EVERY MONOTONE 3-GRAPH PROPERTY IS TESTABLE

CHRISTIAN AVART, VOJTĚCH RÖDL, AND MATHIAS SCHACHT

Abstract. Recently Alon and Shapira [*Every monotone graph property is testable*, Proceedings of the thirty-seventh annual ACM symposium on Theory of computing, ACM Press, 2005, pp. 128–137] have established that every monotone graph property is testable. They raised the question whether their results can be extended to hypergraphs. The aim of this paper is to address this problem. Based on the recent regularity lemma of the last two authors [*Regular partitions of hypergraphs*, Combin. Probab. Comput., to appear.] we prove that any monotone property of 3-uniform hypergraphs is testable answering in part a question of Alon and Shapira. Our approach is similar to the one developed by Alon and Shapira for graphs. The authors believe that based on the general version of the hypergraph regularity lemma the proof presented in this article extends to k-uniform hypergraphs.

1. Introduction

1.1. **Basic definitions.** Let $k \geq 2$ be an integer and $\mathscr{A}$ a property of k-uniform hypergraphs. In other words, $\mathscr{A}$ is a (possibly infinite) family of k-uniform hypergraphs and we say that a given hypergraph $\mathcal{H}$ satisfies $\mathscr{A}$ if $\mathcal{H} \in \mathscr{A}$. In this paper we only consider *decidable properties* $\mathscr{A}$, which are those for which there is an algorithm that decides if $\mathcal{H} \in \mathscr{A}$ or $\mathcal{H} \notin \mathscr{A}$ in finite time (depending on the size of $\mathcal{H}$) for every k-uniform hypergraph $\mathcal{H}$.

For a given constant $\eta > 0$, we say a k-uniform hypergraph $\mathcal{H}$ on n vertices is *η-far* from $\mathscr{A}$ if no k-uniform hypergraph $\mathcal{G}$ on the same vertex set with $|E(\mathcal{G}) \triangle E(\mathcal{H})| \leq \eta n^k$ satisfies $\mathscr{A}$. This is a natural measure of how far the given hypergraph $\mathcal{H}$ is from satisfying the property $\mathscr{A}$.

We consider randomized algorithms which for an input hypergraph $\mathcal{H}$ on the vertex set $\{1, 2, \dots, n\} = [n]$ are able to make queries whether a given k-tuple of vertices spans an edge in $\mathcal{H}$. For a property $\mathscr{A}$ and a constant $\eta > 0$, such an algorithm will be called a *tester* for $\mathscr{A}$ if it can distinguish with, say probability $2/3$, whether $\mathcal{H}$ satisfies $\mathscr{A}$ or is η-far from it. If a property $\mathscr{A}$ has for every $\eta > 0$ a tester whose *query complexity* (i.e., number of queries) is bounded by a function of η and $\mathscr{A}$ but is independent of the number of vertices of the input hypergraph $\mathcal{H}$, the property is called *testable*.

One can observe that some simple properties as connectivity or containing a copy of some fixed hypergraph $\mathcal{F}$ are not testable. Perhaps surprisingly, many other properties, e.g. being $\mathcal{F}$-free, are testable.

1.2. **Testable graph properties.** The general notion of property testing was introduced by Rubinfeld and Sudan in [26]. In [14], Goldreich, Goldwasser and Ron

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initiated the study of property testing for combinatorial structures. In the present paper the combinatorial structures we focus on are hypergraphs. Our work builds on some of the earlier work of Alon et al. In a series of papers [1, 2, 3, 6, 4, 5] Alon and his co-authors investigated testability of graph properties. This line of research culminated in the recent result of Alon and Shapira [4] asserting that every hereditary property $\mathscr{A}$, i.e., $\mathscr{A}$ is closed under taking induced subgraphs, is testable (see also Lovász and Szegedy [19] for an alternative proof). A central tool in the work for graphs is *Szemerédi's regularity lemma* (see Theorem 4) for graphs [28].

Some ideas of property testing for graphs were already present before the notion of a tester was developed. For example if $\mathscr{A}$ consists of all graphs not containing a fixed graph F (as a not necessarily induced subgraph), then the existence of a tester for $\mathscr{A}$ follows from the so-called *removal lemma* for graphs. The removal lemma asserts that for every graph F and every $\eta > 0$ there exists a $c > 0$ such that if G is an n-vertex graph which is η-far from being F-free, then G contains at least $cn^{|V(F)|}$ copies of F. This result was first obtained for F being the triangle K_3 by Ruzsa and Szemerédi [27] and later extended to arbitrary graphs F by Erdős, Frankl, and Rödl [12]. Those results can straightforwardly be generalized to prove the testability of properties $\mathscr{A}$, which can be defined by a *finite* collection $\mathscr{F}$ of forbidden subgraphs, i.e.,

$$\mathscr{A} = \mathrm{Forb}(\mathscr{F}) := \{G\colon\ F \nsubseteq G \text{ for every } F \in \mathscr{F}\} \tag{1}$$

with $|\mathscr{F}| < \infty$ (see discussion in Section 3.1).

For *infinite* families $\mathscr{F}$ it follows for example from a result of Bollobás, Erdős, Simonovits, and Szemerédi [7] that being bipartite is a testable graph property. In [10] answering a question of Erdős (see, e.g., [11]) Duke and Rödl generalized the result from [7] and proved that being h-colorable is a testable for any $h \geq 2$. The proof in [10] is also based on Szemerédi's regularity lemma. Later this result and related results were established by Goldreich, Goldwasser, and Ron [14] and subsequently improved by Alon and Krivelevich [3]. The authors of [14] and [3] could avoid using Szemerédi's regularity lemma and, consequently, obtained much better bounds on the query complexity for the testers.

The general problem for *monotone* graph properties, which are those properties as described in (1) with a possibly *infinite* forbidden family $\mathscr{F}$, was solved by Alon and Shapira [5]. They showed that every monotone graph property is testable and asked if the same holds for hypergraphs. In this paper we answer their question positively for 3-uniform hypergraphs (see Theorem 2 below). Our proof uses the ideas of Alon and Shapira and is based on the recent hypergraph extensions of Szemerédi's regularity lemma [13, 15, 21, 23, 24, 29]. The transition from graphs to hypergraphs leads, however, to some technical difficulties. In this paper we restrict ourself to 3-uniform hypergraphs. This case already reflects the main differences between the graphs and the general case of k-uniform hypergraphs, but allows us to simplify the notation and to improve the presentation of the proof. We believe that the argument can be extended with no major conceptual modification to k-uniform hypergraphs (see also Section 5).

The main result of the present paper is the first general result for 3-uniform hypergraphs which establishes testability for a fairly natural and general class of properties. A few other hypergraph results were already known before, e.g., h-colorability [9], not containing one fixed *induced* sub-hypergraph [17], and not containing one fixed *non-induced* sub-hypergraph [20, 21].

1.3. **Main result.** We now state the main result of the paper. A 3-uniform hypergraph $\mathcal{H}$ on the vertex set V is some family of 3-element subsets of V, i.e., $\mathcal{H} \subseteq \binom{V}{3}$. Note that we identify hypergraphs with its edge set and we write $V(\mathcal{H})$ for the vertex set. We recall that a property $\mathscr{A}$ of 3-uniform hypergraphs is *monotone* if $\mathcal{H} \in \mathscr{A}$ implies that every (not necessarily induced) sub-hypergraph $\mathcal{G} \subseteq \mathcal{H}$ exhibits property $\mathscr{A}$ as well. In other words, $\mathscr{A}$ is closed under removal of vertices and edges. Note that if $\mathscr{A}$ is a monotone property and the hypergraph $\mathcal{H}$ does not satisfy $\mathscr{A}$ then no hypergraph obtained by adding edges to $\mathcal{H}$ will satisfy $\mathscr{A}$. Consequently, for monotontone properties the definition of η-far given earlier is equivalent to the following.

Definition 1. For a monotone property $\mathscr{A}$ we say an n-vertex 3-uniform hypergraph $\mathcal{H}$ is *η-far from $\mathscr{A}$* if every sub-hypergraph $\mathcal{G}$ of $\mathcal{H}$ with $|\mathcal{H} \setminus \mathcal{G}| \leq \eta n^3$ satisfies $\mathcal{G} \notin \mathscr{A}$.

We say a tester has *one-sided error* if it confirms with probability 1 that $\mathcal{H} \in \mathscr{A}$. In other words, whenever $\mathcal{H}$ satisfies $\mathscr{A}$, the algorithm will be correct with probability equal to 1. Moreover, a property $\mathscr{A}$ is testable with one-sided error, if for every $\eta > 0$ there exists a tester with one-sided error.

In [5] Alon and Shapira proved that for any (decidable) monotone graph property $\mathscr{A}$ and any $\eta > 0$ there exists a tester which after a bounded number of random edge queries comes to the following conclusion:

- If $\mathcal{H} \in \mathcal{P}$, then the tester confirms it with probability 1.
- If $\mathcal{H}$ is η-far from $\mathscr{A}$, then the tester outputs with probability 2/3 that $\mathcal{H} \notin \mathcal{P}$.
- Otherwise, if $\mathcal{H} \notin \mathscr{A}$ and $\mathcal{H}$ is not η-far from $\mathscr{A}$, then there are no guarantees for the output of the tester.

In this paper we generalize this result from graphs to 3-uniform hypergraphs.

Theorem 2. *Every decidable and monotone property $\mathscr{A}$ of 3-uniform hypergraphs is testable with one-sided error.*

As discussed earlier, monotone properties can be described by a (possibly infinite) family of forbidden hypergraphs, i.e, for every monotone property $\mathscr{A}$ there exists a family of hypergraphs $\mathscr{F}$ such that $\mathscr{A} = \mathrm{Forb}(\mathscr{F})$ where $\mathrm{Forb}(\mathscr{F})$ is the family of those hypergraphs not containing any element of $\mathscr{F}$ as a (not necessarily induced) sub-hypergraph. Theorem 1 is then a consequence of the following result, as we will show momentarily.

Theorem 3. *Let $\mathscr{F}$ be a family of 3-uniform hypergraphs and $\mathscr{A} = \mathrm{Forb}(\mathscr{F})$. For all $\eta > 0$ there exists $c = c(\mathscr{A}, \eta) > 0$ and there are positive integers $C = C(\mathscr{A}, \eta)$ and $n_0 = n_0(\mathscr{A}, \eta)$ such that the following holds.*

If $\mathcal{H}$ is a 3-uniform hypergraph on $n \geq n_0$ vertices which is η-far from satisfying $\mathscr{A}$, then there exists a hypergraph $\mathcal{F}_0 \in \mathscr{F}$ on $f_0 \leq C$ vertices such that the number of copies of $\mathcal{F}_0$ in $\mathcal{H}$ is at least cn^{f_0}.

Theorem 1 easily follows from Theorem 2.

Proof: Theorem 3 $\Longrightarrow$ Theorem 2. Let a decidable and monotone property $\mathscr{A} = \mathrm{Forb}(\mathscr{F})$ and some $\eta > 0$ be given. By Theorem 2, there is some $c > 0$ and there are integers C and $n_0 \in \mathbb{N}$ such that any 3-uniform hypergraph on $n \geq n_0$ vertices, which is η-far from exhibiting $\mathscr{A}$ contains at least $cn^{|V(\mathcal{F}_0)|}$ copies of some $\mathcal{F}_0 \in \mathcal{F}$ with $|V(\mathcal{F}_0)| \leq C$.

Let $s \in \mathbb{N}$ be such that $(1-c)^{s/C} < 1/3$ and set $m_0 = \max\{s, n_0\}$. We claim that there exists a one-sided tester with query complexity $\binom{m_0}{3}$ for $\mathscr{A}$. For that let $\mathcal{H}$ be a 3-uniform hypergraph on n vertices. If $n \leq m_0$, then the tester simply queries all edges of $\mathcal{H}$ and since $\mathscr{A}$ is decidable, there is an exact algorithm with running time only depending on the fixed m_0, which determines correctly if $\mathcal{H} \in \mathscr{A}$ or not.

Consequently, let $n > m_0$. Then we choose uniformly at random a set S of s vertices from $\mathcal{H}$. Consider the hypergraph $\mathcal{H}[S] = \mathcal{H} \cap \binom{S}{3}$ induced on S. If $\mathcal{H}[S]$ has $\mathscr{A}$, then the tester says "yes" and otherwise "no." Since $\mathscr{A}$ is decidable and s is fixed the algorithm decides whether or not $\mathcal{H}[S]$ is in $\mathscr{A}$ in constant time (constant only depending on s and $\mathscr{A}$).

Clearly, if $\mathcal{H} \in \mathscr{A}$ or $n \leq m_0$, then this tester outputs correctly and hence it is one-sided. On the other hand, if $\mathcal{H}$ is η-far from $\mathscr{A}$ and $n > m_0$, then due to Theorem 3 the random set S spans a copy of $\mathcal{F}_0$ for some $\mathcal{F}_0 \in \mathscr{F}$ on $f_0 \leq C$ vertices, with probability at least

$$cn^{f_0}/\binom{n}{f_0} \geq c. \tag{2}$$

Hence the probability that S does not span any copy of $\mathcal{F}_0$ is at most $(1-c)^{s/f_0} \leq (1-c)^{s/C} < 1/3$. In other words, S spans a copy of $\mathcal{F}_0$ with probability at least $2/3$, which shows that the tester works as specified. $\square$

From now on we are only concerned with the proof of Theorem 3. The main philosophy of the proof of Theorem 3 is similar to the corresponding statement for graphs in [5], which was originally obtained by Alon, Fischer, Krivelevich, and Szegedy in [2]. The proof requires a strengthening of the hypergraph regularity lemma analogous to the modification of Szemerédi's regularity lemma proved in [2]. A similar lemma for 3-uniform hypergraphs was already proved by Kohayakawa, Nagle, and Rödl [17] based on the regularity lemma for 3-uniform hypergraphs of Frankl and Rödl [13] (see also [24]). We give here a different (and simpler) proof, based on a "cleaner" version of the regularity lemma from [13], which was obtained for general k-uniform hypergraphs by the last two authors [23]. We call this auxiliary result the *representative lemma* (see Lemma 16 below).

This paper is organized as follows. In Section 2, we develop the necessary definitions for the regularity method of 3-uniform hypergraphs. In particular we state the *hypergraph regularity lemma* (Theorem 13), the corresponding *counting lemma* (Theorem 8), and the *representative lemma* (Lemma 16). Section 3 is devoted to the proof of Theorem 3 and in Section 4 we prove the representative lemma.

2. Regularity method for hypergraphs

In this section we recall some definitions of the hypergraph regularity method following the approach from [13].

2.1. **Szemerédi's regularity lemma.** We start the discussion with graphs. Given a graph G and disjoint subsets X, $Y \subseteq V(G)$, the *density* of the pair (X,Y) is

$$d_G(X,Y) = \frac{e_G(X,Y)}{|X||Y|}\,, \tag{3}$$

where $e_G(X,Y)$ denotes the number of edges in G with one vertex in X and one vertex in Y. The pair (X,Y) will be called (ε,d)-*regular* if for every $X'\subseteq X$ and $Y'\subseteq Y$ such that $|X'|\ge\varepsilon|X|$ and $|Y'|\ge\varepsilon|Y|$ we have

$$|d_G(X',Y')-d|<\varepsilon\,. \tag{4}$$

We also say (X,Y) is ε-regular if it is (ε,d)-regular for some d. Roughly speaking, an (ε,d)-regular pair (X,Y) behaves in a similar way as a random bipartite graph on the same vertex sets, where each edge appears with probability d. Szemerédi's regularity lemma [28] states that for every $\varepsilon>0$, we can partition the vertex set of any large graph into a bounded number (only depending on ε) of sets such that almost all bipartite graphs between the partition classes are ε-regular.

Theorem 4 (Szemerédi's regularity lemma [28])**.** *For any $\varepsilon>0$ and any integer t_0, there are positive integers $T_0=T_0(\varepsilon,t_0)>t_0$ and $n_0=n_0(\varepsilon,t_0)$ such that for every graph $G=(V,E)$ with $|V|=n\ge n_0$ there exists a partition $\mathscr{P}^{(1)}=\{V_1,V_2,\dots,V_t\}$ of V such that*

(*i*) $t_0\le t\le T_0$,
(*ii*) $||V_i|-|V_j||\le 1$ *for all* $1\le i<j\le t$, *and*
(*iii*) *all but* εt^2 *pairs* (V_i,V_j) *are* ε*-regular, where* $1\le i<j\le t$.

This lemma is a powerful tool in extremal graph theory (see [18] for a survey or many of its applications). It is often used in conjunction with the so-called *counting lemma* for graphs. We will later need the simplest form of that lemma for triangles.

Lemma 5 (Triangle counting lemma [18])**.** *For all constants $\gamma>0$ and $d>0$ there exists $\varepsilon_{\rm tcl}=\varepsilon_{\rm tcl}(\gamma,d)>0$ and $m_{\rm tcl}=m_{\rm tcl}(f,\gamma,d)\in\mathbb{N}$ such that the following holds. If $\mathcal{P}$ is a tripartite graph with vertex classes V_1, V_2, and V_3, of size $|V_1|=|V_2|=|V_3|=m\ge m_{\rm tcl}$ and if, moreover, (V_i,V_j) is $(\varepsilon_{\rm tcl},d)$-regular for all $1\le i<j\le 3$, then the number of triangles K_3 in $\mathcal{P}$ is in the interval $(1\pm\gamma)d^3m^3$.*

An extension of Szemerédi's regularity lemma for 3-uniform hypergraphs has been developed in [13]. More recently extensions to k-uniform hypergraphs were obtained by several authors in [15, 16, 24] and subsequently in [23, 29]. The key feature of all those extensions of Theorem 4 to hypergraphs mentioned above, is that it allows to prove a corresponding extension of the counting lemma, Lemma 5, as shown in [13, 15, 16, 20, 21, 23, 29].

In our proof we will use the regularity lemma and the counting lemma for hypergraphs from [23]. Since, in this paper, we only focus on 3-uniform hypergraphs, we develop the definitions only for that case, following the approach from [13]. Moreover, from now on, by a hypergraph we mean a 3-uniform hypergraph.

2.2. **Regular hypergraphs and the counting lemma for hypergraphs.** Let V_1, V_2, and V_3 be mutually disjoint subsets of some vertex set V. We call a triple $\hat{\mathcal{Q}}=(Q^{12},Q^{13},Q^{23})$ of bipartite graphs with vertex sets $V_1\cup V_2$, $V_2\cup V_3$ and $V_1\cup V_3$ a *triad*. Usually, we will think of a triad $\hat{\mathcal{Q}}=(Q^{12},Q^{13},Q^{23})$ as a tripartite graph with vertex set $V_1\cup V_2\cup V_3$ and edge set $E(Q^{12})\cup E(Q^{13})\cup E(Q^{23})$. For the regularity of hypergraphs, triads play the same role as pairs of vertex sets in Szemerédi's regularity lemma.

For a triad $\hat{\mathcal{Q}}=(Q^{12},Q^{13},Q^{23})$ with vertex set $V_1\cup V_2\cup V_3$ we define $\mathrm{Tr}(\hat{\mathcal{Q}})$ as the set of triples of vertices of $\hat{\mathcal{Q}}$ each inducing a triangle in $\hat{\mathcal{Q}}$

$$\mathrm{Tr}(\hat{\mathcal{Q}})=\big|\big\{\{v_1,v_2,v_3\}\colon\ v_i\in V_i \text{ and } v_iv_j\in E(Q^{ij}) \text{ for all } 1\le i<j\le 3\big\}\big|\,.$$

For a hypergraph $\mathcal{H}$ on some vertex set V and a triad $\hat{\mathcal{Q}}$ with vertex classes V_1, V_2, and $V_3 \subset V$ we define the *density of* $\mathcal{H}$ *on the triad* $\hat{\mathcal{Q}}$ as

$$d_{\mathcal{H}}(\hat{\mathcal{Q}}) = \begin{cases} \frac{|\mathcal{H}\cap \operatorname{Tr}(\hat{\mathcal{Q}})|}{|\operatorname{Tr}(\hat{\mathcal{Q}})|} & \text{if } |\operatorname{Tr}(\hat{\mathcal{Q}})| > 0\,,\\ 0 & \text{otherwise}\,. \end{cases} \tag{5}$$

This is a natural extension of the notion of density from graphs with respect to pairs (see (3)) to hypergraphs w.r.t. triads. We generalize the last definition to the density of an r-tuple of sub-triads of a given triad. We say a tripartite graph $\hat{\mathcal{X}} = (X^{12}, X^{13}, X^{23})$ with vertex sets W_1, W_2, and W_3 is a sub-triad of a triad $\hat{\mathcal{Q}} = (Q^{12}, Q^{13}, Q^{23})$ with vertex sets $V_1 \supseteq W_1$, $V_2 \supseteq W_2$, and $V_3 \supseteq W_3$ if for every $1 \le i < j \le 3$ we have $E(X^{ij}) \subseteq E(Q^{ij})$. For a given triad $\hat{\mathcal{Q}} = (Q^{12}, Q^{13}, Q^{23})$ and a family of (not necessarily disjoint) sub-triads $\hat{\boldsymbol{\mathcal{X}}} = \{\hat{\mathcal{X}}_s = (X_s^{12}, X_s^{13}, X_s^{23})\colon\ s = 1, \dots, r\}$ we define

$$\operatorname{Tr}(\hat{\boldsymbol{\mathcal{X}}}) = \bigcup_{i=1}^{r} \operatorname{Tr}(\hat{\mathcal{X}}_i)$$

and extend (5) by setting

$$d_{\mathcal{H}}(\hat{\boldsymbol{\mathcal{X}}}) = \begin{cases} \frac{|\mathcal{H}\cap \operatorname{Tr}(\hat{\boldsymbol{\mathcal{X}}})|}{|\operatorname{Tr}(\hat{\boldsymbol{\mathcal{X}}})|} & \text{if } |\operatorname{Tr}(\hat{\boldsymbol{\mathcal{X}}})| > 0\,,\\ 0 & \text{otherwise}\,. \end{cases}$$

We now proceed to a central definition and extend the notion of a regular pair to a regular triad.

Definition 6 ((δ, d, r)-regularity)**.** Let $\delta > 0$, $d > 0$ and $r \in \mathbb{N}$. We say a hypergraph $\mathcal{H}$ is (δ, d, r)*-regular* with respect to a triad $\hat{\mathcal{Q}} = (Q^{12}, Q^{13}, Q^{23})$ on the vertex sets V_1, V_2, and $V_3 \subseteq V(\mathcal{H})$ if for any family of r sub-triads $\hat{\boldsymbol{\mathcal{X}}} = \{\hat{\mathcal{X}}_s\colon\ s = 1, \dots, r\}$ satisfying

$$|\operatorname{Tr}(\hat{\boldsymbol{\mathcal{X}}})| > \delta |\operatorname{Tr}(\hat{\mathcal{Q}})|$$

we have

$$|d_{\mathcal{H}}(\hat{\boldsymbol{\mathcal{X}}}) - d| < \delta\,.$$

This notion was introduced in [13] and similarly as Szemerédi's regularity lemma decomposes every graph in a bounded number of "mostly" regular pairs, the hypergraph regularity lemma (upcoming Theorem 13) will partition the edge set of any given hypergraph into "triads" in such a way that most of them are regular in the sense of Definition 6. In order to simplify the notation we sometimes do not specify the density d. We will say a hypergraph is $(\delta, *, r)$*-regular* if it is (δ, d, r)-regular for some density d.

The counting lemma for hypergraphs is a crucial tool in our proof of Theorem 3. It ensures the existence of many copies of a fixed small hypergraph inside a larger, dense and "sufficiently regular" hypergraph $\mathcal{H}$. We need a few more definitions before we give the precise statement.

Let $V_1 \cup V_2 \cup \dots \cup V_f$ be a partition of some vertex set V. We denote by $K_f(V_1, \dots, V_f)$ the complete f-partite graph on that partition. Let R be any f-partite subgraph of $K_f(V_1, \dots, V_f)$ on the same vertex partition and as above, let $\operatorname{Tr}(R)$ be the set of those 3-element subsets of V, which span a K_3 in R. We say R *underlies a hypergraph* $\mathcal{H}$ on the same vertex set V if $\mathcal{H} \subseteq \operatorname{Tr}(R)$. This leads to the notion of a *regular complex*.

Definition 7 (regular complex)**.** Let positive integers f, m, and $r \in \mathbb{N}$ and positive constants δ_2, δ_3, d_2, $d_3 > 0$ be given. Suppose $\mathcal{F}$ is a hypergraph with vertex set $[f] = \{1, 2, \dots, f\}$, $V_1 \cup V_2 \cup \dots \cup V_f$ is a partition of some vertex set V, $R \subseteq K_f(V_1, \dots, V_f)$ and R underlies a hypergraph $\mathcal{H}$ with vertex set $V(\mathcal{H}) = V$. We say the pair $(R, \mathcal{H})$ is a $(\delta_2, \delta_3, d_2, d_3, r)$*-regular* $(m, \mathcal{F})$*-complex* if the following holds

- (i) $|V_i| = m$ for all $i = 1, \dots, f$,
- (ii) for every $1 \leq i < j \leq f$ such that $\{i, j, k\} \in \mathcal{F}$ for some $k \in [f]$ the induced subgraph $R^{ij} = R[V_i, V_j]$ of R on the vertex sets V_i and V_j is (δ_2, d_2)-regular, and
- (iii) for every $\{i, j, k\} \in \mathcal{F}$ the hypergraph $\mathcal{H}$ is (δ_3, d_{ijk}, r)-regular w.r.t. the triad $\hat{\mathcal{R}} = (R^{ij}, R^{ik}, R^{jk})$ for some $d_{ijk} \geq d_3$.

The counting lemma for hypergraphs extends Lemma 5 and gives a bound on the number of copies of a fixed hypergraph $\mathcal{F}$ in $\mathcal{H}$ for sufficiently $(\delta_2, \delta_3, d_2, d_3, r)$-regular $(m, \mathcal{F})$-complexes $(R, \mathcal{H})$.

Theorem 8 (Counting lemma for 3-uniform hypergraphs [20])**.** *For every $f \in \mathbb{N}$ and constants $\gamma > 0$ and $d_3 > 0$, there exist $\delta_3 = \delta_3(f, \gamma, d_3) > 0$ such that for every $d_2 > 0$ there exist $\delta_2 = \delta_2(f, \gamma, d_3, d_2) > 0$ and positive integers $r = r(f, \gamma, d_3, d_2)$ and $m_0 = m_0(f, \gamma, d_3, d_2) \in \mathbb{N}$ such that the following holds.*

Suppose $\mathcal{F}$ is a hypergraph with vertex set $[f] = \{1, \dots, f\}$, $V_1 \cup V_2 \cup \dots \cup V_f$ is a partition of some vertex set V, $R \subseteq K_f(V_1, \dots, V_f)$ and R underlies a hypergraph $\mathcal{H}$ with vertex set $V(\mathcal{H}) = V$. If, moreover, $(R, \mathcal{H})$ is a $(\delta_2, \delta_3, d_2, d_3, r)$-regular $(m, \mathcal{F})$-complex with $m \geq m_0$, then the number of copies of $\mathcal{F}$ in $\mathcal{H}$ is at least

$$(1 - \gamma) d_2^{|\Delta(\mathcal{F})|} d_3^{|\mathcal{F}|} m^f \,, \tag{6}$$

where $\Delta(\mathcal{F})$ is the shadow of $\mathcal{F}$, i.e.,

$$\Delta(\mathcal{F}) = \big\{\{i, j\} \colon 1 \leq i < j \leq f \text{ so that there exists } k \in [f] \text{ with } \{i, j, k\} \in \mathcal{F}\big\}\,.$$

A generalization of this counting lemma to k-uniform hypergraphs can be found in [21] and [23].

2.3. **Regularity lemma for hypergraphs.** In this section we state a variant of the regularity for 3-uniform hypergraphs [13], which was obtained by the last two authors for general k-uniform hypergraphs in [23]. First we generalize the concept of vertex partition present in Szemerédi's regularity lemma.

Definition 9 ((t, ℓ)-partition)**.** Let V be a vertex set, let $\mathscr{P}^{(1)} = \{V_1, V_2, \dots, V_t\}$ be a partition of V, and let $\mathscr{P}^{(2)} = \{P_\alpha^{ij} \colon 1 \leq i < j \leq t \text{ and } 1 \leq \alpha \leq \ell\}$ be a family of $\binom{t}{2}\ell$ bipartite graphs. We say the pair $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ is a (t, ℓ)-*partition*[1] *on* V if for every $1 \leq i < j \leq t$ the family $\{E(P_1^{ij}), E(P_2^{ij}), \dots, E(P_\ell^{ij})\}$ is a partition of the edge set of the complete bipartite graph $K_2(V_i, V_j)$.

We say a (t, ℓ)-partition is T*-bounded* if $\max\{t, \ell\} \leq T$. Moreover, for a (t, ℓ)-partition $\mathscr{P}$, we denote by $\hat{\mathscr{P}}$ the set of all triads of the form $(P_\alpha^{ij}, P_\beta^{ik}, P_\gamma^{jk})$ with $1 \leq \alpha, \beta, \gamma \leq \ell$ and $1 \leq i < j < k \leq t$.

[1] Note that while $\mathscr{P}^{(1)}$ is a partition of the V, the family of bipartite graphs $\mathscr{P}^{(2)}$ is not a partition of $\binom{V}{2}$, but a partition of the edge set of the complete t-partite graph $K_t(V_1, \dots, V_t)$.

We consider such (t, ℓ)-partitions for which the bipartite graphs P_α^{ij} are μ-regular. Moreover, similarly as in Szemerédi's regularity lemma we will require the vertex partition classes to have almost the same size. This leads us to the following definition.

Definition 10 ((μ, t, ℓ)-equitable)**.** We say a (t, ℓ)-partition $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ is (μ, t, ℓ)-*equitable* if

(*i*) $\mathscr{P}^{(1)} = \{V_1, V_2, \dots, V_t\}$ is equitable, i.e., for all $1 \le i < j \le t$ we have $||V_i| - |V_j|| \le 1$ and
(*ii*) for every $1 \le i < j \le t$ and $1 \le \alpha \le \ell$ the bipartite graph $P_\alpha^{ij} \in \mathscr{P}^{(2)}$ is $(\mu, 1/\ell)$-regular on the pair (V_i, V_j).

The regularity lemma from [23] guarantees the existence of a T-bounded (μ, t, ℓ)-equitable partition $\mathscr{P}$ (where $\mu = \mu(t, \ell)$ is any function of t and ℓ) for any hypergraph $\mathcal{H}$ where T is independent of the number of vertices of $\mathcal{H}$. Moreover, the hypergraph $\mathcal{H}$ will be $(\delta, *, r)$-regular with respect to almost all triads of $\hat{\mathscr{P}}$.

Theorem 11 (Regularity lemma for 3-uniform hypergraphs [23])**.** *For every integer $t_0 \in \mathbb{N}$, every constant $\delta_{\mathscr{P}} > 0$ and all functions $\mu_{\mathscr{P}} \colon \mathbb{N}^2 \to (0, 1]$ and $r_{\mathscr{P}} \colon \mathbb{N}^2 \to \mathbb{N}$ there exist positive integers $T_0 = T_0(t_0, \delta_{\mathscr{P}}, \mu_{\mathscr{P}}, r_{\mathscr{P}})$ and $n_0 = n_0(t_0, \delta_{\mathscr{P}}, \mu_{\mathscr{P}}, r_{\mathscr{P}})$ such that for every hypergraph $\mathcal{H}$ with $n \ge n_0$ vertices V there exists a partition $\mathscr{P}$ and there are positive integers $t_{\mathscr{P}}$ and $\ell_{\mathscr{P}}$ such that for $\mu_{\mathscr{P}} = \mu_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ and $r_{\mathscr{P}} = r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ the following holds*

(*i*) *$\mathscr{P}$ is $(\mu_{\mathscr{P}}, t_{\mathscr{P}}, \ell_{\mathscr{P}})$-equitable and T_0-bounded partition on V and*
(*ii*) *$\mathcal{H}$ is $(\delta_{\mathscr{P}}, *, r_{\mathscr{P}})$-regular w.r.t. all but at most $\delta_{\mathscr{P}} t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3$ triads $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$.*

In our proof we will use the regularity lemma twice. First we use it in the form as stated above and in the second application we will refine the given partition $\mathscr{P}$ to obtain a partition $\mathscr{Q}$ with respect to which $\mathcal{H}$ will be "more regular." To state that version we need the notion of a refinement of a partition.

Definition 12 (refinement)**.** We say a partition $\mathscr{Q} = \{\mathscr{Q}^{(1)}, \mathscr{Q}^{(2)}\}$ on V *refines* a partition $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ on V and write $\mathscr{Q} \prec \mathscr{P}$ if

(*i*) for every vertex set $U \in \mathscr{Q}^{(1)}$ there exists $W \in \mathscr{P}^{(1)}$ such that $U \subseteq W$ and
(*ii*) for every bipartite graph $Q \in \mathscr{Q}^{(2)}$ there exists $P \in \mathscr{P}^{(2)}$ such that Q is a subgraph of P.

We now state that "refinement version" of Theorem 11. In fact, Theorem 11 is a simple corollary of the refinement version and a proof of that stronger version can be found in [23]. The lemma roughly states that given a $(\mu, t_{\mathscr{P}}, \ell_{\mathscr{P}})$-equitable partition $\mathscr{P}$ (with $(\mu, 1/\ell_{\mathscr{P}})$-regular auxiliary graphs $P_\alpha^{ij} \in \mathscr{P}^{(2)}$ for sufficiently small μ) any hypergraph $\mathcal{H}$ admits a partition $\mathscr{Q} \prec \mathscr{P}$ for which $\mathcal{H}$ is $(\delta, *, r)$-regular on most triads $\hat{\mathcal{Q}} \in \hat{\mathscr{Q}}$.

Theorem 13 (Refinement version of the regularity lemma [23])**.** *For all positive integers $t_{\mathscr{P}}$, $\ell_{\mathscr{P}} \in \mathbb{N}$, every constant $\delta_{\mathscr{Q}} > 0$, and all functions $\varepsilon_{\mathscr{Q}} \colon \mathbb{N}^2 \to (0, 1]$ and $r_{\mathscr{Q}} \colon \mathbb{N}^2 \to \mathbb{N}$ there exist $\mu_{\mathrm{hrl}} = \mu_{\mathrm{hrl}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}, \delta_{\mathscr{Q}}, \varepsilon_{\mathscr{Q}}, r_{\mathscr{Q}}) > 0$ and positive integers $T_{\mathrm{hrl}} = T_{\mathrm{hrl}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}, \delta_{\mathscr{Q}}, \varepsilon_{\mathscr{Q}}, r_{\mathscr{Q}})$ and $n_{\mathrm{hrl}} = n_{\mathrm{hrl}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}, \delta_{\mathscr{Q}}, \varepsilon_{\mathscr{Q}}, r_{\mathscr{Q}})$ such that the following holds. If*

(*a*) *$\mathcal{H}$ is a hypergraph with $n \ge n_{\mathrm{hrl}}$ vertices V and* (*b*) *$\mathscr{P}$ is a $(\mu_{\mathrm{hrl}}, t_{\mathscr{P}}, \ell_{\mathscr{P}})$-equitable (and hence* $\max\{t_{\mathscr{P}}, \ell_{\mathscr{P}}\}$*-bounded) partition on V,*

then there exists a partition $\mathscr{Q}$ and there are positive integers $t_{\mathscr{Q}}$ and $\ell_{\mathscr{Q}}$ such that the following holds for $t_{\mathscr{P}\mathscr{Q}} = t_{\mathscr{P}} t_{\mathscr{Q}}$, $\ell_{\mathscr{P}\mathscr{Q}} = \ell_{\mathscr{P}} \ell_{\mathscr{Q}}$, $\varepsilon_{\mathscr{Q}} = \varepsilon_{\mathscr{Q}}(t_{\mathscr{P}\mathscr{Q}}, \ell_{\mathscr{P}\mathscr{Q}})$ and $r_{\mathscr{Q}} = r_{\mathscr{Q}}(t_{\mathscr{P}\mathscr{Q}}, \ell_{\mathscr{P}\mathscr{Q}})$

(*i*) *$\mathscr{Q}$ is $(\varepsilon_{\mathscr{Q}}, t_{\mathscr{P}\mathscr{Q}}, \ell_{\mathscr{P}\mathscr{Q}})$-equitable and T_{hrl}-bounded partition on V,*

(*ii*) *$\mathscr{Q} \prec \mathscr{P}$, and*

(*iii*) *$\mathcal{H}$ is $(\delta_{\mathscr{Q}}, *, r_{\mathscr{Q}})$-regular w.r.t. all but at most $\delta_{\mathscr{Q}} t^3_{\mathscr{P}\mathscr{Q}} \ell^3_{\mathscr{P}\mathscr{Q}}$ triads $\hat{\mathcal{Q}} \in \hat{\mathscr{Q}}$.*

2.4. **Statement of the representative lemma for hypergraphs.** We now turn to the key definition of a *representative* of a (t, ℓ)-partition $\mathscr{P}$. Roughly speaking, a representative is a sub-object of a (t, ℓ)-partition $\mathscr{P}$ reflecting the structure of $\mathscr{P}$.

Definition 14 (representative)**.** Let $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ be a (t, ℓ)-partition on V with vertex partition $\mathscr{P}^{(1)} = \{V_1, V_2, \dots, V_t\}$ and $\mathscr{P}^{(2)} = \{P^{ij}_\alpha \colon 1 \le i < j \le t \text{ and } 1 \le \alpha \le \ell\}$. We say $\mathscr{R} = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$, where $\mathscr{R}^{(1)} = \{W_1, W_2, \dots, W_t\}$ and $\mathscr{R}^{(2)} = \{R^{ij}_\alpha \colon 1 \le i < j \le t \text{ and } 1 \le \alpha \le \ell\}$ is a *representative* of $\mathscr{P}$ (or $\mathscr{R}$ *represents* $\mathscr{P}$) if

(*i*) $W_i \subseteq V_i$ for every $1 \le i \le t$ and

(*ii*) R^{ij}_α is a (bipartite) subgraph of P^{ij}_α with vertex classes W_i and W_j for every $1 \le i < j \le t$ and $1 \le \alpha \le \ell$.

Moreover, we define for every triad $\hat{\mathcal{P}} = (P^{ij}_\alpha, P^{ik}_\beta, P^{jk}_\gamma) \in \hat{\mathscr{P}}$ the corresponding triad $\hat{\mathcal{R}}(\hat{\mathcal{P}}) = (R^{ij}_\alpha, R^{ik}_\beta, R^{jk}_\gamma)$ and we let $\hat{\mathscr{R}} = \{\hat{\mathcal{R}}(\hat{\mathcal{P}}) \colon \hat{\mathcal{P}} \in \hat{\mathscr{P}}\}$ be the family of triads of the representative.

In our proof of Theorem 3 the representative $\mathscr{R}$ of the partition $\mathscr{P}$ will be appropriately chosen from an equitable refinement $\mathscr{Q}$ of $\mathscr{P}$ (cf. Theorem 13) and, hence, $\mathscr{R}$ will be equitable in the following sense.

Definition 15 ($(\varepsilon_{\mathscr{R}}, t_{\mathscr{R}}, \ell_{\mathscr{R}})$-representative)**.** Let $\varepsilon_{\mathscr{R}} > 0$ and positive integers $t_{\mathscr{R}}$ and $\ell_{\mathscr{R}} \in \mathbb{N}$ be given. We say a representative $\mathscr{R} = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$ of a $(t_{\mathscr{P}}, \ell_{\mathscr{P}})$-partition $\mathscr{P}$ on n vertices is an $(\varepsilon_{\mathscr{R}}, t_{\mathscr{R}}, \ell_{\mathscr{R}})$*-representative* if

(*i*) $|W| = n/(t_{\mathscr{P}} t_{\mathscr{R}})$ for every $W \in \mathscr{R}^{(1)}$ and

(*ii*) R is $(\varepsilon_{\mathscr{R}}, 1/(\ell_{\mathscr{P}} \ell_{\mathscr{R}}))$*-regular* for every $R \in \mathscr{R}^{(2)}$.

We say the $(\varepsilon_{\mathscr{R}}, t_{\mathscr{R}}, \ell_{\mathscr{R}})$-representative $\mathscr{R}$ of a $(t_{\mathscr{P}}, \ell_{\mathscr{P}})$-partition $\mathscr{P}$ is T*-bounded* for some $T \in \mathbb{N}$ if $\max\{t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}}\} \le T$.

Recall that due to the quantification of the regularity lemma (Theorem 11), which states that for every $\delta_{\mathscr{P}}$ there exists T_0, the resulting T_0-bounded partition $\mathscr{P}$ may satisfy $t_{\mathscr{P}}, \ell_{\mathscr{P}} \gg 1/\delta_{\mathscr{P}}$. This would, however, not suffice to count hypergraphs $\mathcal{F}$ of size comparable to $t_{\mathscr{P}}$ or $\ell_{\mathscr{P}}$, the number of the blocks in the partition $\mathscr{P}$. That is due to the quantification of the counting lemma (Theorem 8), which for a given hypergraph $\mathcal{F}$ of size f ensures the existence of sufficiently small $\delta \ll 1/f$ (δ_3 in the statement).

To circumvent a similar problem arising in the graph case, Alon and Shapira [5] used an iterated version of Szemerédi's regularity lemma, which was first obtained and used by Alon, Fischer, Krivelevich, and Szegedy in [2]. This iterated regularity lemma yields for a given graph G a vertex partition $\mathscr{P}^{(1)} = \{V_1, V_2, \dots, V_t\}$ and a "representative" $\mathscr{R}^{(1)} = \{W_1, W_2, \dots, W_t\}$ with $W_i \subseteq V_i$ for every $i = 1, 2, \dots, t$. In that lemma the representative $\mathscr{R}^{(1)}$ resembles "typically" the density of G w.r.t. $\mathscr{P}$, i.e., $d_G(W_i, W_j) \sim d_G(V_i, V_j)$ for "most" pairs $1 \leq i < j \leq t$. Moreover, the graph G is ε-regular on *every* pair (W_i, W_j) of the representative, and (most importantly) ε can be chosen as an arbitrary function of t, e.g., on the representative one can count graphs of order t, i.e., the size of the partition $\mathscr{P}^{(1)}$.

The representative lemma, Lemma 16 below, is an analogous statement for 3-uniform hypergraphs. For a given hypergraph $\mathcal{H}$ it asserts the existence of a partition $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ and of a representative $\mathscr{R} = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$ of $\mathscr{P}$, so that $\mathcal{H}$ is $(\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}), *, r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}}))$-regular on *every* triad of the representative (see (iv) in Lemma 16). Note that the number of "partition blocks" in $\mathscr{R}$, which is the same as that in the partition $\mathscr{P}$, depends on $t_{\mathscr{P}}$ and $\ell_{\mathscr{P}}$ only, and here $\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ is a function of those parameters. On the other hand, the functions $r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}})$ and $\varepsilon_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}})$ can depend on $t_{\mathscr{P}}$, $t_{\mathscr{R}}$, $\ell_{\mathscr{P}}$, and $\ell_{\mathscr{R}}$, so, in particular, they can depend on $\ell_{\mathscr{P}}\ell_{\mathscr{R}}$, which is the reciprocal of the densities of $R \in \mathscr{R}^{(2)}$. Choosing $\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ and $r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}})$ appropriately as functions of $t_{\mathscr{P}}$, $\ell_{\mathscr{P}}$, $t_{\mathscr{R}}$, and $\ell_{\mathscr{R}}$ will allow us to satisfy the quantification of the counting lemma, Theorem 8, for counting hypergraphs $\mathcal{F}$ whose size depends on $t_{\mathscr{P}}$ and $\ell_{\mathscr{P}}$. Additionally, we will also ensure that $d_{\mathcal{H}}(\hat{\mathcal{R}}(\hat{\mathcal{P}})) \sim d_{\mathcal{H}}(\hat{\mathcal{P}})$ for "most" triads $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ (see (iii) in Lemma 16).

Lemma 16 (Representative lemma)**.** *For every $t_1 \in \mathbb{N}$ and $\delta > 0$ and all functions $\varepsilon_{\mathscr{P}} : \mathbb{N}^2 \to (0,1]$ $\delta_{\mathscr{R}} : \mathbb{N}^2 \to (0,1]$, $\varepsilon_{\mathscr{R}} : \mathbb{N}^4 \to (0,1]$, and $r_{\mathscr{R}} : \mathbb{N}^4 \to \mathbb{N}$ there exist positive integers $T_{\mathscr{P}}$, $T_{\mathscr{P}\mathscr{R}}$ and $n_1 \in \mathbb{N}$ such that the following holds.*

If $\mathcal{H}$ is a hypergraph on at least n_1 vertices, then there exist positive integers $t_{\mathscr{P}}$ and $\ell_{\mathscr{P}}$ and a $T_{\mathscr{P}}$-bounded $(t_{\mathscr{P}}, \ell_{\mathscr{P}})$-partition $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ with $t_1 \leq t_{\mathscr{P}}$ and there is representative $\mathscr{R} = \{\mathscr{R}^{(1)}, \mathscr{R}^{(2)}\}$ of $\mathscr{P}$ such that

- (i) *every graph in $P \in \mathscr{P}^{(2)}$ is $(\varepsilon_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}), 1/\ell_{\mathscr{P}})$-regular,*
- (ii) *$\mathscr{R}$ is a $T_{\mathscr{P}\mathscr{R}}$-bounded $(\varepsilon_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}}), t_{\mathscr{R}}, \ell_{\mathscr{R}})$-representative of $\mathscr{P}$ for some positive integers $t_{\mathscr{R}}$ and $\ell_{\mathscr{R}}$,*
- (iii) *for all but at most $\delta t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3$ triads $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ we have $|d_{\mathcal{H}}(\hat{\mathcal{P}}) - d_{\mathcal{H}}(\hat{\mathcal{R}}(\hat{\mathcal{P}}))| \leq \delta$, and*
- (iv) *$\mathcal{H}$ is $(\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}), *, r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}}))$-regular with respect to $\hat{\mathcal{R}}(\hat{\mathcal{P}}) \in \hat{\mathscr{R}}$ for every triad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$.*

A similar lemma was proved by Kohayakawa, Nagle, and Rödl [17]. We give a different proof of the representative lemma based on Theorem 13 in Section 4.

3. Proof of Theorem 3

A weakened version of Theorem 3 is obtained by restricting to *finite* forbidden families $\mathscr{F}$. In that case the corresponding statement of Theorem 3 has essentially been proved for k-uniform hypergraphs in [15, 21, 25]. We briefly outline that proof in Section 3.1 and discuss its limitations with respect to *infinite* families $\mathscr{F}$. The representative lemma, Lemma 16, will allow us to overcome those difficulties and in Section 3.2 we give a proof of Theorem 3 based on Lemma 16 and the counting lemma, Theorem 8.

3.1. **The finite case.** We now sketch a (straightforwardly adjusted) proof of the removal lemma [15, 21, 25], based on the regularity and the counting lemma for hypergraphs, which yields the restricted version of Theorem 3 for finite forbidden families $\mathscr{F}$.

Let $\mathscr{F}$ be a finite family of hypergraphs and $\eta > 0$ be given and consider an n-vertex hypergraph $\mathcal{H}$ which is η-far from $\mathscr{A} = \mathrm{Forb}(\mathscr{F})$. We apply Theorem 11 with appropriately chosen parameters $\delta_{\mathscr{P}}$ and functions $\mu_{\mathscr{P}}$ and $r_{\mathscr{P}}$ (discussed below). This way we obtain a partition $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$. We then delete those edges H of $\mathcal{H}$ which satisfy one of the following conditions

- (a) H is *non-crossing* in $\mathscr{P}$, i.e., there exist $V_i \in \mathscr{P}^{(1)}$ so that $|H \cap V_i| \geq 2$, or
- (b) H belongs to a *sparse* triad, i.e., $d_{\mathcal{H}}(\hat{\mathcal{P}}) < \eta/3$ for the unique $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ with $H \in \mathrm{Tr}(\hat{\mathcal{P}})$, or
- (c) H belongs to a $(\delta_{\mathscr{P}}, *, r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}))$-*irregular* triad, i.e., the hypergraph $\mathcal{H}$ is not $(\delta_{\mathscr{P}}, *, r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}))$-regular w.r.t. the unique $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ with $H \in \mathrm{Tr}(\hat{\mathcal{P}})$.

We call the resulting hypergraph $\mathcal{H}'$. Choosing $\delta_{\mathscr{P}} < \eta/3$ and provided the regular partition has sufficiently many vertex classes (which implies that only a few tuples, e.g., less than $\eta n^3/3$, are deleted because of (a)) one can show that at most ηn^3 edges of $\mathcal{H}$ were deleted. Since by assumption $\mathcal{H}$ is η-far from $\mathscr{A}$, the hypergraph $\mathcal{H}'$ still does not satisfy $\mathscr{A}$. Consequently, $\mathcal{H}'$ contains a sub-hypergraph $\mathcal{F}_0$ isomorphic to some forbidden hypergraph from $\mathscr{F}$. Due to the construction of $\mathcal{H}'$ all edges of $\mathcal{F}_0$ are crossing w.r.t. the vertex partition $\mathscr{P}^{(1)}$ and belong to dense and regular triads from $\hat{\mathscr{P}}$. Suppose now that the entire copy $\mathcal{F}_0$ is crossing w.r.t. $\mathscr{P}^{(1)}$, i.e., $V(\mathcal{F}_0)$ intersects each vertex partition class $V_i \in \mathscr{P}^{(1)}$ in at most one vertex. (The case when $\mathcal{F}_0$ is not crossing can be handled similarly, as we will show in the general proof for not necessarily finite families $\mathscr{F}$.)

Since each edge of $\mathcal{F}_0$ belongs to a dense and regular triad, the union of those triads defines a dense and regular $(m, \mathcal{F}_0)$-complex (see Definition 7) with $m = n/t_{\mathscr{P}}$. Moreover, $\max_{\mathcal{F} \in \mathscr{F}} |V(\mathcal{F})|$ exists since $|\mathscr{F}| < \infty$. In other words, we can "forecast" the maximum possible size of $\mathcal{F}_0$ we may encounter and we can choose $\delta_{\mathscr{P}}$ and functions $\mu_{\mathscr{P}}$ and $r_{\mathscr{P}}$ at the beginning of the proof appropriately so that the $(m, \mathcal{F}_0)$-complex from above is ready for an application of the counting lemma, Theorem 8. The counting lemma then guarantees $\Omega(n^{|V(\mathcal{F}_0)|})$ copies of $\mathcal{F}_0$ in the $(m, \mathcal{F}_0)$-complex, and, consequently, in $\mathcal{H}' \subseteq \mathcal{H}$, which concludes the proof.

Clearly, this argument breaks down for infinite families $\mathscr{F}$, as we cannot forecast an upper bound on the size of $\mathcal{F}_0$. The representative lemma, Lemma 16, allows us to get around this issue. Given a hypergraph $\mathcal{H}$ we apply Lemma 16 and delete non-crossing edges and edges belonging to triads $\hat{\mathcal{P}}$ for which $\hat{\mathcal{R}}(\hat{\mathcal{P}})$ is sparse, similarly, as in the discussion above (using additionally (iii) of Lemma 16), we will be left with a hypergraph $\mathcal{H}'$ which again contains a hypergraph $\mathcal{F}_0$ from the forbidden family $\mathscr{F}$. In this (infinite) case we have no upper bound on the size of $\mathcal{F}_0$. However, since all edges of $\mathcal{F}_0$ belong to triads of the $(t_{\mathscr{P}}, \ell_{\mathscr{P}})$-partition $\mathscr{P}$ we will argue that $\mathcal{H}'$ also contains some other forbidden hypergraph $\mathcal{F}_1$ of $\mathscr{F}$, whose edges belong to the same triads as the edges of $\mathcal{F}_0$ and, more importantly, the size of $\mathcal{F}_1$ will be bounded by a function only depending on $t_{\mathscr{P}}$ and $\ell_{\mathscr{P}}$. (Roughly speaking, the $\mathcal{F}_0$ and $\mathcal{F}_1$ can both be homomorphically mapped in the "cluster-structure" of the partition $\mathscr{P}$ and the size of $\mathcal{F}_1$ only depends on the number of partition blocks of $\mathscr{P}$.) This will allow us, with appropriately chosen functions $\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ and $r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}})$ for the regularity of the representative to find (using the counting lemma) $\Omega(n^{|V(\mathcal{F}_1)|})$ copies of $\mathcal{F}_1$ in $\mathcal{H}' \subseteq \mathcal{H}$. We now give the details of this outline.

3.2. **The general case.** In this section we make the ideas presented in the outline above precise. For that we need a few more definitions. For positive integers t and ℓ we denote by $M(t,\ell)$ the complete multigraph with vertex set $[t]=\{1,\dots,t\}$ and edge multiplicity ℓ. We can view edges as ordered pairs $(\{i,j\},\alpha)$, where $1\le i<j\le t$ and $\alpha\in[l]$. Denote by $\mathrm{Tr}(M(t,\ell))$ the set of all $\binom{t}{3}\ell^3$ triangles of $M(t,\ell)$. We will identify a triangle on the vertices $1\le i<j<k\le t$ and edges $(\{i,j\},\alpha)$, $(\{i,k\},\beta)$, $(\{j,k\},\gamma)$ with the 6-tuple $((i,j,k),(\alpha,\beta,\gamma))$ and set

$$\mathrm{Tr}\left(M(t,\ell)\right)=\left\{\big((i,j,k),\,(\alpha,\beta,\gamma)\big)\colon\ 1\le i<j<k\le t \text{ and } \alpha,\beta,\gamma\in[l]\right\}.$$

We also consider homomorphisms into sub-multihypergraphs of $\mathrm{Tr}(M(t,\ell))$. Recall that for a hypergraph $\mathcal{F}$ we denote by $\Delta(\mathcal{F})$ the shadow of $\mathcal{F}$, i.e., the family of all pairs of vertices contained in an edge of $\mathcal{F}$.

Definition 17. Let t and ℓ be integers and let $\mathcal{S}\subseteq\mathrm{Tr}(M(t,\ell))$ be a multi-hypergraph. For a 3-uniform hypergraph $\mathcal{F}$ on f vertices, we say a pair of mappings (φ,ψ)

$$\varphi\colon\ V(\mathcal{F})\to V(\mathcal{S})\subseteq[t] \qquad\text{and}\qquad \psi\colon\ \Delta(\mathcal{F})\to[\ell]$$

is a *homomorphism* from $\mathcal{F}$ to the multi-hypergraph $\mathcal{S}$ if φ is onto and if there exists a labeling of $V(\mathcal{F})=\{v_1,\dots,v_f\}$ such that for every edge $\{v_x,v_y,v_z\}\in\mathcal{F}$ with $1\le x<y<z\le f$ we have $\varphi(v_x)<\varphi(v_y)<\varphi(v_z)$ and

$$\Big(\big(\varphi(v_x),\varphi(v_y),\varphi(v_z)\big),\ \big(\psi(v_x,v_y),\psi(v_x,v_z),\psi(v_y,v_z)\big)\Big)\in\mathcal{S}\,.$$

We will abbreviate the existence of a homomorphism from $\mathcal{F}$ to $\mathcal{S}$ by $\mathcal{F}\twoheadrightarrow\mathcal{S}$.

Proof of Theorem 3. Let $\mathscr{A}=\mathrm{Forb}(\mathscr{F})$ for a (possibly infinite) family of forbidden hypergraphs $\mathscr{F}$ and $\eta>0$ be a positive constant. We need a few auxiliary functions, before we reveal the promised constants $c>0$, C and n_0 (see (13) below). Given two positive integers t and ℓ and a multi-hypergraph $\mathcal{S}\subseteq\mathrm{Tr}(M(t,\ell))$ we set

$$\mathscr{F}_{\mathcal{S}}=\{\mathcal{F}\in\mathscr{F}\colon\ \mathcal{F}\twoheadrightarrow\mathcal{S}\}$$

and

$$C_{\mathcal{S}}=\begin{cases}\min\{|V(\mathcal{F})|\colon\ \mathcal{F}\in\mathscr{F}_{\mathcal{S}}\} & \text{if } \mathscr{F}_{\mathcal{S}}\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Let $\Psi\colon\ \mathbb{N}^2\to\mathbb{N}\cup\{0\}$ be defined for two positive integers t and ℓ as

$$\Psi(t,l)=\max\left\{C_{\mathcal{S}}\colon\ \mathcal{S}\subseteq\mathrm{Tr}(M(t,\ell))\right\}.$$

The function $\Psi(t,\ell)$ is designed to "forecast" the maximal size of a witness from $\mathscr{F}$ we may encounter after application of the representative lemma, Lemma 16. Next we introduce the parameters δ and t_1 and the functions $\varepsilon_{\mathscr{P}}$, $\delta_{\mathscr{R}}$, $\varepsilon_{\mathscr{R}}$, and $r_{\mathscr{R}}$ with which we are going to apply Lemma 16. Recall the functions $\delta_3(f,\gamma,d_3)$, $\delta_2(f,\gamma,d_3,d_2)$, $r(f,\gamma,d_3,d_2)$, and $m_0(f,\gamma,d_3,d_2)$ from Theorem 8 and $\varepsilon_{\mathrm{tcl}}(\gamma,d)$ and $m_{\mathrm{tcl}}(\gamma,d)$ from Theorem 5. For the given η from above we set

$$\delta=\frac{\eta}{3}\qquad\text{and}\qquad t_1=\left\lceil\frac{4}{\eta}\right\rceil \tag{8}$$

and define functions in integer variables t, t', ℓ, and ℓ'

$$\varepsilon_{\mathscr{P}}(t,\ell) = \varepsilon_{\mathrm{tcl}}(\gamma = 1/2, d = 1/\ell) \tag{9}$$

$$\delta_{\mathscr{R}}(t,\ell) = \delta_3\big(f = \Psi(t,\ell), \gamma = 1/2, d_3 = \eta/3\big) \tag{10}$$

$$\varepsilon_{\mathscr{R}}(t,t',\ell,\ell') = \delta_2\big(f = \Psi(t,\ell), \gamma = 1/2, d_3 = \eta/3, d_2 = 1/(\ell\ell')\big) \tag{11}$$

$$r_{\mathscr{R}}(t,t',\ell,\ell') = r\big(f = \Psi(t,\ell), \gamma = 1/2, d_3 = \eta/3, d_2 = 1/(\ell\ell')\big)\,. \tag{12}$$

For that choice of δ, t_1, $\varepsilon_{\mathscr{P}}$, $\delta_{\mathscr{R}}$, $\varepsilon_{\mathscr{R}}$, and $r_{\mathscr{R}}$ Lemma 16 yields constants

$$T_{\mathscr{P}}\,, \qquad T_{\mathscr{P}\mathscr{R}}\,, \quad \text{and} \quad n_1\,.$$

Now we define the constants c, C, and n_0 promised by Theorem 3 and we set

$$C = \Psi(T_{\mathscr{P}}, T_{\mathscr{P}})\,, \qquad c = \frac{1}{4C!} \times \left(\frac{1}{T_{\mathscr{P}}T_{\mathscr{P}\mathscr{R}}}\right)^{\binom{C}{2}} \times \left(\frac{\eta}{3}\right)^{\binom{C}{3}} \left(\frac{1}{T_{\mathscr{P}}T_{\mathscr{P}\mathscr{R}}}\right)^{C}$$

and (13)

$$n_0 = \max\big\{n_1,\, T_{\mathscr{P}} \times m_{\mathrm{tcl}}(\gamma = 1/2, d = 1/T_{\mathscr{P}}),\, C^2/c,\\ T_{\mathscr{P}}T_{\mathscr{P}\mathscr{R}} \times m_0(f = C, \gamma = 1/2, d_3 = \eta/3, d_2 = 1/(T_{\mathscr{P}}T_{\mathscr{P}\mathscr{R}}))\big\}\,.$$

This concludes the definition of all constants and functions involved in the proof.

Let $\mathcal{H}$ be a hypergraph on $n \geq n_0$ vertices which is η-far from $\mathscr{A}$. Due to Lemma 16 the hypergraph $\mathcal{H}$ admits a $(t_{\mathscr{P}}, \ell_{\mathscr{P}})$-partition $\mathscr{P}$ (where $t_{\mathscr{P}} \geq t_1$) with a representative $\mathscr{R}$ and there are integers $t_{\mathscr{R}}$ and $\ell_{\mathscr{R}}$ such that (i)–(iv) of the lemma hold. We formally fix

$$\varepsilon_{\mathscr{P}} = \varepsilon_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}), \quad \delta_{\mathscr{R}} = \delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}),\\ \varepsilon_{\mathscr{R}} = \varepsilon_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}}), \quad \text{and} \quad r_{\mathscr{R}} = r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{R}}, \ell_{\mathscr{P}}, \ell_{\mathscr{R}})\,.$$

Now we delete all edges H of $\mathcal{H}$ which satisfy at least one of the two properties below

- (a) H is non-crossing w.r.t. the vertex partition $\mathscr{P}^{(1)}$, i.e., there is some $V_i \in \mathscr{P}^{(1)}$ $(1 \leq i \leq t_{\mathscr{P}})$ such that $|H \cap V_i| \geq 2$ or
- (b) $H \in \mathrm{Tr}(\hat{\mathcal{P}})$ for which $d_{\mathcal{H}}(\hat{\mathcal{R}}(\hat{\mathcal{P}})) < 2\eta/3$.

We call the resulting sub-hypergraph $\mathcal{H}' \subseteq \mathcal{H}$. Next we estimate $\mathcal{H} \setminus \mathcal{H}'$. We first consider the edges deleted due to (a). Recalling the definition of t_1 in (8) and $t_{\mathscr{P}} \geq t_1$ we get that the number of non-crossing triples in $\mathcal{H}$ is at most

$$2\binom{t_{\mathscr{P}}}{2}\binom{n/t_{\mathscr{P}}}{2}\frac{n}{t_{\mathscr{P}}} + t_{\mathscr{P}}\binom{n/t_{\mathscr{P}}}{3} \leq \frac{n^3}{t_{\mathscr{P}}} \leq \frac{\eta}{4}n^3\,. \tag{14}$$

Next we estimate the number of edges deleted because of (b). By property (i) of Lemma 16 all graphs $P^{ij}_{\alpha} \in \mathscr{P}^{(2)}$ are $(\varepsilon_{\mathscr{P}}, 1/\ell_{\mathscr{P}})$-regular and, consequently, Lemma 5 applies to every triad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ (with $\gamma = 1/2$ and $d = 1/\ell_{\mathscr{P}}$). We consider two sub-cases. First, the edge H could belong to a triad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ which is exceptional in the sense of (iii) of Lemma 16. However, the number of those edges cannot exceed

$$\Big|\bigcup\big\{\,\mathrm{Tr}(\hat{\mathcal{P}})\colon\ \hat{\mathcal{P}} \in \hat{\mathscr{P}} \text{ and } |d_{\mathcal{H}}(\hat{\mathcal{P}}) - d_{\mathcal{H}}(\hat{\mathcal{R}}(\hat{\mathcal{P}}))| > \delta\big\}\Big|\\ \leq \delta t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3 \times \max_{\hat{\mathcal{P}} \in \hat{\mathscr{P}}} \big|\,\mathrm{Tr}(\hat{\mathcal{P}})\big| \leq \delta t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3 \times \left(1 + \frac{1}{2}\right)\frac{n^3}{t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3} \leq \frac{\eta}{2}n^3\,. \tag{15}$$

Finally, the number of edges of $\mathcal{H}$ in triads $\hat{\mathcal{P}}$ which are not exceptional in the sense of part (iii) of Lemma 16, but satisfy (b) is at most

$$\left(\frac{2\eta}{3}+\delta\right)\times\max_{\hat{\mathcal{P}}\in\hat{\mathscr{P}}}\left|\operatorname{Tr}(\hat{\mathcal{P}})\right|\times\binom{t_{\mathscr{P}}}{3}\ell_{\mathscr{P}}^3\leq\eta\times\frac{3}{2}\frac{n^3}{t_{\mathscr{P}}^3\ell_{\mathscr{P}}^3}\times\frac{t_{\mathscr{P}}^3}{6}\ell_{\mathscr{P}}^3\leq\frac{\eta}{4}n^3\,. \tag{16}$$

It follows from the considerations above and (14)–(16) that

$$|\mathcal{H}\setminus\mathcal{H}'|\leq\eta n^3\,.$$

Hence, by assumption on $\mathcal{H}$ the hypergraph $\mathcal{H}'\notin\mathscr{A}$ and contains some copy $\mathcal{F}_0$ of some forbidden hypergraph from $\mathscr{F}$. Note that since $\mathcal{H}'$ contains only crossing edges in $\mathscr{P}$, the existence of $\mathcal{F}_0\subseteq\mathcal{H}'$ yields the existence of some homomorphism to $\operatorname{Tr}(M(t_{\mathscr{P}},\ell_{\mathscr{P}}))$. Let $\mathcal{S}\subseteq\operatorname{Tr}(M(t_{\mathscr{P}},\ell_{\mathscr{P}}))$ be a homomorphic image of such a homomorphism. In particular, $\mathscr{F}_{\mathcal{S}}\neq\emptyset$, since $\mathcal{F}_0\in\mathscr{F}_{\mathcal{S}}$. We denote by $\mathcal{F}_1$ some hypergraph in $\mathscr{F}_{\mathcal{S}}$ with the minimum number $C_{\mathcal{S}}$ of vertices (see (7)) and let (φ,ψ) be the homomorphism from $\mathcal{F}_1$ to $\mathcal{S}$. Let

$$f_1=C_{\mathcal{S}}=|V(\mathcal{F}_1)|\leq C$$

and let $V(\mathcal{F}_1)=\{v_1,\dots,v_{f_1}\}$. We are going to show that the number of copies of $\mathcal{F}_1$ in $\mathcal{H}'$ will satisfy

$$\#\{\mathcal{F}_1\subseteq\mathcal{H}'\}\geq cn^{f_1}\,, \tag{17}$$

which clearly implies Theorem 3.

We define the graph

$$R_{\mathcal{S}}=\bigcup_{(\{i,j\},\alpha)\in\mathcal{S}}R_\alpha^{ij}\,,$$

where $R_\alpha^{ij}\in\mathscr{R}^{(2)}$ are graphs of the representative. Assume w.l.o.g. that $V(\mathcal{S})=\{1,2,\dots,s\}$ for some $s\leq t_{\mathscr{P}}$ and thus $V(R_{\mathcal{S}})=W_1\cup W_2\cup\dots\cup W_s$ with $W_i\in\mathscr{R}^{(1)}$. If $f=s$, then $(R_{\mathcal{S}},\mathcal{H}'\cap\operatorname{Tr}(R_{\mathcal{S}}))$ is a $(n/(t_{\mathscr{P}}t_{\mathscr{R}}),\mathcal{F}_1)$-complex (since by definition φ is onto) and we could invoke the counting lemma, Theorem 8, which would yield (17). However, since this does not have to be the case, we will define an auxiliary $(n/(t_{\mathscr{P}}t_{\mathscr{R}}),\mathcal{F}_1)$-complex $(G,\mathcal{G})$ which will satisfy the assumptions of Theorem 8 and due to its construction we will infer (17) from it.

We first define the vertex set of $(G,\mathcal{G})$. For each $x=1,2,\dots,f_1$ let Y_x be a copy of $W_{\varphi(x)}$ such that for all $1\leq x<y\leq f_1$ we have $Y_x\cap Y_y=\emptyset$. Moreover, for every $x=1,2,\dots,f_1$ let $\vartheta_x\colon W_{\varphi(x)}\to Y_x$ be a bijection. Note that if $\{x,y\}\in\Delta(\mathcal{F}_1)$, then $\varphi(x)\neq\varphi(y)$ and, consequently, $(\{\varphi(x),\varphi(y)\},\psi(x,y))\in\mathcal{S}$ and $R_{\psi(x,y)}^{\varphi(x)\varphi(y)}\in\mathscr{R}^2$. Hence, we can define for every $\{x,y\}\in\Delta(\mathcal{F}_1)$ with $x<y$ a bipartite graph G^{xy} with vertex classes Y_x and Y_y and edge set

$$E(G^{xy})=\big\{\{\vartheta(w),\vartheta(w')\}\colon\ \{w,w'\}\in E(R_{\psi(x,y)}^{\varphi(x)\varphi(y)})\big\}\,.$$

It follows from that definition that G^{xy} is an isomorphic copy of $R_{\psi(x,y)}^{\varphi(x)\varphi(y)}$ and that $G=(Y,E_G)$ defined by

$$Y=Y_1\cup\dots\cup Y_{f_1}\quad\text{and}\quad E_G=\bigcup\big\{E(G^{xy})\colon\ \{x,y\}\in\Delta(\mathcal{F}_1)\big\}$$

is an f_1-partite graph satisfying (i) and (ii) of Definition 7 with $m=n/(t_{\mathscr{P}}t_{\mathscr{R}})$, $\delta_2=\varepsilon_{\mathscr{R}}$, and $d_2=1/(\ell_{\mathscr{P}}\ell_{\mathscr{R}})$. Similarly, for every edge $\{x,y,z\}\in\mathcal{F}_1$ there is a triad $\mathcal{R}(x,y,z)\in\hat{\mathscr{R}}$ defined by

$$\mathcal{R}(x,y,z)=(R_{\psi(x,y)}^{\varphi(x)\varphi(y)},\,R_{\psi(x,z)}^{\varphi(x)\varphi(z)},\,R_{\psi(y,z)}^{\varphi(y)\varphi(z)})\,.$$

We set
$$\mathcal{G}^{xyz} = \big\{\{w,w',w''\}\colon \{w,w',w''\} \in \mathcal{H}' \cap \mathrm{Tr}(\mathcal{R}(x,y,z))\big\}$$
and
$$\mathcal{G} = \bigcup \big\{\mathcal{G}^{xyz}\colon \{x,y,z\} \in \mathcal{F}_1\big\}\,.$$
Again it follows from the definition that $\mathcal{G}$ satisfies (iii) of Definition 7 with $\delta_3 = \delta_{\mathscr{R}}$, $d_3 \geq \eta/3$, and $r = r_{\mathscr{R}}$. Summarizing, $(G,\mathcal{G})$ is an $(\varepsilon_{\mathscr{R}}, \delta_{\mathscr{R}}, 1/(\ell_{\mathscr{P}}\ell_{\mathscr{R}}), \eta/3, r_{\mathscr{R}})$-regular $(n/(t_{\mathscr{P}}t_{\mathscr{R}}), \mathcal{F}_1)$-complex. By the choices in (10)–(12), and (13) we can apply the counting lemma, Theorem 8, and, hence,
$$\#\{\mathcal{F}_1 \subseteq \mathcal{G}\} \geq \left(1-\frac{1}{2}\right)\left(\frac{1}{\ell_{\mathscr{P}}\ell_{\mathscr{R}}}\right)^{|\Delta(\mathcal{F}_1)|}\left(\frac{\eta}{3}\right)^{|\mathcal{F}_1|}\left(\frac{n}{t_{\mathscr{P}}t_{\mathscr{R}}}\right)^{f_1} \overset{(13)}{\geq} 2f_1!cn^{f_1}\,.$$
Observe that almost every copy of $\mathcal{F}_1$ in $\mathcal{G}$ corresponds to a labeled copy of $\mathcal{F}_1$ in $\mathcal{H}'$ (with $\vartheta_x\colon Y_x \to W_{\varphi(x)}$ defining the isomorphism). The only possible exceptions are those copies of $\mathcal{F}_1$ with image of size less then f_1. This may happen since although for each $x = 1,\dots,f_1$ the map ϑ_x is a bijection, but $\vartheta_x^{-1}(u) = \vartheta_y^{-1}(w)$ for two different vertices u and w of a copy of $\mathcal{F}_1$ in $\mathcal{G}$ if, e.g., x and y are such that $W_{\varphi(x)} = W_{\varphi(y)}$. The number of those copies of $\mathcal{F}_1$ in $\mathcal{G}$ is however bounded from above by $\binom{f_1}{2}(n/t_{\mathscr{P}}t_{\mathscr{R}})^{f_1-1}$. Consequently, we can find $2f_1!cn^{f_1} - \binom{f_1}{2}(n/t_{\mathscr{P}}t_{\mathscr{R}})^{f_1-1})$ labeled copies of $\mathcal{F}_1$ in $\mathcal{H}'$ and by the choice of $n_0 \geq C^2/c$ at least cn^{f_1} unlabeled copies. Hence, we verified (17), which concludes the proof of Theorem 3. □

4. Proof of the Representative Lemma

The proof of Lemma 16 is based on two successive applications of the regularity lemma (first in form of Theorem 11 and the second time in form of Theorem 13).

Proof of Lemma 16. First we recall the quantification of the representative lemma, Lemma 16. Let constants $t_1 \in \mathbb{N}$ and $\delta > 0$ and functions $\varepsilon_{\mathscr{P}}\colon \mathbb{N}^2 \to (0,1]$, $\delta_{\mathscr{R}}\colon \mathbb{N}^2 \to (0,1]$, $\varepsilon_{\mathscr{R}}\colon \mathbb{N}^4 \to (0,1]$, and $r_{\mathscr{R}}\colon \mathbb{N}^4 \to \mathbb{N}$ be given. We are supposed to define positive integers $T_{\mathscr{P}}$, $T_{\mathscr{P}\mathscr{R}}$, and n_1 and we are going to define them in (23). First, however, we need some preparations.

Our proof of Lemma 16 will rely on the regularity lemma for hypergraphs in form of Theorem 13 and Theorem 11 and the counting lemma for graphs, Lemma 5. Below we will use the functions $\mu_{\mathrm{hrl}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}, \delta_{\mathscr{Q}}, \varepsilon_{\mathscr{Q}}, r_{\mathscr{Q}})$, $T_{\mathrm{hrl}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}, \delta_{\mathscr{Q}}, \varepsilon_{\mathscr{Q}}, r_{\mathscr{Q}})$, and $n_{\mathrm{hrl}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}, \delta_{\mathscr{Q}}, \varepsilon_{\mathscr{Q}}, r_{\mathscr{Q}})$ given by Theorem 13, the functions $T_0(t_0, \delta_{\mathscr{P}}, \mu_{\mathscr{P}}, r_{\mathscr{P}})$ and $n_0(t_0, \delta_{\mathscr{P}}, \mu_{\mathscr{P}}, r_{\mathscr{P}})$ given by Theorem 11, and the functions $\varepsilon_{\mathrm{tcl}}(\gamma, d)$ and $m_{\mathrm{tcl}}(\gamma, d)$ given by the triangle counting lemma, Lemma 5.

We define auxiliary functions $\mu_{\mathrm{aux}}\colon \mathbb{N}^2 \to (0,1]$, $T_{\mathrm{aux}}\colon \mathbb{N}^2 \to \mathbb{N}$, and $n_{\mathrm{aux}}\colon \mathbb{N}^2 \to \mathbb{N}$ and we set for positive integers t and ℓ and $x \in \{\mu, T, n\}$
$$\begin{aligned} x_{\mathrm{aux}}(t,\ell) = x_{\mathrm{hrl}}\Big(t_{\mathscr{P}} = t,\ \ell_{\mathscr{P}} = \ell,\ \delta_{\mathscr{Q}} = \min\big\{\delta_{\mathscr{R}}(t,\ell),\, (t\ell)^{-3}/3\big\},& \\ \varepsilon_{\mathscr{Q}}(t',\ell') = \min\big\{\varepsilon_{\mathscr{R}}(t,t',\ell,\ell'),\, \varepsilon_{\mathrm{tcl}}\big(\gamma = \tfrac{1}{2},\, d = \tfrac{1}{\ell\ell'}\big)\big\},& \\ r_{\mathscr{Q}}(t',\ell') = r_{\mathscr{R}}(t,t',\ell,\ell')\Big)\,.& \end{aligned} \tag{18}$$
In other words, for fixed t and ℓ the values $\mu_{\mathrm{aux}}(t,\ell)$, $T_{\mathrm{aux}}(t,\ell)$, and $n_{\mathrm{aux}}(t,\ell)$ are defined by the corresponding constants μ_{hrl}, T_{hrl}, and n_{hrl} given by Theorem 13 for those parameters $t_{\mathscr{P}}$, $\ell_{\mathscr{P}}$, $\delta_{\mathscr{Q}}$ and functions $\varepsilon_{\mathscr{Q}}$ and $r_{\mathscr{Q}}$ displayed in (18). Note that the choice in (18) is such that for fixed integers t and ℓ, the parameters $t_{\mathscr{P}}$,

$\ell_{\mathscr{P}}$, and $\delta_{\mathscr{Q}}$ are constants, while $\varepsilon_{\mathscr{Q}}\colon \mathbb{N}^2 \to (0,1]$ and $r_{\mathscr{Q}}\colon \mathbb{N}^2 \to \mathbb{N}$ are functions in the variables t' and ℓ', which matches the quantification of Theorem 13.

With those auxiliary functions at hand, we define the parameters and constants with which we will apply the "simple" regularity lemma, Theorem 11, later. For that we fix constants

$$t_0 = t_1 \qquad \text{and} \qquad \delta_{\mathscr{P}} = \frac{\delta}{9} \tag{19}$$

and functions $\mu_{\mathscr{P}}\colon \mathbb{N}^2 \to (0,1]$ and $r_{\mathscr{P}}\colon \mathbb{N}^2 \to \mathbb{N}$ defined for all positive integers t and ℓ by

$$\mu_{\mathscr{P}}(t,\ell) = \min\left\{\mu_{\mathrm{aux}}(t,\ell),\ \varepsilon_{\mathscr{P}}(t,\ell),\ \varepsilon_{\mathrm{tcl}}(\gamma = 1/2,\, d = 1/\ell)\right\}, \tag{20}$$

$$r_{\mathscr{P}}(t,\ell) = \left(T_{\mathrm{aux}}(t,\ell)\right)^6, \tag{21}$$

where t_1, δ, and $\varepsilon_{\mathscr{P}}$ are input parameters of Lemma 16. Given t_0, $\delta_{\mathscr{P}}$, $\mu_{\mathscr{P}}$, and $r_{\mathscr{P}}$ from above, Theorem 11 yields positive integers

$$T_0 = T_0(t_0, \delta_{\mathscr{P}}, \mu_{\mathscr{P}}, r_{\mathscr{P}}) \quad \text{and} \quad n_0(t_0, \delta_{\mathscr{P}}, \mu_{\mathscr{P}}, r_{\mathscr{P}}). \tag{22}$$

Now we are able to determine the promised constants $T_{\mathscr{P}}$, $T_{\mathscr{P}\mathscr{R}}$, and n_1 of Lemma 16 and we set

$$T_{\mathscr{P}} = T_0, \qquad T_{\mathscr{P}\mathscr{R}} = \max_{1\le t,\ell\le T_0} T_{\mathrm{aux}}(t,\ell)$$

$$\text{and} \quad n_1 = \max\Big\{\max_{1\le t,\ell\le T_0} n_{\mathrm{aux}}(t,\ell),\ n_0,\ T_{\mathscr{P}} \times m_{\mathrm{tcl}}(\gamma = \tfrac{1}{2},\, d = \tfrac{1}{T_{\mathscr{P}}}),\ T_{\mathscr{P}}T_{\mathscr{P}\mathscr{R}} \times m_{\mathrm{tcl}}(\gamma = \tfrac{1}{2},\, d = \tfrac{1}{T_{\mathscr{P}}T_{\mathscr{P}\mathscr{R}}})\Big\}. \tag{23}$$

Having defined those constants, let $\mathcal{H}$ be a given hypergraph on $n \ge n_1$ vertices. Since $n_1 \ge n_0$ we can apply Theorem 11 with constants t_0 and $\delta_{\mathscr{P}}$ and functions $\mu_{\mathscr{P}}$ and $r_{\mathscr{P}}$ defined in (19)–(21). Theorem 11 yields a partition $\mathscr{P} = \{\mathscr{P}^{(1)}, \mathscr{P}^{(2)}\}$ and positive integers $t_{\mathscr{P}}$ and $\ell_{\mathscr{P}}$ such that (i) and (ii) of Theorem 11 hold, i.e.,

- $(P1)$ $\mathscr{P}$ is $(\mu_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}), t_{\mathscr{P}}, \ell_{\mathscr{P}})$-equitable, T_0-bounded, and $t_{\mathscr{P}} \ge t_0$, and
- $(P2)$ $\mathcal{H}$ is $(\delta_{\mathscr{P}}, *, r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}))$-regular w.r.t. all but at most $\delta_{\mathscr{P}} t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3$ triads $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$.

Next we will apply the "refining version" of the regularity lemma, Theorem 13 to $\mathcal{H}$ and $\mathscr{P}$, with parameters $t_{\mathscr{P}}$, $\ell_{\mathscr{P}}$, $\delta_{\mathscr{Q}} = \min\{\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}),\, (3t_{\mathscr{P}}^3\ell_{\mathscr{P}}^3)^{-1})\}$, $\varepsilon_{\mathscr{Q}}(t',\ell') = \min\{\varepsilon_{\mathscr{R}}(t_{\mathscr{P}}, t', \ell_{\mathscr{P}}, \ell'), \varepsilon_{\mathrm{tcl}}(\gamma = \frac{1}{2}, d = \frac{1}{\ell_{\mathscr{P}}\ell'})\}$, and $r_{\mathscr{Q}}(t',\ell') = r_{\mathscr{R}}(t_{\mathscr{P}}, t', \ell_{\mathscr{P}}, \ell')$. In other words, we will apply Theorem 13 with precisely the same parameters as those in (18). Therefore, we have to check that $\mathcal{H}$ and $\mathscr{P}$ satisfy the assumptions (a) and (b) of Theorem 13 for $n_{\mathrm{aux}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ and $\mu_{\mathrm{aux}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$. However, due to (23) and $(P1)$ we have $n \ge n_{\mathrm{aux}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$ and due to the choice in (20) we have $\mu_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}) \le \mu_{\mathrm{aux}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$. Consequently, assumptions (a) and (b) of Theorem 13 hold and Theorem 13 yields a partition $\mathscr{Q} = \{\mathscr{Q}^{(1)}, \mathscr{Q}^{(2)}\}$ and positive integers $t_{\mathscr{Q}}$ and $\ell_{\mathscr{Q}}$ such that with

$$\delta_{\mathscr{Q}} = \min\{\delta_{\mathscr{R}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}),\, (t_{\mathscr{P}}\ell_{\mathscr{P}})^{-3}/3\}, \tag{24}$$

$$\varepsilon_{\mathscr{Q}} = \min\{\varepsilon_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{Q}}, \ell_{\mathscr{P}}, \ell_{\mathscr{Q}}), \varepsilon_{\mathrm{tcl}}(\gamma = \tfrac{1}{2}, d = \tfrac{1}{\ell_{\mathscr{P}}\ell_{\mathscr{Q}}})\}, \tag{25}$$

and

$$r_{\mathscr{Q}} = r_{\mathscr{R}}(t_{\mathscr{P}}, t_{\mathscr{Q}}, \ell_{\mathscr{P}}, \ell_{\mathscr{Q}}). \tag{26}$$

we have

($Q1$) $\mathscr{Q}$ is $(\varepsilon_{\mathscr{Q}}, t_{\mathscr{P}}t_{\mathscr{Q}}, \ell_{\mathscr{P}}\ell_{\mathscr{Q}})$-equitable and $T_{\mathrm{aux}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})$-bounded partition on V,
($Q2$) $\mathscr{Q} \prec \mathscr{P}$, and
($Q3$) $\mathcal{H}$ is $(\delta_{\mathscr{Q}}, *, r_{\mathscr{Q}})$-regular w.r.t. all but at most $\delta_{\mathscr{Q}} t_{\mathscr{P}}^3 t_{\mathscr{Q}}^3 \ell_{\mathscr{P}}^3 \ell_{\mathscr{Q}}^3$ triads $\hat{\mathcal{Q}} \in \hat{\mathscr{Q}}$.

It follows directly from ($P1$), (20), and the choice of $T_{\mathscr{P}}$ in (23) that

$$\mathscr{P} \text{ is } T_{\mathscr{P}}\text{-bounded and satisfies } (i) \text{ of Lemma 16.} \tag{27}$$

Below we will select the representative $\mathscr{R}$ from the finer partition $\mathscr{Q}$. For property (iii) of Lemma 16 the following claim will be useful.

Claim 18. *If $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ is such that $\mathcal{H}$ is $(\delta_{\mathscr{P}}, *, r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$, then all but at most $2\delta_{\mathscr{P}} t_{\mathscr{Q}}^3 \ell_{\mathscr{Q}}^3$ triads $\hat{\mathcal{Q}} \in \hat{\mathscr{Q}}$ with $\hat{\mathcal{Q}} \subseteq \hat{\mathcal{P}}$ satisfy*

$$\left| d_{\mathcal{H}}(\hat{\mathcal{P}}) - d_{\mathcal{H}}(\hat{\mathcal{Q}}) \right| \le \delta\,. \tag{28}$$

Proof. Let $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ as in the claim be given. We set

$$\mathcal{B}_+ = \{\hat{\mathcal{Q}} \in \hat{\mathscr{Q}} \colon\ \hat{\mathcal{Q}} \subseteq \hat{\mathcal{P}} \text{ and } d_{\mathcal{H}}(\hat{\mathcal{Q}}) > d_{\mathcal{H}}(\hat{\mathcal{P}}) + \delta\}$$
$$\mathcal{B}_- = \{\hat{\mathcal{Q}} \in \hat{\mathscr{Q}} \colon\ \hat{\mathcal{Q}} \subseteq \hat{\mathcal{P}} \text{ and } d_{\mathcal{H}}(\hat{\mathcal{Q}}) < d_{\mathcal{H}}(\hat{\mathcal{P}}) - \delta\}\,.$$

We first consider $|\mathcal{B}_+|$. Observe that by definition of $\mathcal{B}_+$

$$d_{\mathcal{H}}\Big(\bigcup_{\hat{\mathcal{Q}} \in \mathcal{B}_+} \hat{\mathcal{Q}}\Big) > d_{\mathcal{H}}(\hat{\mathcal{P}}) + \delta \overset{(19)}{>} d_{\mathcal{H}}(\hat{\mathcal{P}}) + \delta_{\mathscr{P}}\,. \tag{29}$$

On the other hand, recalling that by ($Q1$) every $Q \in \mathscr{Q}^{(2)}$ is an $(\varepsilon_{\mathscr{Q}}, 1/(\ell_{\mathscr{P}}\ell_{\mathscr{Q}}))$-regular bipartite graph and that by (25) $\varepsilon_{\mathscr{Q}} \le \varepsilon_{\mathrm{tcl}}(\gamma = \frac{1}{2}, d = \frac{1}{\ell_{\mathscr{P}}\ell_{\mathscr{Q}}})$, we infer from Lemma 5 that the total number of triangles contained in some $\hat{\mathcal{Q}} \in \mathcal{B}_+ \subseteq \hat{\mathscr{Q}}$ does not exceed

$$|\mathcal{B}_+| \times \frac{3}{2}\left(\frac{1}{\ell_{\mathscr{P}}\ell_{\mathscr{Q}}}\right)^3 \left(\frac{n}{t_{\mathscr{P}}t_{\mathscr{Q}}}\right)^3 . \tag{30}$$

Since $\mathcal{H}$ is $(\delta_{\mathscr{P}}, *, r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}$ and by (21)

$$r_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}) \ge \big(T_{\mathrm{aux}}(t_{\mathscr{P}}, \ell_{\mathscr{P}})\big)^6 \overset{(Q1)}{\ge} t_{\mathscr{Q}}^3 \ell_{\mathscr{Q}}^3 \ge |\mathcal{B}_+|$$

we infer from (29) that the quantity from (30) is smaller than

$$|\mathcal{B}_+| \times \frac{3}{2}\left(\frac{1}{\ell_{\mathscr{P}}\ell_{\mathscr{Q}}}\right)^3 \left(\frac{n}{t_{\mathscr{P}}t_{\mathscr{Q}}}\right)^3 \le \delta_{\mathscr{P}} \big|\operatorname{Tr}(\hat{\mathcal{P}})\big| \le \delta_{\mathscr{P}} \times \frac{3}{2}\left(\frac{1}{\ell_{\mathscr{P}}}\right)^3 \left(\frac{n}{t_{\mathscr{P}}}\right)^3 ,$$

where we used the $(\mu_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}), 1/\ell_{\mathscr{P}})$-regularity of the bipartite subgraphs of every $\hat{\mathcal{P}} \in \hat{\mathscr{P}}$ (see ($P1$)), $\mu_{\mathscr{P}}(t_{\mathscr{P}}, \ell_{\mathscr{P}}) \le \varepsilon_{\mathrm{tcl}}(\gamma = \frac{1}{2}, d = \frac{1}{\ell_{\mathscr{P}}})$ (see (20)), and Lemma 5 for the last inequality. Consequently,

$$|\mathcal{B}_+| \le \delta_{\mathscr{P}} t_{\mathscr{Q}}^3 \ell_{\mathscr{Q}}^3\,.$$

Repeating the same argument for $|\mathcal{B}_-|$ yields $|\mathcal{B}_+| + |\mathcal{B}_-| \le 2\delta_{\mathscr{P}} t_{\mathscr{Q}}^3 \ell_{\mathscr{Q}}^3$ and the claim follows. □

In what follows we will select the representative $\mathscr{R}$ from $\mathscr{Q}$ randomly and show that with positive probability such an $\mathscr{R}$ satisfies properties (ii)–(iv) of Lemma 16. For that the following notation will be useful. Recall that $\mathscr{P}^{(1)} = \{V_1, V_2, \dots, V_{t_{\mathscr{P}}}\}$ and $\mathscr{P}^{(2)} = \{P_\alpha^{ij} \colon\ 1 \le i < j \le t_{\mathscr{P}}, \alpha \in [\ell_{\mathscr{P}}]\}$, where $V(P_\alpha^{ij}) = V_i \cup V_j$. Let the vertex partition classes of $\mathscr{Q}^{(1)}$ be labeled in such a way that $\mathscr{Q}^{(1)} = \{W_{i,i'} \colon\ (i,i') \in$

$[t_{\mathscr{P}}]\times[t_{\mathscr{Q}}]\}$ and $V_i = W_{i,1}\cup W_{i,2}\cup\cdots\cup W_{i,t_{\mathscr{Q}}}$ for every $i = 1,2,\ldots,t_{\mathscr{P}}$. Furthermore, let the graphs of $\mathscr{Q}^{(2)}_{\mathscr{P}} = \{Q\in\mathscr{Q}^{(2)}\colon\ Q\subseteq P \text{ for some } P\in\mathscr{P}^{(2)}\}$ be labeled

$$\mathscr{Q}^{(2)}_{\mathscr{P}} = \left\{Q^{(i,i'),(j,j')}_{\alpha,\alpha'}\colon\ (\alpha,\alpha')\in[\ell_{\mathscr{P}}]\times[\ell_{\mathscr{Q}}],(i,i'),(j,j')\in[t_{\mathscr{P}}]\times[t_{\mathscr{Q}}],\ \text{and } i<j\right\}$$

such that for every $(i,i'),(j,j')\in[t_{\mathscr{P}}]\times[t_{\mathscr{Q}}]$ with $i<j$ and $(\alpha,\alpha')\in[\ell_{\mathscr{P}}]\times[\ell_{\mathscr{Q}}]$

$$V(Q^{(i,i'),(j,j')}_{\alpha,\alpha'}) = W_{i,i'}\cup W_{j,j'}$$
$$E(P^{ij}_{\alpha}[W_{i,i'}\cup W_{j,j'}]) = \bigcup_{\alpha'\in[\ell_{\mathscr{Q}}]} E(Q^{(i,i'),(j,j')}_{\alpha,\alpha'})\,.$$

Now consider a pair of random mappings

$$\varphi\colon\ [t_{\mathscr{P}}]\to[t_{\mathscr{Q}}] \quad\text{and}\quad \psi\colon\ \binom{[t_{\mathscr{P}}]}{2}\times[\ell_{\mathscr{P}}]\to[\ell_{\mathscr{Q}}]$$

both mappings are chosen independently and uniformly at random from the set of all $t_{\mathscr{Q}}^{t_{\mathscr{P}}}$ or $\ell_{\mathscr{Q}}^{\binom{t_{\mathscr{P}}}{2}\ell_{\mathscr{P}}}$ mappings. To each such pair of mappings we associate a random representative $\mathscr{R} = \mathscr{R}(\varphi,\psi) = \{\mathscr{R}^{(1)},\mathscr{R}^{(2)}\}$ defined by

$$\mathscr{R}^{(1)} = \{W_{i,\varphi(i)}\colon\ i\in[t_{\mathscr{P}}]\}$$

and

$$\mathscr{R}^{(2)} = \left\{Q^{(i,\varphi(i)),(j,\varphi(j))}_{\alpha,\psi(\{i,j\},\alpha)}\colon\ 1\le i<j\le t_{\mathscr{P}},\alpha\in[\ell_{\mathscr{P}}]\right\}. \tag{31}$$

It is easy to check that $\mathscr{R}(\varphi,\psi)$ indeed is a representative of $\mathscr{P}$ for every choice of φ and ψ. Moreover, we infer from (*Q1*), (25) and the choice of $T_{\mathscr{P}\mathscr{R}}$ in (23) that setting

$$t_{\mathscr{R}} = t_{\mathscr{Q}} \qquad\text{and}\qquad \ell_{\mathscr{R}} = \ell_{\mathscr{Q}} \tag{32}$$

yields

$$\mathscr{R}(\varphi,\psi) \text{ satisfies } (ii) \text{ of Lemma 16 for every choice of } \varphi \text{ and } \psi. \tag{33}$$

We are going to show that there is a choice of mappings φ and ψ such that the representative $\mathscr{R} = \mathscr{R}(\varphi,\psi)$ satisfies properties (*iii*) and (*iv*) of Lemma 16, as well.

Due to Claim 18 we have that if $\hat{\mathcal{P}}\in\hat{\mathscr{P}}$ is such that $\mathcal{H}$ is $(\delta_{\mathscr{P}},*,r_{\mathscr{P}}(t_{\mathscr{P}},\ell_{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}$, then at most $2\delta_{\mathscr{P}}t^3_{\mathscr{Q}}\ell^3_{\mathscr{Q}}$ sub-triads $\hat{\mathcal{Q}}\subseteq\hat{\mathcal{P}}$ from $\hat{\mathscr{Q}}$ violate (28). Moreover, by (*P2*) the number of $(\delta_{\mathscr{P}},*,r_{\mathscr{P}}(t_{\mathscr{P}},\ell_{\mathscr{P}}))$-irregular triads $\hat{\mathcal{P}}\in\hat{\mathscr{P}}$ does not exceed $\delta_{\mathscr{P}}t^3_{\mathscr{P}}\ell^3_{\mathscr{P}}$ and, consequently, the total number of sub-triads $\hat{\mathcal{Q}}\in\hat{\mathscr{Q}}$ with $\hat{\mathcal{Q}}\subseteq\hat{\mathcal{P}}\in\hat{\mathscr{P}}$, where $\hat{\mathcal{P}}$ is $(\delta_{\mathscr{P}},*,r_{\mathscr{P}}(t_{\mathscr{P}},\ell_{\mathscr{P}}))$-irregular is at most $\delta_{\mathscr{P}}t^3_{\mathscr{P}}\ell^3_{\mathscr{P}}\times t^3_{\mathscr{Q}}\ell^3_{\mathscr{Q}}$.

We say a triad $\hat{\mathcal{Q}}\in\hat{\mathscr{Q}}$ is *bad* if there exists some $\hat{\mathcal{P}}\in\hat{\mathscr{P}}$ such that $\hat{\mathcal{P}}\supseteq\hat{\mathcal{Q}}$ and either (28) is violated or $\hat{\mathcal{P}}$ is $(\delta_{\mathscr{P}},*,r_{\mathscr{P}}(t_{\mathscr{P}},\ell_{\mathscr{P}}))$-irregular. From the discussion above we clearly infer that

$$\left|\{\hat{\mathcal{Q}}\in\hat{\mathscr{Q}}\colon\ \hat{\mathcal{Q}}\text{ is bad}\}\right| \le 2\delta_{\mathscr{P}}t^3_{\mathscr{Q}}\ell^3_{\mathscr{Q}}\times|\hat{\mathscr{P}}| + \delta_{\mathscr{P}}t^3_{\mathscr{P}}\ell^3_{\mathscr{P}}t^3_{\mathscr{Q}}\ell^3_{\mathscr{Q}} \le 3\delta_{\mathscr{P}}(t_{\mathscr{P}}\ell_{\mathscr{P}}t_{\mathscr{Q}}\ell_{\mathscr{Q}})^3\,.$$

Since each $\hat{\mathcal{Q}}\in\hat{\mathscr{Q}}$ which is a sub-triad of some $\hat{\mathcal{P}}\in\hat{\mathscr{P}}$ is contained in the same number $(t_{\mathscr{Q}}^{t_{\mathscr{P}}-3}\ell_{\mathscr{Q}}^{\binom{t_{\mathscr{P}}}{2}\ell_{\mathscr{P}}-3})$ of all representatives $\mathscr{R}(\varphi,\psi)$, the expected number of bad triads contained in a random representative $\mathscr{R}(\varphi,\psi)$ is smaller than $3\delta_{\mathscr{P}}t^3_{\mathscr{P}}\ell^3_{\mathscr{P}}$. Let B_1 be the event that the random representative contains more than $9\delta_{\mathscr{P}}t^3_{\mathscr{P}}\ell^3_{\mathscr{P}} =$

$\delta t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3$ (see (19)) bad triads $\hat{\mathcal{Q}} \in \hat{\mathscr{Q}}$. From Markov's inequality we infer $\mathbb{P}(\mathscr{R}(\varphi,\psi) \in B_1) \le 1/3$. In other words,

$$\mathbb{P}\big(\mathscr{R}(\varphi,\psi) \text{ satisfies } (iii) \text{ of Lemma 16}\big) \ge \frac{2}{3}\,. \tag{34}$$

Similarly, due to $(Q3)$ combined with (24), (26), and (32) we have that the number of $(\delta_{\mathscr{R}}(t_{\mathscr{P}},\ell_{\mathscr{P}}), *, r_{\mathscr{R}}(t_{\mathscr{P}},t_{\mathscr{R}},\ell_{\mathscr{P}},\ell_{\mathscr{R}}))$-irregular triads $\hat{\mathcal{Q}} \in \hat{\mathscr{Q}}$ is at most $\delta_{\mathscr{Q}} t_{\mathscr{P}}^3 t_{\mathscr{Q}}^3 \ell_{\mathscr{P}}^3 \ell_{\mathscr{Q}}^3$. Hence, the expected number of such irregular triads contained in the random representative $\mathscr{R}(\varphi,\psi)$ is at most

$$\delta_{\mathscr{Q}} t_{\mathscr{P}}^3 \ell_{\mathscr{P}}^3 \overset{(24)}{\le} \frac{1}{3}\,.$$

Let B_2 be the event that the random representative $\mathscr{R}(\varphi,\psi)$ contains at least one $(\delta_{\mathscr{R}}(t_{\mathscr{P}},\ell_{\mathscr{P}}), *, r_{\mathscr{R}}(t_{\mathscr{P}},t_{\mathscr{R}},\ell_{\mathscr{P}},\ell_{\mathscr{R}}))$-irregular triad. Thus again by Markov's inequality we infer $\mathbb{P}(\mathscr{R}(\varphi,\psi) \in B_2) \le 1/3$, i.e.,

$$\mathbb{P}\big(\mathscr{R}(\varphi,\psi) \text{ satisfies } (iv) \text{ of Lemma 16}\big) \ge \frac{2}{3}\,. \tag{35}$$

Combining (34) and (35) implies that the probability that $\mathscr{R}(\varphi,\psi)$ satisfies (iii) and (iv) of Lemma 16 is at least 1/3. Hence, there exist a representative satisfying properties (iii) and (iv) and Lemma 16 follows from (27) and (33). □

5. Concluding remarks

We close this paper with a few remarks concerning extensions of Theorem 2 to monotone properties of general k-uniform hypergraphs and hereditary properties of hypergraphs.

5.1. **Monotone properties of k-uniform hypergraphs.** As we mentioned earlier, the proof of Theorem 2 presented in this paper extends without any major modification from 3-uniform to k-uniform hypergraphs. This is because the two main tools used in proof, namely the hypergraph regularity lemma (Theorem 11 and Theorem 13) and the hypergraph counting lemma (Theorem 8), were already proved for general k-uniform hypergraphs (see [23]). While the philosophy of the regularity method for general uniform hypergraphs and its application in this context stays the same, the general case of k-uniform hypergraphs brings a more complicated and technical notation. For example the concept of a (t,ℓ)-partition extends to a family of partitions of the vertices, pairs, triples, ..., and $(k-1)$-tuples of vertices. Due to this more complicated structure of the partition provided by the general regularity lemma, the notion of an appropiate representative is more involved. In particular, it cannot be described through such explicit labels as e.g. used in (ii) of Definition 14 or (31).

We believe that the special (and more explicit) case of 3-uniform hypergraphs provides a good balance between generality and clarity and that due to the less complex notation, the proof is more readable. Therefore we restricted ourselves to 3-uniform hypergraphs here.

5.2. **Hereditary properties of hypergraphs.** Another interesting generalization of Theorem 2 is the extension from monotone to hereditary properties. A hypergraph property is called hereditary if it is closed under taking induced sub-hypergraphs. Monotone properties are a special case of hereditary properties. Recently Alon and Shapira [4] and later Lovász and Szegedy [19] (see also [8]) proved that every hereditary graph property is testable. In particular Alon and Shapira use a strengthened version of Szemerédi's regularity lemma from [2], which in some sense corresponds to the representative lemma, Lemma 16, from our proof (see also [17] for a similar lemma). We believe that the proof of Alon and Shapira can be adapted to k-uniform hypergraphs by using the extension of the representative lemma given in this paper. Here again the main obstacles seem to be of technical order. In particular dealing with edges which are "not crossing" in the partition seems to present additional problems of technical nature.

Inspired by the work of Lovázs and Szegedy, the last two authors [22] found a way to merge some ideas from [19] with that of Alon et al. [2]. This yields a somewhat different proof, which circumvents the technical issues mentioned above.

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Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

E-mail address: cavart@mathcs.emory.edu

Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA

E-mail address: rodl@mathcs.emory.edu

Institut für Informatik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099 Berlin, Germany

E-mail address: schacht@informatik.hu-berlin.de