) > 1 - \mathrm{e}^{-be_F p}.$$

The proof of Theorem 14 is by double induction on r and e_G. The case $e_G = 1$, or more generally, $\Delta(G) = 1$, is practically the same as in the proof of Theorem 9, while the case $r = 1$, relies on Lemma 8 and Janson's inequality (4).

The proof of the induction step boils down to showing analogs of Facts 10 and 11, except that now also Fact 10 must hold with probability exponentially close to 1. The most difficult part is then to prove that (20) holds with probability exponentially close to 1, for which we apply a technique for bounding upper tails of subgraph counts called the deletion method (see Lemma 2.51 in [12] and also [13]), combined with Lemma 5.

We also employ Lemma 5, as before, inside the proof of the analog of Fact 11. This is no longer preceded by Janson's inequality, but instead, the induction's hypothesis with $r-1$ colors. In a sense, Janson's inequality is equivalent to Theorem 14 for $r = 1$.

5. Open Problems

The main problem which remains open is to prove Theorem 2 for arbitrary (not necessarily k-partite) k-uniform hypergraphs G. To do so, we need to find the right notion of a *dense* hypergraph F, for which, on the one hand, an extension of Lemma 12 holds, while on the other hand, it could be proved that Γ_χ (cf. Section 4.1) is dense in the sense of that new concept.

Another, related problem is to find threshold probabilities for the Turán properties of $\mathbb{G}^{(k)}(n,p)$. For a k-uniform hypergraph G, let

$$\mathrm{ex}(n,G) = \max\{\mathrm{e}(F)\colon\ F \text{ is a } k\text{-uniform hypergraph}, G \nsubseteq F, \text{ and } v(F) = n\}$$

and let $\pi(G) = \lim_{n\to\infty} \mathrm{ex}(n,F)/\binom{n}{k}$. It is well-known that the limit $\pi(G)$ exists for every G (see, e.g., [14]). For example, Lemma 8 implies that $\pi(G) = 0$ for every k-partite, k-uniform hypergraph G.

Given a hypergraph G, we say that a family of hypergraphs $\mathcal{F}$ has the *Turán property* if for every $\delta > 0$ every sufficiently large hypergraph $F \in \mathcal{F}$ has the property that every sub-hypergraph F' of F with $e(F') \geq (\pi(G)+\delta)e(F)$ contains a copy of G. In the case of random graphs, i.e., $\mathcal{F} = \{\mathbb{G}(n,p)\colon\ n \in \mathbb{N}\}$, thresholds for Turán properties were established so far only for very few cases, including odd and even cycles [10, 9], and small cliques K_4 and K_5 [8, 15] (see also [22, 16] for weaker bounds for general graphs G). This experience with random graphs suggests that Turán thresholds should coincide with those for Ramsey properties.

As opposed to Ramsey properties, the 0-statements for Turán properties are rather easy. Indeed, we know that for $p = o(n^{-1/m_G^{(k)}})$, there are in $\mathbb{G}^{(k)}(n,p)$ a.a.s. $o(n^k p)$ copies of the least likely (the densest) subgraph H of G. These copies, and thus, all copies of G in $\mathbb{G}^{(k)}(n,p)$, can be destroyed by removing $o(n^k p)$ edges. This shows that for $p = o(n^{-1/m_G^{(k)}})$, the Turán property with respect to G does not hold a.a.s. (see [12, Section 8.1] for the case $k = 2$).

The real challenge is the 1-statement, but, in view of Lemma 8, we believe that similarly to Ramsey properties, the case of k-partite G is somewhat easier. In particular, the following conjecture seems to be true.

Conjecture 15. *For all integers $k \geq 2$, for every k-partite k-uniform hypergraph G, and for all $\delta > 0$ there exists $C > 0$ such that if $p \geq Cn^{-1/m_G^{(k)}}$, then a.a.s. every sub-hypergraph F of $\mathbb{G}^{(k)}(n,p)$ with $e(F) \geq \delta e(\mathbb{G}^{(k)}(n,p))$ contains a copy of G. In particular, if $pn^{1/m_G^{(k)}} \to \infty$, then a.a.s. $\mathbb{G}^{(k)}(n,p)$ has the Turán property with respect to G.*

For $k = 2$ (graphs), the conjecture was proved only for even cycles in [9]. It would be most interesting to settle it for $G = K_{3,3}$. For $k \geq 3$ nothing is known, except that for p constant, Lemma 8 implies the conclusion of the above conjecture for all k-partite k-uniform hypergraphs G, $k \geq 2$.

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