INTEGER AND FRACTIONAL PACKINGS OF HYPERGRAPHS

V. RÖDL, M. SCHACHT, M. H. SIGGERS, AND N. TOKUSHIGE

Abstract. Let F_0 be a fixed k-uniform hypergraph. The problem of finding the integer F_0-packing number $\nu_{F_0}(\mathcal{H})$ of a k-uniform hypergraph $\mathcal{H}$ is an NP-hard problem. Finding the fractional F_0-packing number $\nu^*_{F_0}(\mathcal{H})$ however can be done in polynomial time. In this paper we give a lower bound for the integer F_0-packing number $\nu_{F_0}(\mathcal{H})$ in terms of $\nu^*_{F_0}(\mathcal{H})$ and show that $\nu_{F_0}(\mathcal{H}) \geq \nu^*_{F_0}(\mathcal{H}) - o(|V(\mathcal{H})|^k)$.

1. Introduction

For positive integer ℓ, we denote by $[\ell]$ the set $\{1,\dots,\ell\}$. For set V and integer $k \geq 1$, we denote by $\binom{V}{k}$ the set of all k-element subsets of V. By $y = x \pm \varepsilon$ we mean $|y-x| < \varepsilon$. A subset $\mathcal{H} \subset \binom{V(\mathcal{H})}{k}$ is called a **k-uniform hypergraph** on vertex set $V(\mathcal{H})$. Notice that we are identifying a hypergraph $\mathcal{H}$ with its edges, so $|\mathcal{H}|$ will be the number of edges in the hypergraph. For $U \subset V(\mathcal{H})$, we denote by $\mathcal{H}[U]$ the subhypergraph of $\mathcal{H}$ induced by U (i.e. $\mathcal{H}[U] = \mathcal{H} \cap \binom{U}{k}$).

For fixed hypergraphs F_0 and $\mathcal{H}$, a subhypergraph $F \subset \mathcal{H}$ is a **copy of** F_0 if there exists a bijection of the vertex sets $\psi\colon V(F_0) \to V(F) \subset V(\mathcal{H})$ such that $\{\psi(u_1),\dots,\psi(u_k)\}$ is an edge in F if and only if $\{u_1,\dots,u_k\}$ is an edge in F_0. Denote the set of copies of F_0 in $\mathcal{H}$ by $\binom{\mathcal{H}}{F_0}$.

A map $\varphi^*\colon \binom{\mathcal{H}}{F_0} \to [0,1]$ such that for any edge $e \in \mathcal{H}$

$$\sum \left\{ \varphi^*(F)\colon\ F \in \binom{\mathcal{H}}{F_0} \quad \text{and} \quad e \in F \right\} \leq 1, \tag{1}$$

is called a **fractional F_0-packing** of $\mathcal{H}$. A fractional F_0-packing φ of $\mathcal{H}$ with image $\{0,1\}$ is called an **integer F_0-packing** of $\mathcal{H}$. The weight of a fractional F_0-packing φ^* of $\mathcal{H}$ is defined

$$w(\varphi^*) = \sum_{F \in \binom{\mathcal{H}}{F_0}} \varphi^*(F).$$

The maximum weight of a fractional F_0-packing of $\mathcal{H}$ is denoted $\nu^*_{F_0}(\mathcal{H})$ and the maximum weight of an integer F_0-packing of $\mathcal{H}$ is denoted $\nu_{F_0}(\mathcal{H})$.

Obviously, $\nu^*_{F_0}(\mathcal{H})$ is an upper bound of $\nu_{F_0}(\mathcal{H})$. The objective of this paper is to prove the following theorem, which provides a lower bound on $\nu_{F_0}(\mathcal{H})$ in terms of $\nu^*_{F_0}(\mathcal{H})$.

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Theorem 1.1 (Main Theorem)**.** *For every k-uniform hypergraph F_0, and for all $\eta > 0$, there exists $N \in \mathbb{N}$, such that for all $n > N$ and all k-uniform hypergraphs $\mathcal{H}$ on n vertices,*

$$\nu^*_{F_0}(\mathcal{H}) - \nu_{F_0}(\mathcal{H}) < \eta n^k.$$

For graphs, Theorem 1.1 was first proved in [14]. There the authors also provided a deterministic algorithm constructing an integer F_0-packing achieving the bound of the theorem in polynomial time. The proof was based on the algorithmic version of Szemerédi's Regularity Lemma [1] and on the algorithmic version of the matching result from [5] (due to Grable [11]). In [12], Theorem 1.1 was proved for 3-uniform hypergraphs. While the general philosophy of that proof is very similar to that of the graph case, the authors had to overcome many technical problems arising from the application of the Regularity Lemma from [6] for 3-uniform hypergraphs. Recent results of [13] can be used to give a deterministic algorithm in this case. In [21], Yuster gave an alternative proof of Theorem 1.1 in the graph case. Although the main approach (i.e., combined application of Szemerédi's Regularity Lemma with the matching result of [5]) is the same, his proof is simpler and allows him to replace F_0 by a family of graphs. On the other hand, these simplifications yield a randomized, rather than a deterministic algorithm to find such an integer packing.

Our proof of Theorem 1.1 for all $k \geq 2$ also follows the same general approach. So in particular we will use a Regularity Lemma for k-uniform hypergraphs from [18] (see Theorem 2.20) and an improved version of the matching result from [5] due to Pippenger and Spencer [17] (see Theorem 2.1). The Regularity Lemma we use here differs from that in [6] (and its extension for k-uniform hypergraphs from [19]). Rather than regularizing the given hypergraph with a constant ε (independent of the partition provided by the Regularity Lemma), the Regularity Lemma used here yields a slightly changed regular hypergraph, but allows ε to depend on the size of the partition. While the small "change" has no effect on our result this "improved" regularity significantly simplifies the argument for 3-uniform hypergraphs and allows the proof for general k.

Related Results. It follows from the result of Dor and Tarsi [4] that finding $\nu_{F_0}(\mathcal{H})$ is an NP-hard problem for all connected graphs F_0 with at least 3 edges. Since $\nu^*_{F_0}(\mathcal{H})$ is the solution of a linear program, it can be computed in polynomial time. Therefore, Theorem 1.1 shows that $\nu_{F_0}(\mathcal{H})$ can be approximated in polynomial time by a factor of $(1-\eta/c)$ for every $\eta > 0$ and for every k-uniform hypergraphs $\mathcal{H}$ with $\nu_{F_0}(\mathcal{H}) \geq c|V(\mathcal{H})|^k$. Thus this problem is an example of an NP-hard problem which has a polynomial time approximation algorithm for appropriately defined "dense case" (see [2, 3, 7, 8] for other examples).

Finally, we mention a consequence of Theorem 1.1 based on a nice result of Yuster [20]. Yuster proved a sufficient condition under which a hypergraph $\mathcal{H}$ admits fractional F_0-decomposition, i.e., a fractional F_0-packing φ^* which satisfies (1) with equality for every $e \in \mathcal{H}$. For a real $0 \leq \gamma \leq 1$ we say a k-uniform hypergraph $\mathcal{H}$ on n vertices is γ-dense if for every $i = 1, \dots, k-1$

$$\min_{I \in \binom{[n]}{i}} \big|\{e \in \mathcal{H} \colon\, e \supset I\}\big| \geq \gamma \binom{n-i}{k-i}.$$

Theorem 1.2 (Yuster [20])**.** *For every k-uniform hypergraph F_0 there exists an $\alpha > 0$ and some $N \in \mathbb{N}$, such that for all $n > N$ every k-uniform, $(1-\alpha)$-dense hypergraph $\mathcal{H}$ on n vertices admits a fractional F_0-decomposition.*

The corollary below follows from a combined application of Theorem 1.1 and Theorem 1.2.

Corollary 1.3. *For every k-uniform hypergraph F_0, and for all $\eta > 0$, there exists an $\alpha > 0$ and some $N \in \mathbb{N}$, such that for all $n > N$ every k-uniform, $(1-\alpha)$-dense hypergraphs $\mathcal{H}$ on n vertices admit an (integer) F_0-packing that covers $(1-\eta)|\mathcal{H}|$ of the edges, where $|\mathcal{H}|$ denotes the number of edges in $\mathcal{H}$.*

Outline of the paper. In Section 2 we introduce some results that will be used in the proof of Theorem 1.1. In Section 3 we state some technical lemmas, and prove the main theorem, Theorem 1.1, from these lemmas. Finally, in Section 4, we prove the lemmas.

2. Preliminary Results

In this section we introduce the main tools we use in the proof of Theorem 1.1. In Section 2.1 we state a theorem of Pippenger and Spencer. Section 2.2 and Section 2.3 are devoted to describe the setup for the Hypergraph Regularity Lemma[1] of the first two authors, Theorem 2.20, which will be an essential tool in our proof.

2.1. **A Matching Result for Hypergraphs.** Let $\mathcal{H}$ be a k-uniform hypergraph and let u be a vertex in $V(\mathcal{H})$, we denote by $\deg_{\mathcal{H}}(u)$ the **degree** of u, i.e., the number of edges in $\mathcal{H}$ which contain u. For two distinct vertices u, $w \in V(\mathcal{H})$ we write $\text{co-deg}_{\mathcal{H}}(u,w)$ for the **co-degree**, which is the number of edges that contain both vertices u and w. Recall that a **matching** $\mathcal{M} \subset \mathcal{H}$ is a subset of the edges of $\mathcal{H}$ such that no vertex occurs in more than one edge of $\mathcal{M}$ and a **perfect matching** is a matching that covers every vertex of $\mathcal{H}$. A theorem ensuring an almost perfect matching in a regular hypergraph of bounded co-degree appeared in [5]. The following extension, due to Pippenger and Spencer, is from [17].

Theorem 2.1 (Matching [17]). *For every real $\zeta > 0$, and real $C \geq 1$ there exist $\gamma_{\text{Mat}} = \gamma_{\text{Mat}}(\zeta, C) > 0$ and $N_{\text{Mat}} = N_{\text{Mat}}(\zeta, C)$ such that for every $n > D > N_{\text{Mat}}$ the following holds.*

If $\mathcal{H}$ is a k-uniform hypergraph on n vertices such that

(i) $\deg_{\mathcal{H}}(u) = (1 \pm \gamma_{\text{Mat}})D$ *for all but at most $\gamma_{\text{Mat}} n$ vertices $u \in V(\mathcal{H})$,*
(ii) $\deg_{\mathcal{H}}(u) \leq CD$ *for all $u \in V(\mathcal{H})$, and*
(iii) $\text{co-deg}(u,w) \leq \gamma_{\text{Mat}} D$ *for all distinct vertices u, $w \in V(\mathcal{H})$,*

then $\mathcal{H}$ contains a matching with at least $(1-\zeta)\frac{n}{k}$ edges.

2.2. **Regular Complexes.** In this section we develop the notation necessary for the statements of Theorem 2.7 and Lemma 2.15, both of which are needed in the proof of Theorem 1.1.

A k-uniform **clique of order** j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \geq k$ vertices consisting of all $\binom{j}{k}$ different k-tuples of the j vertices. Note that we will sometimes use the parentheses superscript to emphasize the uniformity of a hypergraph.

Given disjoint vertex sets $V_1, \dots, V_\ell$, we denote by $K_\ell^{(i)}(V_1, \dots, V_\ell)$ the **complete** ℓ-partite, i-uniform hypergraph (i.e. the family of all i-element subsets $I \subset \bigcup_{\lambda \in [\ell]} V_\lambda$

[1] There are different regularity lemmas for hypergraphs (see, e.g., [9, 18, 19]). The one we use here is from [18] and there it is called 'Regular Approximation Lemma'. However, since this is the only one we use here, we will call it the 'regularity lemma'.

satisfying $|V_\lambda \cap I| \le 1$ for every $\lambda \in [\ell]$). Any subset $\mathcal{H}^{(i)} \subset K_\ell^{(i)}(V_1, \dots, V_\ell)$ is called an (ℓ, i)**-hypergraph** on $V_1 \cup \dots \cup V_\ell$. If $m \le |V_\lambda| \le m+1$ for every $\lambda \in [\ell]$ then such $\mathcal{H}^{(i)}$ is further specified as an (m, ℓ, i)**-hypergraph**. Given integer j such that $i \le j \le \ell$, j element subset J of $[\ell]$, and (m, ℓ, i)-hypergraph $\mathcal{H}^{(i)}$, we denote by $\mathcal{H}^{(i)}[J] = \mathcal{H}^{(i)}[\bigcup_{\lambda \in J} V_\lambda]$ the (m, j, i)-subhypergraph of $\mathcal{H}^{(i)}$ induced by vertex set $\bigcup_{\lambda \in J} V_\lambda$.

For (m, ℓ, i)-hypergraph $\mathcal{H}^{(i)}$ and integer j with $i \le j \le \ell$, we denote by $\mathcal{K}_j(\mathcal{H}^{(i)})$ the set of j element subsets J of $V(\mathcal{H}^{(i)})$ for which every $I \in \binom{J}{i}$ is an edge of $\mathcal{H}^{(i)}$ (i.e. $\mathcal{K}_j(\mathcal{H}^{(i)})$ is the family of vertex sets of elements of $\binom{\mathcal{H}^{(i)}}{K_j^{(i)}}$).

Given $(m, \ell, i-1)$-hypergraph $\mathcal{H}^{(i-1)}$ and (m, ℓ, i)-hypergraph $\mathcal{H}^{(i)}$, we say an edge I of $\mathcal{H}^{(i)}$ **belongs to** $\mathcal{H}^{(i-1)}$ if $I \in \mathcal{K}_i(\mathcal{H}^{(i-1)})$, i.e. I is the vertex set of a copy of $K_i^{(i-1)}$ in $\mathcal{H}^{(i-1)}$. Moreover; $\mathcal{H}^{(i-1)}$ **underlies** $\mathcal{H}^{(i)}$ if $\mathcal{H}^{(i)} \subset \mathcal{K}_i(\mathcal{H}^{(i-1)})$.

Definition 2.2 ((m, ℓ, j)**-complex**). *Let $m \ge 1$ and $\ell \ge j \ge 1$ be integers. An (m, ℓ, j)***-complex** $\boldsymbol{\mathcal{H}}$ *is a collection of (m, ℓ, i)-hypergraphs $\{\mathcal{H}^{(i)}\}_{i=1}^{j}$ such that*

- *$\mathcal{H}^{(1)}$ is an $(m, \ell, 1)$-hypergraph, i.e. $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_\ell$ with $m \le |V_\lambda| \le m+1$ for $\lambda \in [\ell]$, and*
- *$\mathcal{H}^{(i-1)}$ underlies $\mathcal{H}^{(i)}$ for $2 \le i \le j$, i.e. $\mathcal{H}^{(i)} \subset \mathcal{K}_i(\mathcal{H}^{(i-1)})$.*

Szemerédi's Regularity Lemma decomposes the edge set of a graph so that 'most' edges belong to random-like (or ε-regular) subgraphs. In the Regularity Lemma for hypergraphs (see Theorem 2.20 below) the ε-regular pairs are replaced by $(\varepsilon, \boldsymbol{d})$-regular (m, k, k)-complexes (see Definition 2.6 below).

Many applications of Szemerédi's Regularity Lemma are based on the result that in an ℓ-partite graph with vertex partition $V_1 \cup \dots \cup V_\ell$ and all pairs (V_i, V_j), $1 \le i < j \le \ell$, being ε-regular of density at least $d \gg \varepsilon$, one can find 'many' copies of K_ℓ. The corresponding result for hypergraphs in the context of Theorem 2.20 is Theorem 2.7.

In order to describe Theorem 2.7 we first introduce the notion of *relative density* of an (m, i, i)-hypergraph with respect to an underlying $(m, i, i-1)$-hypergraph.

Definition 2.3 (**relative density**). *Let $\mathcal{H}^{(i)}$ be an i-uniform hypergraph and let $\mathcal{H}^{(i-1)}$ be an $(i-1)$-uniform hypergraph on the same vertex set. We define the* **density of $\mathcal{H}^{(i)}$ w.r.t. $\mathcal{H}^{(i-1)}$** *as*

$$d\big(\mathcal{H}^{(i)} \big| \mathcal{H}^{(i-1)}\big) = \begin{cases} \frac{|\mathcal{H}^{(i)} \cap \mathcal{K}_i(\mathcal{H}^{(i-1)})|}{|\mathcal{K}_i(\mathcal{H}^{(i-1)})|} & \text{if} \quad \big|\mathcal{K}_i(\mathcal{H}^{(i-1)})\big| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We now define the concept of regularity of an (m, i, i)-hypergraph with respect to an underlying hypergraph.

Definition 2.4. *Let positive real ε and non-negative real d_i be given along with an (m, i, i)-hypergraph $\mathcal{H}^{(i)}$ and an underlying $(m, i, i-1)$-hypergraph $\mathcal{H}^{(i-1)}$. We say $\mathcal{H}^{(i)}$ is* (ε, d_i)**-regular w.r.t.** $\mathcal{H}^{(i-1)}$ *if whenever $\mathcal{Q}^{(i-1)} \subset \mathcal{H}^{(i-1)}$ satisfies*

$$\big|\mathcal{K}_i(\mathcal{Q}^{(i-1)})\big| \ge \varepsilon \big|\mathcal{K}_i(\mathcal{H}^{(i-1)})\big|, \quad \text{then} \quad d\big(\mathcal{H}^{(i)} \big| \mathcal{Q}^{(i-1)}\big) = d_i \pm \varepsilon.$$

We extend the notion of (ε, d_i)-regularity from (m, i, i)-hypergraphs to (m, ℓ, i)-hypergraphs $\mathcal{H}^{(i)}$ for arbitrary $\ell > i$.

Definition 2.5 (**(ε, d_i)-regular hypergraph**). *Let positive real ε and non-negative real d_i be given along with an (m,ℓ,i)-hypergraph $\mathcal{H}^{(i)}$ and an underlying $(m,\ell,i-1)$-hypergraph $\mathcal{H}^{(i-1)}$. We say $\mathcal{H}^{(i)}$ is (ε, d_i)*-**regular w.r.t.** *$\mathcal{H}^{(i-1)}$ if the induced subhypergraph $\mathcal{H}^{(i)}[I]$ of $\mathcal{H}^{(i)}$ is (ε, d_i)-regular w.r.t. $\mathcal{H}^{(i-1)}[I]$ for all $I \in \binom{[\ell]}{i}$.*

We sometimes write ε-**regular** *to mean $\big(\varepsilon, d\big(\mathcal{H}^{(i)}\big|\mathcal{H}^{(i-1)}\big)\big)$-regular.*

Finally, we arrive at the notion of a regular complex.

Definition 2.6 (**$(\varepsilon, \boldsymbol{d})$-regular complex**). *Let ε be a positive real and $\boldsymbol{d} = (d_i)_{i=2}^{j}$ be a vector of non-negative reals. We say an (m,ℓ,j)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(i)}\}_{i=1}^{j}$, for $\ell \geq j$, is $(\varepsilon, \boldsymbol{d})$*-**regular** *if $\mathcal{H}^{(i)}$ is (ε, d_i)-regular w.r.t. $\mathcal{H}^{(i-1)}$ for every $i = 2, \ldots, j$.*

With these definitions, we can state the following theorem of Kohayakawa, Skokan, and Rödl [15].

Theorem 2.7 (Dense Counting Lemma [15, Thm. 6.5]). *For all $k \geq 2$ and positive reals ξ and d_0 there exist $\delta_{\mathrm{DCL}} = \delta_{\mathrm{DCL}}(k, \xi, d_0) > 0$ and integer $m_{\mathrm{DCL}} = m_{\mathrm{DCL}}(k, \xi, d_0)$ so that the following holds.*

If $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(i)}\}_{i=1}^{k-1}$ is a $(\delta_{\mathrm{DCL}}, \boldsymbol{d})$-regular $(m,k,k-1)$-complex with $\boldsymbol{d} = (d_i)_{i=2}^{k-1}$ satisfying $d_i > d_0$ for every $i = 2, \ldots, k-1$ and $m > m_{\mathrm{DCL}}$, then

$$\big|\mathcal{K}_k\big(\mathcal{H}^{(k-1)}\big)\big| = (1 \pm \xi) m^k \prod_{i=2}^{k-1} d_i^{\binom{k}{i}} . \tag{2}$$

Remark 2.8. *Without loss of generality we can assume that $m_{\mathrm{DCL}}(k, \xi, d_0)$ is monotone decreasing in d_0.*

Note that (2) coincides with that of the random setting. More precisely, suppose $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_k$ is a given vertex partition and $\mathcal{H}^{(2)}$ is randomly chosen from $K_k^{(2)}(V_1, \ldots, V_k) = \mathcal{K}_2(\mathcal{H}^{(1)})$ with probability d_2, and for every $i = 2, \ldots, k-1$ suppose $\mathcal{H}^{(i)}$ is a random subhypergraph of $\mathcal{K}_i(\mathcal{H}^{(i-1)})$ with relative density d_i, then with high probability the number of $K_k^{(k-1)}$'s in $\mathcal{H}^{(k-1)}$ would match (2). Thus $(\varepsilon, \boldsymbol{d})$-regularity ensures that the number of $K_k^{(k-1)}$'s in an $(\varepsilon, \boldsymbol{d})$-regular complex is approximately the same as in the corresponding random complex.

Since we will need to count not only cliques, but copies of an arbitrary fixed k-uniform hypergraph F_0, we appropriately generalize the concepts developed earlier.

Definition 2.9 (**(m, F)-hypergraph**). *Let F be a j-uniform hypergraph with v vertices, and $\mathcal{F}^{(j)}$ be an (m, v, j)-hypergraph on vertex set $V = \bigcup_{\lambda \in [v]} V_\lambda$.*

Then $\mathcal{F}^{(j)}$ is an (m, F)-**hypergraph** *if there exists a labeling $\{x_1, \ldots, x_v\}$ of the vertices of F such that the map $f \colon V \to \{x_1, \ldots, x_v\}$ defined $f(V_\lambda) = x_\lambda$ for $\lambda \in [v]$, is edge preserving.*

Note that a $(m, K_\ell^{(j)})$-hypergraph is just a (m, ℓ, j)-hypergraph.

Definition 2.10. *Given k-uniform hypergraph F_0, and $i \in [k]$, the i^{th}* **shadow** *$\Delta_i(F_0)$* **of** *F_0 is defined by*

$$\Delta_i(F_0) = \bigcup_{e \in F_0} \binom{e}{i}.$$

Definition 2.11 (**(m, F_0)-complex**). *Let F_0 be a k-uniform hypergraph with v vertices, and $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^{k}$ be an (m, v, k)-complex on vertex set $V = \bigcup_{i \in [v]} V_{\lambda_i}$.*

Then $\boldsymbol{\mathcal{F}}$ *is an* (m, F_0)**-complex** *if there is a labeling* $\{x_1, \dots x_v\}$ *of the vertices of* F_0 *such that the map* $f\colon V \to \{x_1, \dots, x_v\}$ *defined* $f(V_\lambda) = x_\lambda$ *for* $\lambda \in [v]$, *preserves edges as a map from* $\mathcal{F}^{(j)}$ *to* $\Delta_j(F_0)$, *for* $j = 2, \dots, k$.

Note that every layer $\mathcal{F}^{(j)}$ of an (m, F_0)-complex $\boldsymbol{\mathcal{F}}$ is an $(m, \Delta_j(F_0))$-hypergraph. Below we extend the notion of regularity from (m, ℓ, i)-hypergraphs and (m, ℓ, j)-complexes to (m, F)-hypergraphs and (m, F_0)-complexes.

Definition 2.12 ((ε, d_j, F)**-regular hypergraph**)**.** *Let a positive real* ε *and a non-negative real* d_i *be given. Let* F *be a* j*-uniform hypergraph, and* $\mathcal{F}^{(j)}$ *be an* (m, F)*-hypergraph with underlying* $(m, \Delta_{j-1}(F))$*-hypergraph* $\mathcal{F}^{(j-1)}$.

Then $\mathcal{F}^{(j)}$ *is* (ε, d_j, F)**-regular w.r.t** $\mathcal{F}^{(j-1)}$ *if the induced subhypergraph* $\mathcal{F}^{(j)}[J]$ *of* $\mathcal{F}^{(j)}$ *is* (ε, d_j)*-regular w.r.t* $\mathcal{F}^{(j-1)}[J]$ *for all edges* $J \in F$.

Definition 2.13 ($(\varepsilon, \boldsymbol{d}, F_0)$**-regular complex**)**.** *Let* ε *be a positive real and let* $\boldsymbol{d} = (d_i)_{i=2}^k$ *be a vector of non-negative reals. Let* F_0 *be a* k*-uniform hypergraph, and* $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$ *be an* (m, F_0)*-complex. Then* $\boldsymbol{\mathcal{F}}$ *is* $(\varepsilon, \boldsymbol{d}, F_0)$**-regular**, *if the* $(m, \Delta_j(F_0))$*-hypergraph* $\mathcal{F}^{(j)}$ *is* $(\varepsilon, d_j, \Delta_j(F_0))$*-regular w.r.t* $\mathcal{F}^{(j-1)}$ *for all* $j = 2, \dots, k$.

Again, note that in view of Definition 2.6 an $(\varepsilon, \boldsymbol{d}, K_\ell^{(k)})$-regular complex recovers the notion of an $(\varepsilon, \boldsymbol{d})$-regular (m, ℓ, k)-complex.

Definition 2.14. *Let* F_0 *be a* k*-uniform hypergraph with* v *vertices, and let* $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^k$ *be an* (m, F_0)*-complex with vertex set* $V = \bigcup_{\lambda=1}^v V_\lambda$*. A copy* F *of* F_0 *in* $\mathcal{F}^{(k)}$ *is crossing if* $|V_\lambda \cap F| = 1$ *for every* $\lambda = 1, \dots, v$.

Let $\operatorname{ext}_{\boldsymbol{\mathcal{F}}}(e)$ *denote the number of (unlabeled) crossing copies* $F \subseteq \mathcal{F}^{(k)}$ *of* F_0 *that contain the edge* e.

The following lemma asserts that for most edges e in a regular (m, F_0)-complex the number of crossing copies of F_0 that contain e is the same as in the corresponding random object.

Lemma 2.15 (Extension Lemma [18])**.** *For every* k*-uniform hypergraph* F_0*, and all positive reals* γ *and* d_0 *there exist* $\delta_{\mathrm{Ext}} = \delta_{\mathrm{Ext}}(F_0, \gamma, d_0) > 0$ *and an integer* $m_{\mathrm{Ext}} = m_{\mathrm{Ext}}(F_0, \gamma, d_0)$ *so that the following holds.*

If $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(i)}\}_{i=1}^k$ *is a* $(\delta_{\mathrm{Ext}}, \boldsymbol{d}, F_0)$*-regular* (m, F_0)*-complex with* $\boldsymbol{d} = (d_i)_{i=2}^k$ *satisfying* $d_i > d_0$ *for every* $i = 2, \dots, k$ *and* $m > m_{\mathrm{Ext}}$*, then*

$$\operatorname{ext}_{\boldsymbol{\mathcal{F}}}(e) = (1 \pm \gamma) m^{|\Delta_1(F_0)| - k} \prod_{i=2}^k d_i^{|\Delta_i(F_0)| - \binom{k}{i}},$$

for all but at most $\gamma|\mathcal{F}^{(k)}|$ *edges* $e \in \mathcal{F}^{(k)}$.

Lemma 2.15 can be derived from Theorem 2.7 and a proof is given in [18].

2.3. **Regularity Lemma for hypergraphs.** Let k be a fixed integer and V be a set of vertices. Throughout this paper we require a family of partitions $\mathscr{P} = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ on V to satisfy properties which we are going to describe below (see Definition 2.16).

Let $\mathscr{P}^{(1)} = V_1 \cup \dots \cup V_{|\mathscr{P}^{(1)}|}$ be a partition of V. For every $1 \le j \le k$ let

$$\mathrm{Cross}_j = \mathrm{Cross}_j(\mathscr{P}^{(1)}) = K^{(j)}_{|\mathscr{P}^{(1)}|}(V_1, \dots, V_{|\mathscr{P}^{(1)}|})$$

be the family of all crossing j-tuples J.

For $j = 2, \ldots, k-1$, we will require that $\mathscr{P}^{(j)}$ be a partition of Cross_j, each partition class will be a (j,j)-hypergraph – thus it seems appropriate to denote a partition class of $\mathscr{P}^{(j)}$ by $\mathcal{P}^{(j)}$. We denote the partition class containing $J \in \mathrm{Cross}_j$ by $\mathcal{P}^{(j)}(J)$.

There is a natural interaction between the partitions $\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}$ of a family. Every j-set $J \in \mathrm{Cross}_j$ uniquely defines, for $i = 1, \ldots, j$, a disjoint union

$$\hat{\mathcal{P}}^{(i)}(J) = \bigcup_{I \in \binom{J}{i}} \mathcal{P}^{(i)}(I) \tag{3}$$

of $\binom{j}{i}$ partition classes of $\mathscr{P}^{(i)}$. Note that $\hat{\mathcal{P}}^{(i)}(J)$ is a (j,i)-hypergraph. The use of '^' is to emphasize the fact that the corresponding hypergraph is not a single partition class of $\mathscr{P}^{(i)}$, but a union of them. In the case where $i = j-1$, we call the $(j, j-1)$-hypergraph $\hat{\mathcal{P}}^{(j-1)}(J)$ a **j-polyad**; often, context will allow us to drop the specification and refer to a j-polyad simply as a polyad.

We denote by $\hat{\mathscr{P}}^{(j-1)}$ the family of all j-polyads.

$$\hat{\mathscr{P}}^{(j-1)} = \big\{\hat{\mathcal{P}}^{(j-1)}(J)\colon\ J \in \mathrm{Cross}_j\big\}.$$

Note that $\hat{\mathscr{P}}^{(j-1)}$ induces a partition $\big\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon\ \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\big\}$ of Cross_j. This allows us to develop one of the properties that we will require of our family of partitions. We say that the partitions $\mathscr{P}^{(j-1)}$ and $\mathscr{P}^{(j)}$ are **cohesive** if $\mathscr{P}^{(j)}$ refines the partition induced by $\mathscr{P}^{(j-1)}$, i.e. if

$$\mathscr{P}^{(j)} \prec \big\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\colon\ \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\big\},$$

where $\prec$ is partition refinement. As well as having cohesion between consecutive partitions, we will want to control the number of partition classes in each partition. We accomplish this with the following definition.

Definition 2.16 (**family of partitions $\mathscr{P}(k-1, \boldsymbol{a})$**). *Suppose V is a set of vertices, $k \geq 2$ is an integer, and $\boldsymbol{a} = (a_j)_{j=1}^{k-1}$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(j)}\}_{j=1}^{k-1}$ is a* **family of partitions on** *V if it satisfies the following:*

- $\big|\mathscr{P}^{(1)}\big| = a_1$,
- *$\mathscr{P}$ is cohesive, i.e. for $j = 2, \ldots, k-1$, $\mathscr{P}^{(j-1)}$ and $\mathscr{P}^{(j)}$ are cohesive, and*
- $\big|\big\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}\colon\ \mathcal{P}^{(j)} \subset \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\big\}\big| = a_j$ *for every* $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$.

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is **L-bounded**, *if* $\max\{a_1, \ldots, a_{k-1}\} \leq L$.

Note that the requirement that a family $\mathscr{P}(k-1, \boldsymbol{a})$ be cohesive implies that for $1 < j \leq k$ and $J \in \mathrm{Cross}_j$, the structure

$$\hat{\boldsymbol{\mathcal{P}}}^{(j-1)}(J) = \{\hat{\mathcal{P}}^{(i)}(J)\}_{i=1}^{j-1}$$

is a complex. Such a complex is uniquely determined by its top layer, the polyad $\hat{\mathcal{P}}^{(j-1)}(J)$. Thus it is appropriate to call it a **j-polyad complex** or a polyad complex for short. Denote by

$$\mathrm{Com}_{j-1} = \mathrm{Com}_{j-1}(\mathscr{P}) = \big\{\hat{\boldsymbol{\mathcal{P}}}^{(j-1)}(J)\colon\ J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})\big\}$$

the set of all j-polyad complexes. In other words, polyad complexes are those $(n/a_1, \ell, i)$-complexes, where $\ell = j$ and $i = j-1$, which naturally arise in a family of partitions $\mathscr{P}$.

Before we state the Regularity Lemma for hypergraphs, we must define a few more conditions on families of partitions.

Definition 2.17 (**$(\eta, \varepsilon, \boldsymbol{a})$-equitable**). *Suppose V is a set of n vertices, η and ε are positive reals, and $\boldsymbol{a} = (a_j)_{j=1}^{k-1}$ is a vector of positive integers.*

*We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V is **$(\eta, \varepsilon, \boldsymbol{a})$-equitable** if it satisfies the following:*

- $\left|\binom{V}{k} \setminus \mathrm{Cross}_k\right| \le \eta\binom{n}{k}$,
- $\mathscr{P}^{(1)} = \{V_\lambda \colon \lambda \in [a_1]\}$ *is an equitable vertex partition, i.e.* $|V_1| \le \cdots \le |V_{a_1}| \le |V_1| + 1$,
- *every polyad-complex* $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)} = \left\{\hat{\mathcal{P}}^{(j)}\right\}_{j=1}^{k-1} \in \mathrm{Com}_{k-1}(\mathscr{P})$ *is an* $(\varepsilon, \boldsymbol{d})$*-regular* $(\lfloor n/a_1 \rfloor, k, k-1)$*-complex, where* $\boldsymbol{d} = (1/a_j)_{j=2}^{k-1}$.

Remark 2.18. *From now on we will drop floors and ceilings, since they have no effect on the arguments. Similarly, we will assume that $|V_\lambda| = n/a_1$ for every $\lambda \in [a_1]$.*

Definition 2.19 (**perfectly ε-regular**). *Suppose ε is some positive real. Let $\mathcal{G}$ be a k-uniform hypergraph and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on $V(\mathcal{G})$. We say $\mathcal{G}$* **is perfectly ε-regular w.r.t.** *$\mathscr{P}$, if for every polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ we have that $\mathcal{G} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$ is ε-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$.*

Theorem 2.20 (Hypergraph Regularity Lemma [18])**.** *Let $k \ge 2$ be a fixed integer. For all positive constants η and γ, and every function $\varepsilon \colon \mathbb{N}^{k-1} \to (0,1]$ there are integers L and n_0 so that the following holds.*

For every k-uniform hypergraph $\mathcal{H}$ with $|V(\mathcal{H})| = n \ge n_0$ there exist a k-uniform hypergraph $\mathcal{G}$ on the same vertex set and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ so that

(i) *$\mathscr{P}$ is $(\eta, \varepsilon(\boldsymbol{a}), \boldsymbol{a})$-equitable and L-bounded,*
(ii) *$\mathcal{G}$ is perfectly $\varepsilon(\boldsymbol{a})$-regular w.r.t. $\mathscr{P}$, and*
(iii) *$|\mathcal{H} \triangle \mathcal{G}| \le \gamma n^k$.*

Let us briefly compare Theorem 2.20 for $k = 2$ with Szemerédi's Regularity Lemma for graphs. Note that as discussed in [16, Section 1.8] there are graphs with irregular pairs in any partition. Therefore, due to the "perfectness" in (ii) of Theorem 2.20 one has to alter $\mathcal{H}$ to obtain $\mathcal{G}$.

The main difference between Theorem 2.20 for $k = 2$ and Szemerédi's Regularity Lemma, however, is in the choice of ε being a function of a_1. It follows from the work of Gowers in [10] that it is not possible to regularize a graph $\mathcal{H}$ with an ε in such a way that, for example, $\varepsilon < 1/a_1$ can be ensured, where $a_1 = |\mathscr{P}^{(1)}|$ is the number of vertex classes. Properties (i) and (iii) of Theorem 2.20 assert, however, that by adding or deleting at most γn^2 edges from $\mathcal{H}$ one can obtain a graph $\mathcal{G}$ which admits an $\varepsilon(a_1)$ regular partition, with $\varepsilon(a_1) < 1/a_1$. This will allow us to simplify the proof of Theorem 1.1 for 3-uniform hypergraphs from [12].

Remark 2.21. *Recall that in Szemerédi's Regularity Lemma it can be assumed that the regular partition $\mathscr{P}^{(1)}$ refines an initially given equitable partition of a fixed number of parts. The same can be assumed in the context of Theorem 2.20,* *i.e., that the vertex partition $\mathscr{P}^{(1)}$ of the family of partitions $\mathscr{P}$ refines an initial partition of fixed size. (In this case L and n_0 then also depend on the number of parts of the initial partition.) In fact, such a lemma is a special case of the more general lemma* DRL(k) *in* [18].

3. Proof of Main Theorem

Now we sketch the idea of the proof of Theorem 1.1. The Matching Theorem, Theorem 2.1, can be used to find large F_0-packings in a hypergraph that has the property that most edges occur in about the same number of copies of F_0. The hypergraph $\mathcal{H}$, however, does not, in general, have this property. Applying the regularity lemma allows us to decompose $\mathcal{H}$ into several subhypergraphs each having the property that each edge is in approximately the same number of copies of F_0. We then apply the Matching Theorem to each of these subhypergraphs separately.

The problem with this approach (which was already used in [14, 12, 21] to prove Theorem 1.1 for graphs and 3-uniform hypergraphs) is that the densities of the subhypergraphs provided by the regularity lemma can be 'very small' and may depend on the number of parts in the regular partition $\mathscr{P}$. (In fact, using this approach, we will have to deal with densities that depend on the number of F_0-complexes occurring in the partition $\mathscr{P}$, this clearly depends on size of $\mathscr{P}$.)

The regularity lemma of Szemerédi, as well as its earlier extensions to hypergraphs in [6, 19, 9], output a partition with the number of partition classes may be much bigger than $1/\varepsilon$. This results in a situation in which the densities of the aforementioned F_0-complexes may be smaller than ε. This is not an environment where regularity gives any information or control. Nevertheless, in each of [14, 12, 21], this problem was resolved in a different way.

The approach taken in this paper is novel in the sense that we use Theorem 2.20. This new regularity lemma allows us to regularize with an ε being an arbitrary function of the number of partition classes of $\mathscr{P}$. Even though Theorem 2.20 achieves this at the expense of having to slightly change the hypergraph, this can easily be overcome, and the stronger regularity properties allow us to give a simpler proof of the result for 3-uniform hypergraphs in [12], which extends to all k.

3.1. **A tailored Regularity Lemma.**

As a first step in the proof of Theorem 1.1 we will apply the Regularity Lemma for hypergraphs, Theorem 2.20. In order to simplify the presentation of the main proof we derive a variation (see Lemma 3.6 below) of Theorem 2.20, which is tailored to our situation.

Recall that in a typical application of Szemerédi's Regularity Lemma the edges belonging to sparse or irregular pairs are usually deleted (see, e.g., [16, Section 1.4]). After application of Theorem 2.20 there are no irregular polyads (though this can be said only of the slightly altered hypergraph $\mathcal{G}$), but we still have to deal with "sparse polyads" $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$. In our application the "sparseness" appears not only in the form of few edges, i.e., $d(\mathcal{G}|\hat{\mathcal{P}})$ is "small", but also concerns a given fractional F_0-packing. Below we first develop the notation necessary to describe the notion of sparse polyads w.r.t. a fractional packing (see Definition 3.5) and then we state the variation of Theorem 2.20 tailored to our application, Lemma 3.6.

Definition 3.1. *A k-uniform hypergraph $\mathcal{G}$ is γ-**density-separated** w.r.t. a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ if for every $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$ the density $d(\mathcal{G}|\hat{\mathcal{P}})$ is either 0 or greater than γ.*

Definition 3.2. *A copy F of F_0 in $\mathcal{G}$ is* **crossing** *w.r.t. family of partitions $\mathscr{P}$ on $V(\mathcal{G})$ if $|V(F)\cap V_\lambda|\le 1$ for every $\lambda=1,\dots,|\mathscr{P}^{(1)}|$.*

The following characterizes those (m,F_0)-complexes (see Definition 2.11) that occur naturally in a family of partitions $\mathscr{P}$ and a k-uniform hypergraph $\mathcal{G}$ on the same vertex set.

Definition 3.3 (**$(F_0,\mathcal{G},\mathscr{P})$-complex**)**.** *Given k-uniform hypergraphs F_0 and $\mathcal{G}$, a family of partitions $\mathscr{P}=\mathscr{P}(k-1,\boldsymbol{a})$ on $V(\mathcal{G})$, and a copy F of F_0 in $\mathcal{G}$ that is crossing w.r.t. $\mathscr{P}$, an $(F_0,\mathcal{G},\mathscr{P})$-***complex** *$\boldsymbol{\mathcal{F}}=\boldsymbol{\mathcal{F}}(F)=\{\mathcal{F}^{(i)}\}_{i=1}^k$ is defined by*

- *$\mathcal{F}^{(i)}=\bigcup_{I\in\Delta_i(F)}\mathcal{P}^{(i)}(I)$ for $i=1,\dots,k-1$, and*
- *$\mathcal{F}^{(k)}=\bigcup_{e\in F}\Big(\mathcal{G}\cap\mathcal{K}_k\big(\hat{\mathcal{P}}^{(k-1)}(e)\big)\Big)$.*

Moreover, let $\mathscr{C}=\mathscr{C}(F_0,\mathcal{G},\mathscr{P})$ be the set of all $(F_0,\mathcal{G},\mathscr{P})$-complexes. Given polyad $\hat{\mathcal{P}}\in\hat{\mathscr{P}}^{(k-1)}$, let $\mathscr{C}_{\hat{\mathcal{P}}}\subseteq\mathscr{C}$ be the set of $(F_0,\mathcal{G},\mathscr{P})$-complexes $\boldsymbol{\mathcal{F}}=\{\mathcal{F}^{(i)}\}_{i=1}^k$ for which $\hat{\mathcal{P}}\subseteq\mathcal{F}^{(k-1)}$.

Remark 3.4. *Note that every $(F_0,\mathcal{G},\mathscr{P})$-complex $\boldsymbol{\mathcal{F}}\in\mathscr{C}(F_0,\mathcal{G},\mathscr{P})$ is an (m,F_0)-complex with $m=|V(\mathcal{G})|/a_1$. Moreover, if*

- *$\mathscr{P}$ is $(\eta,\varepsilon,\boldsymbol{a})$-equitable for some constants η, ε, and vector $\boldsymbol{a}=(a_i)_{i=1}^{k-1}$, and*
- *$\mathcal{F}^{(k)}$ is (ε,d,F_0)-regular w.r.t. $\mathcal{F}^{(k-1)}$,*

then $\boldsymbol{\mathcal{F}}$ is an $\big(\varepsilon,(\frac{1}{a_2},\dots,\frac{1}{a_{k-1}},d),F_0\big)$-regular (m,F_0)-complex.

Definition 3.5. *Let F_0 and $\mathcal{G}$ be k-uniform hypergraphs, $\mathscr{P}=\mathscr{P}(k-1,\boldsymbol{a})$ be a family of partitions, and $\varphi^*_{\mathcal{G}}$ be an F_0-packing of $\mathcal{G}$.*

(a) *Call $\varphi^*_{\mathcal{G}}$* **crossing** *w.r.t. $\mathscr{P}$ if $\varphi^*_{\mathcal{G}}(F)=0$ for any copy F of F_0 in $\mathcal{G}$ that is not crossing w.r.t. $\mathscr{P}$ (cf. Definition 3.2).*

(b) *For an $(F_0,\mathcal{G},\mathscr{P})$-complex $\boldsymbol{\mathcal{F}}=\{\mathcal{F}^{(i)}\}_{i=1}^k\in\mathscr{C}(F_0,\mathcal{G},\mathscr{P})$ set*

$$\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})=\frac{\sum\big\{\varphi^*_{\mathcal{G}}(F)\colon\ F\text{ is a copy of }F_0\text{ in }\mathcal{F}^{(k)}\big\}}{\max\big\{|\mathcal{K}_k(\hat{\mathcal{P}})|\colon\ \hat{\mathcal{P}}\in\hat{\mathscr{P}}^{(k-1)}\big\}}.$$

(c) *For a positive real γ, we say $\varphi^*_{\mathcal{G}}$ is γ-***separated** *w.r.t. $\mathscr{P}$ if for every $(F_0,\mathcal{G},\mathscr{P})$-complex $\boldsymbol{\mathcal{F}}\in\mathscr{C}(F_0,\mathcal{G},\mathscr{P})$ either*

$$\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})=0\qquad\text{or}\qquad\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})\ge\gamma\prod_{i=1}^{k-1}\left(\frac{1}{a_i}\right)^{|\Delta_i(F_0)|-\binom{k}{i}}.$$

Observe that $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})$ is normalized so that for any $\hat{\mathcal{P}}\in\hat{\mathscr{P}}^{(k-1)}$ we have

$$\begin{aligned}\sum_{\boldsymbol{\mathcal{F}}\in\mathscr{C}_{\hat{\mathcal{P}}}(F_0,\mathcal{G},\mathscr{P})}\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})&\le\frac{\sum_{\boldsymbol{\mathcal{F}}\in\mathscr{C}_{\hat{\mathcal{P}}}}\sum\big\{\varphi^*_{\mathcal{G}}(F)\colon\ F\text{ is a copy of }F_0\text{ in }\mathcal{F}^{(k)}\ \big\}}{|\mathcal{K}_k(\hat{\mathcal{P}})|}\\&\le\frac{\sum_{e\in\mathcal{K}_k(\hat{\mathcal{P}})}\sum_{F\ni e}\big\{\varphi^*_{\mathcal{G}}(F)\colon\ F\text{ is a copy of }F_0\text{ in }\mathcal{G}\big\}}{|\mathcal{K}_k(\hat{\mathcal{P}})|}\\&\le\frac{\sum_{e\in\mathcal{G}\cap\mathcal{K}_k(\hat{\mathcal{P}})}1}{|\mathcal{K}_k(\hat{\mathcal{P}})|}=d(\mathcal{G}|\hat{\mathcal{P}})\,.\end{aligned}\tag{4}$$

Finally, we can state the variation of Theorem 2.20 mentioned earlier.

Lemma 3.6 (Tailored Regularity Lemma)**.** *For all $\mu > 0$, all k-uniform hypergraphs F_0, and all positive real-valued functions $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$, there exist $n_{\mathrm{Reg}} = n_{\mathrm{Reg}}(\varepsilon(\cdot,\dots,\cdot),\mu,F_0)$ and $L_{\mathrm{Reg}} = L_{\mathrm{Reg}}(\varepsilon(\cdot,\dots,\cdot),\mu,F_0)$ such that the following holds.*

For k-uniform hypergraph $\mathcal{H}$ with $|V(\mathcal{H})| = n \geq n_{\mathrm{Reg}}$, there exists a k-uniform hypergraph $\mathcal{G}$ with $V(\mathcal{G}) = V(\mathcal{H})$, and a family of partitions $\mathscr{P} = \mathscr{P}(k-1,\boldsymbol{a})$ on $V(\mathcal{G})$, such that

- (i) *$\mathscr{P}$ is $(\mu,\varepsilon(\boldsymbol{a}),\boldsymbol{a})$-equitable and L-bounded,*
- (ii) *$\mathcal{G}$ is perfectly $\varepsilon(\boldsymbol{a})$-regular w.r.t. $\mathscr{P}$,*
- (iii) *$\mathcal{G}$ is $\frac{\mu}{5}$-density-separated w.r.t. $\mathscr{P}$, and*
- (iv) *$|\mathcal{H}\triangle\mathcal{G}| < \mu n^k$.*

Moreover; if $\varphi^*_{\mathcal{H}}$ is a fractional F_0-packing of $\mathcal{H}$ with weight $w(\varphi^*_{\mathcal{H}}) = \alpha n^k$ for some $\alpha > \mu$, then we can choose $\mathscr{P}$ and $\mathcal{G}$, and find a fractional F_0-packing $\varphi^*_{\mathcal{G}}$ of $\mathcal{G}$, such that, in addition to the above properties,

- (v) *$\varphi^*_{\mathcal{G}}$ is crossing w.r.t. $\mathscr{P}$,*
- (vi) *$\varphi^*_{\mathcal{G}}$ is $\frac{\mu}{5}$-separated w.r.t. $\mathscr{P}$, and*
- (vii) *$w(\varphi^*_{\mathcal{G}}) > (\alpha-\mu)n^k$.*

We briefly compare Lemma 3.6 and Theorem 2.20. Note that properties (i), (ii), and (iv) are the conclusion of Theorem 2.20 and (iii) is easily obtained by removing those edges which belong to sparse polyads. The fractional F_0-packing $\varphi^*_{\mathcal{G}}$ is obtained by adjusting $\varphi^*_{\mathcal{H}}$ appropriately. We give the formal but straightforward proof of the existence of such a $\varphi^*_{\mathcal{G}}$ satisfying (v)–(vii) in Section 4.1.

3.2. **Decomposition Lemma.** In our proof of Theorem 1.1 we will first apply the Tailored Regularity Lemma, Lemma 3.6, from the last section. In the second step we select for each $(F_0,\mathcal{G},\mathscr{P})$-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(i)}\}_{i=1}^k$ with $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) > 0$ (c.f. Definition 3.5 (b) and Lemma 3.6 (vi)), an (m,F_0)-subhypergraph ($m = |V(\mathcal{G})|/a_1$) $\mathcal{G}_{\boldsymbol{\mathcal{F}}} \subseteq \mathcal{F}^{(k)}$ which is $(\varepsilon,\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}),F_0)$-regular w.r.t. $\mathcal{F}^{(k-1)}$. Then the Extension Lemma, Lemma 2.15, will imply that the auxiliary $|F_0|$-uniform hypergraph $\mathcal{L}_{\boldsymbol{\mathcal{F}}}$ with $V(\mathcal{L}_{\boldsymbol{\mathcal{F}}})$ equal to the edges set of $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ and $E(\mathcal{L}_{\boldsymbol{\mathcal{F}}})$ corresponding to the crossing copies of F_0 in $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$, satisfies the assumptions of the Matching Lemma, Lemma 2.1. Consequently, we will be able to infer that $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ contains an integer F_0-packing with weight ‘close’ to the weight of the fractional packing $\varphi^*_{\mathcal{G}}$ restricted to $\mathcal{F}^{(k)}$. Repeating this process over all $(F_0,\mathcal{G},\mathscr{P})$-complexes $\boldsymbol{\mathcal{F}} \in \mathscr{C}(F_0,\mathcal{G},\mathscr{P})$ and ensuring that $\mathcal{G}_{\boldsymbol{\mathcal{F}}} \cap \mathcal{G}_{\boldsymbol{\mathcal{F}}'} = \emptyset$ for all distinct $\boldsymbol{\mathcal{F}}$, $\boldsymbol{\mathcal{F}}' \in \mathscr{C}$ will yield the integer F_0-packing satisfying the conclusion of Theorem 1.1.

Below we formally define such a desired decomposition of $\mathcal{G}$ into regular (m,F_0)-subhypergraph's $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$. Then we state Lemma 3.8 which guarantees the existence of such a decomposition in an environment provided by the Tailored Regularity Lemma, Lemma 3.6.

Definition 3.7. *Given k-uniform hypergraphs F_0 and $\mathcal{G}$, and family of partitions $\mathscr{P} = \mathscr{P}(k-1,\boldsymbol{a})$, we have the set $\mathscr{C} = \mathscr{C}(F_0,\mathcal{G},\mathscr{P})$ of all $(F_0,\mathcal{G},\mathscr{P})$-complexes. For each $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(i)}\}_{i=1}^k \in \mathscr{C}$, let $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ be a subset of $\mathcal{F}^{(k)}$. If*

$$\mathcal{G}_{\boldsymbol{\mathcal{F}}} \cap \mathcal{G}_{\boldsymbol{\mathcal{F}}'} = \emptyset$$

for all pairs of distinct $\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{F}}' \in \mathscr{C}$, *then the set* $\{\mathcal{G}_{\boldsymbol{\mathcal{F}}} \colon \boldsymbol{\mathcal{F}} \in \mathscr{C}\} \cup \{\mathcal{T}\}$, *where*

$$\mathcal{T} = \mathcal{G} \setminus \bigcup_{\boldsymbol{\mathcal{F}} \in \mathscr{C}} \mathcal{G}_{\boldsymbol{\mathcal{F}}},$$

is called a $\mathscr{C}$-**decomposition** *of* $\mathcal{G}$.

Moreover, we say a $\mathscr{C}$*-decomposition of* $\mathcal{G}$ *is* $(\varepsilon, \varphi^*_{\mathcal{G}})$-**regular** *w.r.t.* $\mathscr{P}$ *for a fractional* F_0*-packing* $\varphi^*_{\mathcal{G}}$ *of* $\mathcal{G}$, *if for all* $\boldsymbol{\mathcal{F}} \in \mathscr{C}$,

$$\mathcal{G}_{\boldsymbol{\mathcal{F}}} \text{ is } (\varepsilon, \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}), F_0)\text{-regular w.r.t. } \mathcal{F}^{(k-1)}\,,$$

where $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})$ *is the quantity defined in Definition 3.5 (b).*

Lemma 3.8 (Decomposition Lemma)**.** *For all* k*-uniform hypergraphs* F_0, *and* $\mu > 0$, *there exists* $\varepsilon_\mu : \mathbb{N}^{k-1} \to (0,1]$ *such that for all functions* $\varepsilon : \mathbb{N}^{k-1} \to (0,1]$ *with* $\varepsilon(\cdot, \dots, \cdot) < \varepsilon_\mu(\cdot, \dots, \cdot)$ *pointwise, and all* L, *there exists* $n_{\mathrm{Dec}} = n_{\mathrm{Dec}}(\varepsilon(\cdot, \dots, \cdot), L)$ *such that the following holds.*

For k*-uniform hypergraph* $\mathcal{G}$ *with* $|V(\mathcal{G})| = n \geq n_{\mathrm{Dec}}$, *constants* $\boldsymbol{a}$, *family* $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ *of partitions on* $V(\mathcal{G})$ *, and* F_0*-packing* $\varphi^*_{\mathcal{G}}$ *of* $\mathcal{G}$, *meeting properties (i), (ii), (iii), (v), and (vi) of Lemma 3.6,*[2] *there exists a* $\mathscr{C}$*-decomposition of* $\mathcal{G}$ *that is* $(3\varepsilon(\boldsymbol{a}), \varphi^*_{\mathcal{G}})$*-regular w.r.t.* $\mathscr{P}$.

The lemma is proved in Section 4.2.

3.3. **Proof of Theorem 1.1.** Let k-uniform hypergraph F_0 and real $0 < \eta < 1$ be given. Since the theorem is trivial for a single edge, we can assume that F_0 has more than one edge. For $i = 1, \dots, k$, let $\triangle_i = |\Delta_i(F_0)|$. Let $\boldsymbol{A} = (A_i)_{i=1}^{k-1}$ be a vector of formal variables, and

$$f(\boldsymbol{A}) = \frac{15}{\eta} \prod_{i=1}^{k-1} A_i^{\triangle_i - \binom{k}{i}}$$

be a function of $\boldsymbol{A}$. Note that when $A_1, \dots, A_{k-1}$ are positive integers, then

$$f(\boldsymbol{A}) > A_i \quad \text{for every} \quad i = 1, \dots, k-1\,, \tag{5}$$

since $|F_0| > 1$. Below we fix all constants and functions crucial for our proof.

(i) Let $C\colon\ \mathbb{N}^{k-1} \to \mathbb{R}$ be such that

$$C(\boldsymbol{A}) > \prod_{i=2}^{k-1} \left(\frac{1}{A_i}\right)^{\binom{k}{i} - \triangle_i} \times \left(\frac{1}{f(\boldsymbol{A})}\right)^{1-\triangle_k}.$$

(ii) Define $\gamma\colon\ \mathbb{N}^{k-1} \to (0,1]$ by

$$\gamma(\boldsymbol{A}) = \gamma_{\mathrm{Mat}}(\eta/100, C(\boldsymbol{A})),$$

where γ_{Mat} is from Theorem 2.1 with $\zeta = \eta/100$ and $C = C(\boldsymbol{A})$.

(iii) Define $\varepsilon\colon\ \mathbb{N}^{k-1} \to (0,1]$ by letting $\varepsilon(\boldsymbol{A})$ be the pointwise minimum of
- $\frac{\eta}{100} \frac{1}{f(\boldsymbol{A})}$,
- $\varepsilon_{\eta/3}(\boldsymbol{A})$, (given by Lemma 3.8 with $\mu = \eta/3$)
- $\frac{1}{3} \cdot \delta_{\mathrm{Ext}}\big(F_0, \gamma(\boldsymbol{A}), \frac{1}{f(\boldsymbol{A})}\big)$, and (given by Lemma 2.15)
- $\delta_{\mathrm{DCL}}\big(k, \frac{\eta}{100}, \min_{2 \leq i < k} \frac{1}{A_i}\big)$. (given by Theorem 2.7)

(iv) Set $L = L_{\mathrm{Reg}}(\varepsilon(\cdot, \dots, \cdot), \eta/3, F_0)$, from Lemma 3.6.

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[2]Note that properties (iv) and (vii) of Lemma 3.6 are not applicable here, since the hypergraph $\mathcal{H}$ and the quantity α are not quantified here.

(v) Let $m_1\colon \mathbb{N}^{k-1} \to \mathbb{N}$ be a componentwise increasing function such that
- $m_1(\boldsymbol{A}) \geq m_{\mathrm{Ext}}\big(F_0, \gamma(\boldsymbol{A}), \frac{1}{f(\boldsymbol{A})}\big)$ (given by Lemma 2.15), and
- that is large enough that $N_{\mathrm{Mat}}\big(\frac{\eta}{100}, C(\boldsymbol{A})\big)$, from Theorem 2.1, is less than
$$|F_0|\left(1-\frac{\eta}{25}\right)\frac{\eta}{15}\cdot m_1(\boldsymbol{A})^k \prod_{i=1}^{k-1}\left(\frac{1}{A_i}\right)^{\Delta_i}. \tag{6}$$

(vi) Let N be an integer greater than the maximum of
- $(15L^{2^{\Delta_1}})^{|F_0|}\Big(\eta^{|F_0|}\cdot \gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(L,\dots,L))\Big)^{-1}$,
- $L\cdot m_1\big(L,\dots,L\big)$, (defined in (v))
- $L\cdot m_{\mathrm{DCL}}\big(k, \frac{\eta}{100}, \frac{1}{L}\big)$, (given by Theorem 2.7)
- $n_{\mathrm{Dec}}(\varepsilon(\cdot,\dots,\cdot), L)$, and (given by Lemma 3.8)
- $n_{\mathrm{Reg}}(\varepsilon(\cdot,\dots,\cdot), \eta/3, F_0)$. (given by Lemma 3.6)

Now let $\mathcal{H}$ be a k-uniform hypergraph on $n > N$ vertices, with maximum fractional packing $\varphi^*_{\mathcal{H}}$ of weight $w(\varphi^*_{\mathcal{H}}) = \alpha n^k$. We may assume that $\alpha > \eta$, since otherwise we are done.

Tailored Regularity Lemma. Since $n > N > n_{\mathrm{Reg}}(\varepsilon(\cdot,\dots,\cdot), \eta/3, F_0)$, we can apply the Tailored Regularity Lemma, Lemma 3.6, to $\mathcal{H}$ and $\varphi^*_{\mathcal{H}}$ with $\mu = \eta/3$, $\varepsilon(\cdot,\dots,\cdot)$, and α. This yields a hypergraph $\mathcal{G}$, a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$, and fractional F_0-packing $\varphi^*_{\mathcal{G}}$ of $\mathcal{G}$, that satisfy properties (i)–(vii) of Lemma 3.6.

By choice of $\varepsilon(\cdot,\dots,\cdot)$ and $n > N$, we have
$$\begin{aligned} &\varepsilon(\boldsymbol{a}) \leq \delta_{\mathrm{DCL}}\big(k, \tfrac{\eta}{100}, \min_{2\leq i<k} \tfrac{1}{a_i}\big) \quad \text{and} \\ &\frac{n}{a_1} > \frac{N}{L} > m_{\mathrm{DCL}}\big(k, \tfrac{\eta}{100}, \tfrac{1}{L}\big) \geq m_{\mathrm{DCL}}\big(k, \tfrac{\eta}{100}, \min_{2\leq i<k} \tfrac{1}{a_i}\big)\,, \end{aligned} \tag{7}$$
where the last inequality follows from Remark 2.8. Consider any polyad-complex $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)}$ of Com_{k-1}. Since $\mathscr{P}$ is $\big(\eta/3, \varepsilon(\boldsymbol{a}), \boldsymbol{a}\big)$-equitable by property (i) of Lemma 3.6 the complex $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)}$ is an $\big(\varepsilon(\boldsymbol{a}), (\frac{1}{a_2},\dots,\frac{1}{a_{k-1}})\big)$-regular $(n/a_1, k, k-1)$-complex. Thus in view of (7) we can apply Theorem 2.7 with $\xi = \frac{\eta}{100}$ and $d_0 = \min_{2\leq i<k}\frac{1}{a_i}$ to $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)} = \big\{\hat{\mathcal{P}}^{(j)}\big\}_{j=1}^{k-1}$ to show that
$$\big|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\big| = (1 \pm \frac{\eta}{100})\left(\frac{n}{a_1}\right)^k \cdot \prod_{i=2}^{k-1}\left(\frac{1}{a_i}\right)^{\binom{k}{i}}. \tag{8}$$

Decomposition Lemma. By choice of $\varepsilon(\cdot,\dots,\cdot) < \varepsilon_{\eta/3}(\cdot,\dots,\cdot)$ and choice of $n > N > n_{\mathrm{Dec}}(\varepsilon(\cdot,\dots,\cdot), L)$ we can apply the Decomposition Lemma, Lemma 3.8, to $\mathcal{G}$, $\mathscr{P}(k-1, \boldsymbol{a})$, and $\varphi^*_{\mathcal{G}}$ with $\mu = \eta/3$, and L as chosen in (iv). Let $\{\mathcal{T}\} \cup \{\mathcal{G}_{\boldsymbol{\mathcal{F}}}\colon \boldsymbol{\mathcal{F}} \in \mathscr{C}\}$ be a $(3\varepsilon(\boldsymbol{a}), \varphi^*_{\mathcal{G}})$-regular $\mathscr{C}(F_0, \mathcal{G}, \mathscr{P})$-decomposition that is given by Lemma 3.8.

Observations. Note that by the definition of the function f we have
$$\frac{1}{f(\boldsymbol{a})} = \frac{\eta}{15}\prod_{i=1}^{k-1}\left(\frac{1}{a_i}\right)^{\Delta_i - \binom{k}{i}}. \tag{9}$$

Now $\overline{\varphi}^*_{\mathcal{G}}$ was provided by Lemma 3.6 with $\mu = \eta/3$, so by property (vi) of that lemma, $\overline{\varphi}^*_{\mathcal{G}}$ is $\frac{\eta}{15}$-separated. By definition (see Definition 3.5(c)) this means that for any $(F_0, \mathcal{G}, \mathscr{P})$-complex $\boldsymbol{\mathcal{F}}$ in $\mathscr{C}$ such that $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) \neq 0$,

$$\frac{1}{f(\boldsymbol{a})} \leq \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}), \tag{10}$$

and in view of (5),

$$\tfrac{1}{f(\boldsymbol{a})} \leq \min\{\tfrac{1}{a_1} \dots, \tfrac{1}{a_{k-1}}, \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})\}. \tag{11}$$

Let $\mathscr{C}^{>0}$ be the subset of $\mathscr{C}$ of these $(F_0, \mathcal{G}, \mathscr{P})$-complexes, i.e.,

$$\mathscr{C}^{>0} = \{\boldsymbol{\mathcal{F}} \in \mathscr{C} \colon \ \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) > 0\}.$$

Later we want to apply the Matching Lemma, Theorem 2.1, to find an integer packing in $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}$ and for the verification of the assumptions we will need the following observations.

Fix some $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}$ and let $\boldsymbol{d}_{\boldsymbol{\mathcal{F}}}$ denote its density vector $\big(\frac{1}{a_1}, \dots, \frac{1}{a_{k-1}}, \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})\big)$. We note the following:

(a) Since the $\mathscr{C}$-decomposition is $(3\varepsilon(\boldsymbol{a}), \varphi^*_{\mathcal{G}})$-regular w.r.t. $\mathscr{P}$ (see Definition 3.7), each decomposition class $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ is $(3\varepsilon(\boldsymbol{a}), \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}), F_0)$-regular w.r.t. $\mathcal{F}^{(k-1)}$ (see Definition 2.12).

(b) Since $\boldsymbol{\mathcal{F}}$ is an $(F_0, \mathcal{G}, \mathscr{P})$-complex and since $\mathscr{P}$ is $(\eta/3, \varepsilon(\boldsymbol{a}), \boldsymbol{a})$-equitable, it follows from (a) that $\boldsymbol{\mathcal{F}}$ is a $\big(3\varepsilon(\boldsymbol{a}), \boldsymbol{d}_{\boldsymbol{\mathcal{F}}}, F_0\big)$-regular $(n/a_1, F_0)$-complex (see Remark 3.4).

(c) Recall that the function ε was chosen so that $\varepsilon(\boldsymbol{a}) \leq \frac{\eta}{100}\frac{1}{f(\boldsymbol{a})} \leq \frac{\eta}{100}\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})$. Thus

$$\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) - 3\varepsilon(\boldsymbol{a}) > \big(1 - \tfrac{3\eta}{100}\big)\, \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}), \tag{12}$$

and consequently, we infer from (a) that

$$\begin{aligned}
|\mathcal{G}_{\boldsymbol{\mathcal{F}}}| \quad &> \quad |F_0|(\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) - 3\varepsilon(\boldsymbol{a})) \min_{\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}} |\mathcal{K}_k(\hat{\mathcal{P}})| \\
&\overset{(8-10)}{>} \quad |F_0| \left(1 - \frac{3\eta}{100}\right) \frac{\eta}{15} \prod_{i=1}^{k-1} \left(\frac{1}{a_i}\right)^{\Delta_i - \binom{k}{i}} \left(1 - \frac{\eta}{100}\right) \left(\frac{n}{a_1}\right)^k \cdot \prod_{i=2}^{k-1} \left(\frac{1}{a_i}\right)^{\binom{k}{i}} \\
&> \quad |F_0| \left(1 - \frac{\eta}{25}\right) \frac{\eta}{15} n^k \prod_{i=1}^{k-1} \left(\frac{1}{a_i}\right)^{\Delta_i} \\
&\overset{(6)}{>} \quad N_{\mathrm{Mat}}(\eta/100, C(\boldsymbol{a})),
\end{aligned}$$

where we used the monotonicity of m_1 for the last inequality.

(d) From the choice of the function ε and $n > N$ in (iii) and (vi) we infer that

$$3\varepsilon(\boldsymbol{a}) < \delta_{\mathrm{Ext}}\big(F_0, \gamma(\boldsymbol{a}), \tfrac{1}{f(\boldsymbol{a})}\big) \quad \text{and} \quad \frac{n}{a_1} > m_{\mathrm{Ext}}\big(F_0, \gamma(\boldsymbol{a}), \tfrac{1}{f(\boldsymbol{a})}\big).$$

Hence, by (b) and (11) we can apply the Extension Lemma, Lemma 2.15, with $\gamma = \gamma(\boldsymbol{a})$ and $d_0 = \frac{1}{f(\boldsymbol{a})}$ to $\boldsymbol{\mathcal{F}}$. This way we infer that all but at most $\gamma(\boldsymbol{a}) = \gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(\boldsymbol{a}))$ proportion of the edges in $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ occur in $\big(1 \pm \gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(\boldsymbol{a}))\big)D$ crossing copies of F_0 in $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$, where

$$D = \left(\frac{n}{a_1}\right)^{\Delta_1 - k} \prod_{i=2}^{k-1} \left(\frac{1}{a_i}\right)^{\Delta_i - \binom{k}{i}} \cdot \big(\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})\big)^{\Delta_k - 1}.$$

(e) An edge of $\mathcal{G}_{\mathcal{F}}$ can occur in at most $(\frac{n}{a_1})^{\Delta_1-k}$ crossing copies of F_0, and by the choice of the function C in (i) and equation (10) we have

$$\left(\frac{n}{a_1}\right)^{\Delta_1-k} \leq C(\boldsymbol{a})D\,.$$

(f) Two different edges of $\mathcal{G}_{\mathcal{F}}$ can occur together in the most crossing copies of F_0 if they share $k-1$ vertices, i.e. if the two edges are spanned by $k+1$ vertices. In this case they can occur together in at most $\left(\frac{n}{a_1}\right)^{\Delta_1-k-1}$ copies. Due to the choice of

$$n > N \geq \frac{15^{|F_0|}L^{|F_0|\times 2^{\Delta_1}}}{\eta^{|F_0|}\times\gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(L,\dots,L))} \geq \frac{15^{|F_0|}L^{|F_0|\times 2^{\Delta_1}}}{\eta^{|F_0|}\times\gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(\boldsymbol{a}))}$$

in (vi) we have that

$$\left(\frac{n}{a_1}\right)^{\Delta_1-k-1} \leq \gamma_{\mathrm{Mat}}\big(\tfrac{\eta}{100}, C(\boldsymbol{a})\big)D\,.$$

Matching Theorem. After these preperations we head to the application of the Matching Theorem, Theorem 2.1. Now for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}$ we construct an auxillary $|F_0|$-uniform hypergraph $\mathcal{L}_{\mathcal{F}}$ defined by

$$V(\mathcal{L}_{\mathcal{F}}) = E(\mathcal{G}_{\mathcal{F}}), \qquad \text{and}$$
$$E(\mathcal{L}_{\mathcal{F}}) = \big\{E(F)\colon\ F \in \tbinom{\mathcal{G}_{\mathcal{F}}}{F_0}\big\}.$$

Since we verified properties (a)–(f) for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}$ we infer that $\mathcal{L}_{\mathcal{F}}$ has the following properties:

(c′) $|V(\mathcal{L}_{\mathcal{F}})| > N_{\mathrm{Mat}}(\frac{\eta}{100}, C(\boldsymbol{a}))$,
(d′) all but at most $\gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(\boldsymbol{a}))|V(\mathcal{L}_{\mathcal{F}})|$ vertices $x \in V(\mathcal{L}_{\mathcal{F}})$, have degree $\deg_{\mathcal{L}_{\mathcal{F}}}(x) \leq \big(1 \pm \gamma_{\mathrm{Mat}}(\frac{\eta}{100}, C(\boldsymbol{a}))\big)D$,
(e′) $\deg_{\mathcal{L}_{\mathcal{F}}}(x) \leq C(\boldsymbol{a})D$ for all $x \in V(\mathcal{L}_{\mathcal{F}})$, and
(f′) $\text{co-deg}_{\mathcal{L}_{\mathcal{F}}}(x,y) \leq \gamma_{\mathrm{Mat}}\big(\frac{\eta}{100}, C(\boldsymbol{a})\big)D$ for all distinct x, $y \in V(\mathcal{L}_{\mathcal{F}})$.

Thus we can apply the Matching Theorem, Theorem 2.1, with $\zeta = \frac{\eta}{100}$ and $C = C(\boldsymbol{a})$ to get an edge-packing of $\mathcal{L}_{\mathcal{F}}$ using at least $(1-\frac{\eta}{100})\frac{|V(\mathcal{L}_{\mathcal{F}})|}{|F_0|} = (1-\frac{\eta}{100})\frac{|\mathcal{G}_{\mathcal{F}}|}{|F_0|}$ edges. This corresponds to a set of at least $(1-\frac{\eta}{100})\frac{|\mathcal{G}_{\mathcal{F}}|}{|F_0|}$ copies of F_0 in $\mathcal{G}_{\mathcal{F}}$, no two of which share an edge. Thus the edge packing of $\mathcal{L}_{\mathcal{F}}$ corresponds to an integer F_0-packing $\varphi_{\mathcal{G}_{\mathcal{F}}}$ of $\mathcal{G}_{\mathcal{F}}$ with weight

$$w(\varphi_{\mathcal{G}_{\mathcal{F}}}) > \left(1-\frac{\eta}{100}\right)\frac{|\mathcal{G}_{\mathcal{F}}|}{|F_0|}\,. \tag{13}$$

Since the number of edges of $\mathcal{G}_{\mathcal{F}}$ belonging to any of the $|F_0|$ underlying hypergraphs $\hat{\mathcal{P}}$ is $|\mathcal{K}_k(\hat{\mathcal{P}})|$ times the density of $\mathcal{G}_{\mathcal{F}}$ with respect to $\mathcal{K}_k(\hat{\mathcal{P}})$, we infer from

$(3\varepsilon(\boldsymbol{a}), \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}), F_0)$-regularity of $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ w.r.t. $\mathcal{F}^{(k-1)}$ (see (a)) that

$$\begin{aligned}\frac{|\mathcal{G}_{\boldsymbol{\mathcal{F}}}|}{|F_0|} &> \big(\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) - 3\varepsilon(\boldsymbol{a})\big) \times \min_{\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}} |\mathcal{K}_k(\hat{\mathcal{P}})| \\ &\overset{(12)}{>} \left(1 - \frac{3\eta}{100}\right) \frac{\sum_{F \in \binom{\mathcal{F}^{(k)}}{F_0}} \varphi^*_{\mathcal{G}}(F)}{\max_{\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}} |\mathcal{K}_k(\hat{\mathcal{P}})|} \times \min_{\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}} |\mathcal{K}_k(\hat{\mathcal{P}})| \\ &\overset{(8)}{>} \left(1 - \frac{\eta}{30}\right) \frac{\left(1 - \frac{\eta}{100}\right)}{\left(1 + \frac{\eta}{100}\right)} \cdot \sum_{F \in \binom{\mathcal{F}^{(k)}}{F_0}} \varphi^*_{\mathcal{G}}(F) \\ &> \left(1 - \frac{\eta}{30}\right)^2 \sum_{F \in \binom{\mathcal{F}^{(k)}}{F_0}} \varphi^*_{\mathcal{G}}(F)\,.\end{aligned}$$

We then repeat the above for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}$ and set $\varphi_{\mathcal{G}} = \sum_{\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}} \varphi_{\mathcal{G}_{\boldsymbol{\mathcal{F}}}}$. Now, by the properties of a $\mathscr{C}$-decomposition every edge of $\mathcal{G}$ is in at most one $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ so $\varphi_{\mathcal{G}}$ is indeed an integer F_0-packing of $\mathcal{G}$. The weight of $\varphi_{\mathcal{G}}$ is

$$\begin{aligned}w(\varphi_{\mathcal{G}}) = \sum_{\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}} \varphi_{\mathcal{G}_{\boldsymbol{\mathcal{F}}}} &\overset{(13)}{\geq} \left(1 - \frac{\eta}{100}\right) \sum_{\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}} \frac{|\mathcal{G}_{\boldsymbol{\mathcal{F}}}|}{|F_0|} \\ &\geq \left(1 - \frac{\eta}{100}\right)\left(1 - \frac{\eta}{30}\right)^2 \sum_{\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}} \sum_{F \in \binom{\mathcal{F}^{(k)}}{F_0}} \varphi^*_{\mathcal{G}}(F)\,.\end{aligned}$$

Moreover, since $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) = 0$ for every $\boldsymbol{\mathcal{F}} \in \mathscr{C} \setminus \mathscr{C}^{>0}$ we further infer that the right-hand side of the last inequality equals

$$\begin{aligned}\left(1 - \frac{\eta}{100}\right)\left(1 - \frac{\eta}{30}\right)^2 \sum_{\boldsymbol{\mathcal{F}} \in \mathscr{C}} \sum_{F \in \binom{\mathcal{F}^{(k)}}{F_0}} \varphi^*_{\mathcal{G}}(F) &\geq \left(1 - \frac{\eta}{3}\right) w(\varphi^*_{\mathcal{G}}) \\ &\geq \left(1 - \frac{\eta}{3}\right)\left(\alpha - \frac{\eta}{3}\right) n^k\,,\end{aligned}$$

where the first inequality uses that $\varphi^*_{\mathcal{G}}$ is crossing w.r.t. $\mathscr{P}$, and the last inequality follows from property (vii) of Lemma 3.6. Consequently,

$$w(\varphi_{\mathcal{G}}) \geq \left(\alpha - \frac{2\eta}{3}\right) n^k\,.$$

Finally, by property (iv) of Lemma 3.6 we have $|\mathcal{H} \triangle \mathcal{G}| < \frac{\eta}{3} n^k$ and, hence, the restriction of $\varphi_{\mathcal{G}}$ to copies of F_0 in $\mathcal{H} \cap \mathcal{G}$ has weight greater than $(\alpha - \eta)n^k$. This completes the proof of the theorem. □

4. Proof of Lemmas

4.1. **Proof of the Tailored Regularity Lemma.** Recall that for given hypergraph $\mathcal{H}$ and fractional F_0-packing $\varphi^*_{\mathcal{H}}$, the Tailored Regularity Lemma, Lemma 3.6, outputs a hypergraph $\mathcal{G}$, a family of partitions $\mathscr{P}$, and a fractional F_0-packing $\varphi^*_{\mathcal{G}}$ which satisfy (i)–(vii) of the lemma. The proof, which is based on a straightforward application of Theorem 2.20 splits into three steps:

- To satisfy condition (v) and (vii) we first consider an auxiliary partition of the vertices so that 'most' of the weight of $\varphi^*_{\mathcal{H}}$ is in crossing copies of F_0.

- Then we apply Theorem 2.20 which outputs a family of partitions $\mathscr{P}$ and a perfectly regular hypergraph $\mathcal{G}$ (which is a small perturbation of $\mathcal{H}$).
- In the last step we adjust $\varphi^*_{\mathcal{H}}$ to a fractional packing of $\mathcal{G}$ which satisfies (v)–(vii).

Proof of Lemma 3.6. We first fix the constants and functions involved in the proof of Lemma 3.6. Let a real $\mu > 0$, a k-uniform hypergraph F_0 with $v_0 = |V(F_0)|$ vertices, and a function $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ be given. The main tool of the proof is the regularity lemma for hypergraphs, Theorem 2.20. For technical reasons we will apply Theorem 2.20 with a slightly smaller 'ε-function', $\varepsilon_{2.20} : \mathbb{N}^{k-1} \to (0,1]$ defined for every $\boldsymbol{A} = (A_i)_{i=1}^{k-1} \in \mathbb{N}^{k-1}$ by

$$\varepsilon_{2.20}(\boldsymbol{A}) = \min\Big\{\varepsilon(\boldsymbol{A}), \delta_{\mathrm{DCL}}\big(k, \tfrac{1}{4}, \min_{2\le i<k} \tfrac{1}{A_i}\big)\Big\},$$

where δ_{DCL} is given by Theorem 2.7. Moreover, fix an integer ℓ in so that

$$\ell > \frac{4v_0^2}{\mu}.$$

Next we apply the variation of Theorem 2.20 discussed in Remark 2.21 with constants $\eta = \mu$ and $\gamma = \mu/5$, the function $\varepsilon_{2.20}$, and the integer ℓ which is the number of vertex classes of the initial vertex partition. Theorem 2.20 yields integers L and n_0 and we fix the constants L_{Reg} and n_{Reg}, promised by Lemma 3.6

$$L_{\mathrm{Reg}} = L \quad\text{and}\quad n_{\mathrm{Reg}} = \max\big\{n_0, L\cdot m_{\mathrm{DCL}}(k, \tfrac{1}{4}, \tfrac{1}{L})\big\},$$

where m_{DCL} is given by Theorem 2.7.

Having defined all constants involved in the proof, let $\mathcal{H}$ be a k-uniform hypergraph with $|V(\mathcal{H})| = n \ge n_{\mathrm{Reg}}$ and $\varphi^*_{\mathcal{H}}$ be a fractional F_0-packing of $\mathcal{H}$ with weight $w(\varphi^*_{\mathcal{H}}) = \alpha n^k$. We have to find a k-uniform hypergraph $\mathcal{G}$ and a fractional F_0-packing $\varphi^*_{\mathcal{G}}$ of $\mathcal{G}$ which satisfy properties (i)–(vii) of Lemma 3.6.

Initial vertex partition. In view of (v) we first define an auxiliary vertex partition of V for which the weight of $\varphi^*_{\mathcal{H}}$ restricted to crossing copies of F_0 is 'close' to αn^k. For that consider a random equipartition of V into ℓ parts of cardinality $\frac{n}{\ell}$.

It follows from the choice of ℓ that

$$\Big(1 - \frac{v_0}{\ell}\Big)^{v_0} > \Big(1 - \frac{\mu}{4v_0}\Big)^{v_0} > 1 - \frac{\mu}{4}.$$

Hence, for every subset $X \subseteq V$ of cardinality v_0 the probability that X is crossing in the random partition can be bounded from below by

$$\mathbb{P}(X \text{ is crossing}) = \frac{\binom{\ell}{v_0}\left(\frac{n}{\ell}\right)^{v_0}}{\binom{n}{v_0}} > \Big(\frac{\ell - v_0}{\ell}\Big)^{v_0} > 1 - \frac{\mu}{4}.$$

Consequently, the expectation of the weight of the fractional packing $\varphi^*_{\mathcal{H}}$ restricted to the random equipartition is

$$\begin{aligned}\mathbb{E}\big[\textstyle\sum\{\varphi^*_{\mathcal{H}}(F)\colon F \in \binom{\mathcal{H}}{F_0} \text{ and } F \text{ is crossing}\}\big] &> \big(1 - \tfrac{\mu}{4}\big)\textstyle\sum\{\varphi^*_{\mathcal{H}}(F)\colon F \in \binom{\mathcal{H}}{F_0}\}\\ &= \big(1 - \tfrac{\mu}{4}\big)\alpha n^k\end{aligned}$$

Thus there is some equipartition $V = W_1 \cup \dots \cup W_\ell$ for which

$$\textstyle\sum\{\varphi^*_{\mathcal{H}}(F)\colon F \in \binom{\mathcal{H}}{F_0} \text{ and } |V(F) \cap W_i| \le 1,\ i = 1, \dots, \ell\} > \big(1 - \tfrac{\mu}{4}\big)\alpha n^k. \tag{14}$$

Regularization. Since $n \geq n_{\mathrm{Reg}} \geq n_0$ we can apply Theorem 2.20 to $\mathcal{H}$ and initial partition $V = W_1 \cup \dots \cup W_\ell$ with constants $\eta = \mu$, $\gamma = \mu/5$, and $\varepsilon_{2.20}$. Theorem 2.20 then yields a k-uniform hypergraph $\mathcal{G}'$ and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ satisfying properties (i)–(iii) of Theorem 2.20. Moreover, the vertex partition $\mathscr{P}^{(1)}$ refines the initial partition $W_1 \cup \dots \cup W_\ell$ (cf. Remark 2.21).

Since the family of partitions $\mathscr{P}$ is our final family of partitions, conclusion (i) of Theorem 2.20 yields property (i) of Lemma 3.6.

Removing sparse polyads and defining $\mathcal{G}$. We obtain $\mathcal{G}$ from $\mathcal{G}'$ by deleting those edges from $\mathcal{G}'$ which belong to a polyad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$ with $d(\mathcal{G}'|\hat{\mathcal{P}}) \leq \frac{\mu}{5}$. Clearly, $\mathcal{G}$ defined this way satisfies properties (ii) and (iii) of Lemma 3.6. Next we verify (iv). We infer from the definition of $\mathcal{G}$ that $|\mathcal{G}' \triangle \mathcal{G}| = |\mathcal{G}' \setminus \mathcal{G}| \leq \frac{\mu}{5} n^k$ and, hence, conclusion (iii) of Theorem 2.20 (with $\mathcal{G}'$ for $\mathcal{G}$) implies

$$|\mathcal{H} \triangle \mathcal{G}| \leq |\mathcal{H} \triangle \mathcal{G}'| + |\mathcal{G}' \triangle \mathcal{G}| \leq \left(\frac{\mu}{5} + \frac{\mu}{5}\right) n^k < \frac{\mu}{2} n^k\,, \tag{15}$$

yielding property (iv) of Lemma 3.6. There remains only to find an appropiate fractional packing of $\mathcal{G}$ which satisfies (v)–(vii).

Defining the fractional packing $\varphi^*_{\mathcal{G}}$. Below for two copies F and F' of F_0 in $\mathcal{G}$ we write $F \sim_{\mathscr{P}} F'$ if their $(F_0, \mathcal{G}, \mathscr{P})$-complex (see Definition 3.3) is the same, i.e.,

$$F \sim_{\mathscr{P}} F' \quad \iff \quad \boldsymbol{\mathcal{F}}(F, \mathcal{G}, \mathscr{P}) = \boldsymbol{\mathcal{F}}(F', \mathcal{G}, \mathscr{P})\,.$$

Then define fractional packing $\varphi^*_{\mathcal{G}}$ on a copy F of F_0 in $\mathcal{G}$ as follows. Set $\varphi^*_{\mathcal{G}}(F) = 0$ if one of the following holds

(a) $F \notin \binom{\mathcal{H} \cap \mathcal{G}}{F_0}$,
(b) F is not crossing w.r.t. $\mathscr{P}$,
(c) $\sum\{\varphi^*_{\mathcal{H}}(F')\colon\ F' \in \binom{\mathcal{H} \cap \mathcal{G}}{F_0} \text{ and } F' \sim_{\mathscr{P}} F\} <$
$< \frac{\mu}{5} \prod_{i=1}^{k-1} \left(\frac{1}{a_i}\right)^{|\Delta_i(F_0)| - \binom{k}{i}} \times \max\left\{|\mathcal{K}_k(\hat{\mathcal{P}})|\colon\ \hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}\right\}$

and set $\varphi^*_{\mathcal{G}}(F) = \varphi^*_{\mathcal{H}}(F)$ otherwise. It follows straight from the definition of $\varphi^*_{\mathcal{G}}(F)$ above, that properties (v) and (vi) of Lemma 3.6 hold.

We need only to verify (vii). The fractional packing $\varphi^*_{\mathcal{G}}$ differs from $\varphi^*_{\mathcal{H}}$ on copies F of F_0 satisfying one of the conditions (a)–(c). Consequently,

$$w(\varphi^*_{\mathcal{H}}) - w(\varphi^*_{\mathcal{G}}) < A + B + C\,, \tag{16}$$

where

$$A = \sum\left\{\varphi^*_{\mathcal{H}}(F)\colon\ F \notin \tbinom{\mathcal{H} \cap \mathcal{G}}{F_0}\right\},$$

$$B = \sum\left\{\varphi^*_{\mathcal{H}}(F)\colon\ F \in \tbinom{\mathcal{H}}{F_0} \text{ and } F \text{ is not crossing w.r.t. } \mathscr{P}\right\}, \text{ and}$$

$$C = \frac{\mu}{5} \prod_{i=1}^{k-1} \left(\frac{1}{a_i}\right)^{|\Delta_i(F_0)| - \binom{k}{i}} \times \max\left\{|\mathcal{K}_k(\hat{\mathcal{P}})|\colon\ \hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}\right\} \times |\mathscr{C}(F_0, \mathcal{H}, \mathscr{P})|\,,$$

where $|\mathscr{C}(F_0, \mathcal{H}, \mathscr{P})|$ is the number of $(F_0, \mathcal{H}, \mathscr{P})$-complexes (see Definition 3.3).

The quantity A can be bounded by

$$A \leq \sum_{e \in \mathcal{H} \setminus \mathcal{G}} \sum\left\{\varphi^*_{\mathcal{H}}(F)\colon\ F \in \tbinom{\mathcal{H}}{F_0} \text{ and } e \in F\right\} \leq \sum_{e \in \mathcal{H} \setminus \mathcal{G}} 1 \overset{(15)}{\leq} \frac{\mu}{2} n^k\,, \tag{17}$$

and since $\mathscr{P}^{(1)}$ refines $W_1 \cup \dots \cup W_\ell$ it follows from (14) that

$$B \le \frac{\mu}{4}\alpha n^k < \frac{\mu}{4} n^k\,. \tag{18}$$

Finally, we consider the quantity C. Note that by the choice of the function $\varepsilon_{2.20}$ we have $\varepsilon_{2.20}(\boldsymbol{a}) \le \delta_{\mathrm{DCL}}\big(k, \frac{1}{4}, \min_{2\le i<k} \frac{1}{a_i}\big)$ and the choice of $n > N$ yields, by remark 2.8,

$$\frac{n}{a_1} > \frac{n}{L_{\mathrm{Reg}}} > m_{\mathrm{DCL}}\Big(k, \tfrac{1}{4}, \min_{2\le i<k} \tfrac{1}{L}\Big) > m_{\mathrm{DCL}}\Big(k, \tfrac{1}{4}, \min_{2\le i<k} \tfrac{1}{a_i}\Big)\,.$$

Hence, we can apply Theorem 2.7 with $\xi = \frac{1}{4}$ and $d_0 = \min_{2\le i<k} \frac{1}{a_i}$ to every polyad-complex in $\mathrm{Com}_{k-1}(\mathscr{P})$ to get that

$$\max\big\{|\mathcal{K}_k(\hat{\mathcal{P}})|\colon\ \hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}\big\} \le \frac{5}{4}\left(\frac{n}{a_1}\right)^k \cdot \prod_{i=2}^{k-1}\left(\frac{1}{a_i}\right)^{\binom{k}{i}}\,.$$

Moreover, the number of $(F_0, \mathcal{H}, \mathscr{P})$-complexes is bounded from above by

$$|\mathscr{C}(F_0, \mathcal{H}, \mathscr{P})| \le \frac{a_1!}{(a_1 - v_0)!}\prod_{i=2}^{k-1} a_i^{|\Delta_i(F_0)|} < a_1^{v_0}\prod_{i=2}^{k-1} a_i^{|\Delta_i(F_0)|}\,.$$

Since $v_0 = |\Delta_1(F_0)|$ we infer that

$$C < \frac{\mu}{4} n^k\,. \tag{19}$$

Therefore, property (vii) of Lemma 3.6 follows from (16) combined with (17), (18), and (19) which finishes the proof. □

4.2. **Proof of the Decomposition Lemma.** The proof of the Decomposition Lemma, Lemma 3.8, relies on the so called Slicing Lemma, which ensures that random subhypergraphs of regular hypergraphs are again regular.

Lemma 4.1 (Slicing Lemma)**.** *Let d and ε be positive real numbers such that $0 < \varepsilon, d \le 1$. Let $\hat{\mathcal{P}}$ be a $(m, k, k-1)$-hypergraph satisfying $|\mathcal{K}_k(\hat{\mathcal{P}})| \ge m^k/\ln m$ and $\mathcal{G}_{\hat{\mathcal{P}}}$ be an (m, k, k)-hypergraph which is (ε, d)- regular w.r.t. $\hat{\mathcal{P}}$. Then, for every $0 < p_1, \dots, p_u < 1$ such that*

- $\sum_{i=1}^u p_i \le 1$,
- $k(\ln m)/m \le \varepsilon^3/5$,
  and for all $i = 1, \dots, u$,
- $3\varepsilon < p_i d$,

the following holds:

There exists a partition $\mathcal{G}_{\hat{\mathcal{P}}} = \mathcal{T}_{\hat{\mathcal{P}}} \cup \mathcal{G}_{\hat{\mathcal{P}}_1} \cup \dots \cup \mathcal{G}_{\hat{\mathcal{P}}_u}$ such that $\mathcal{G}_{\hat{\mathcal{P}}_i}$ is $(3\varepsilon, p_i d)$-regular w.r.t. $\hat{\mathcal{P}}$ for every $i = 1, \dots, u$.

The proof of Lemma 3.8 is based on the Chernoff inequality, and is along the lines of [19, Lemma 11.3]. We omit the details here.

Let us briefly recall the Decomposition Lemma. Roughly speaking, for a given k-polyad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$ the Decomposition Lemma guarantees that for every $(F_0, \mathcal{G}, \mathscr{P})$-complex $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(i)}\}_{i=1}^k$ with $\hat{\mathcal{P}} \subseteq \mathcal{F}^{(k-1)}$ (i.e., $\boldsymbol{\mathcal{F}} \in \mathscr{C}_{\hat{\mathcal{P}}}$) there is a $(3\varepsilon(\boldsymbol{a}), \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}))$-regular (w.r.t. $\hat{\mathcal{P}}$) subhypergraph $\mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})}$ of $\mathcal{G} \cap \mathcal{K}_k(\hat{\mathcal{P}})$ such that $\mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})} \cap \mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}}')} = \emptyset$ for all distinct $\boldsymbol{\mathcal{F}}$, $\boldsymbol{\mathcal{F}}' \in \mathscr{C}_{\hat{\mathcal{P}}}$. Since $\mathcal{G}$ is perfectly $\varepsilon(\boldsymbol{a})$-regular w.r.t. the given family of partitions $\mathscr{P}$ such a decomposition will be ensured by a straightforward application of the Slicing Lemma. We give the formal proof below.

Proof of Lemma 3.8. Given F_0 and $\mu > 0$ let $\varepsilon_\mu\colon \mathbb{N}^{k-1} \to (0,1]$ be such that for formal variables $\boldsymbol{A} = (A_i)_{i=1}^k$, $\varepsilon_\mu(\boldsymbol{A})$ is less than

- $\dfrac{\mu}{15}\displaystyle\prod_{i=1}^{k-1}\left(\frac{1}{A_i}\right)^{|\Delta_i(F_0)|-\binom{k}{i}}$, and
- $\delta_{\mathrm{DCL}}\big(k, \frac{1}{4}, \min_{2\le i<k} \frac{1}{A_i}\big)$.

Let $\varepsilon\colon \mathbb{N}^{k-1} \to (0,1]$ be such that $\varepsilon(\boldsymbol{A}) < \varepsilon_\mu(\boldsymbol{A})$, and L be given. Without loss of generality we may assume that $\varepsilon(\cdot,\dots,\cdot)$ is componentwise decreasing. Now fix an auxiliary constant m_{Dec} large enough that

- $\frac{3}{4}\ln m_{\mathrm{Dec}} > L^{2^k}$,
- $\frac{k\ln m_{\mathrm{Dec}}}{m_{\mathrm{Dec}}} \le \frac{\varepsilon(L,\dots,L)^3}{5}$, and
- $m_{\mathrm{Dec}} > m_{\mathrm{DCL}}\big(k, \frac{1}{4}, \frac{1}{L}\big)$.

Finally, set

$$n_{\mathrm{Dec}} = L \cdot m_{\mathrm{Dec}}\,.$$

Let $\mathcal{G}$ be a k-uniform hypergraph with vertex set V and $|V| = n > n_{\mathrm{Dec}}$. Moreover, let $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on V, and $\varphi^*_{\mathcal{G}}$ be an F_0-packing of $\mathcal{G}$ meeting properties (i),(ii),(iii),(v), and (vi) of Lemma 3.6. Note that $\mathscr{P}^{(1)} = V_1 \cup \dots \cup V_{a_1}$ where for $\lambda \in [a_1]$, V_λ has size

$$m = \frac{n}{a_1} > \frac{n}{L} > \frac{n_{\mathrm{Dec}}}{L} = m_{\mathrm{Dec}}\,. \tag{20}$$

For each polyad $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$ we use Lemma 4.1 to partition the edges of $\mathcal{G} \cap \mathcal{K}_k(\hat{\mathcal{P}})$ into partition classes $\mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})}$ where $\boldsymbol{\mathcal{F}}$ runs over

$$\mathscr{C}_{\hat{\mathcal{P}}} = \left\{\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(i)}\}_{i=1}^k\colon\ \boldsymbol{\mathcal{F}} \in \mathscr{C}(F_0, \mathcal{G}, \mathscr{P}) \text{ and } \mathcal{F}^{(k-1)} \subseteq \hat{\mathcal{P}}\right\},$$

(see Definition 3.3). We then join the partition classes corresponding to each $\boldsymbol{\mathcal{F}} \in \mathscr{C}$, to get

$$\mathcal{G}_{\boldsymbol{\mathcal{F}}} = \bigcup\left\{\mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})}\colon\ \hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } \hat{\mathcal{P}} \subseteq \mathcal{F}^{(k-1)}\right\}.$$

These classes $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ will define the required $(3\varepsilon(\boldsymbol{a}), \varphi^*_{\mathcal{G}})$-regular $\mathscr{C}$-decomposition of $\mathcal{G}$.

More precisely, let $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$ with $d(\mathcal{G}|\hat{\mathcal{P}}) > 0$. Set

$$\mathscr{C}^{>0}_{\hat{\mathcal{P}}} = \{\boldsymbol{\mathcal{F}} \in \mathscr{C}_{\hat{\mathcal{P}}}\colon\ \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) > 0\}$$

and for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}_{\hat{\mathcal{P}}}$ set

$$p_{(\boldsymbol{\mathcal{F}},\hat{\mathcal{P}})} = \frac{\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})}{d(\mathcal{G}|\hat{\mathcal{P}})}.$$

We now verify the assumptions of Lemma 4.1 for $\mathcal{G} \cap \mathcal{K}_k(\hat{\mathcal{P}})$:

- $\mathscr{P}$ is $(\mu, \varepsilon(\boldsymbol{a}), \boldsymbol{a})$-equitable, so polyad-complex $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)}$, corresponding to polyad $\hat{\mathcal{P}}$ is an $\big(\varepsilon(\boldsymbol{a}), (1/a_2, \dots, 1/a_{k-1})\big)$-regular $(m, k, k-1)$-complex. By the earlier choice of the function ε we have $\varepsilon(\boldsymbol{a}) < \delta_{\mathrm{DCL}}\big(k, \frac{1}{4}, \min_{2\le i<k} \frac{1}{a_i}\big)$. Moreover, due to (20), the choice of m_{Dec} and Remark 2.8 we have and
$$n/a_1 = m > m_{\mathrm{Dec}} > m_{\mathrm{DCL}}\Big(k, \frac{1}{4}, \frac{1}{L}\Big) \ge m_{\mathrm{DCL}}\Big(k, \frac{1}{4}, \min_{2\le i<k} \frac{1}{a_i}\Big)\,.$$

Consequently we can apply Theorem 2.7 to $\hat{\boldsymbol{\mathcal{P}}}^{(k-1)}$ with $\xi = \frac{1}{4}$ and $d_0 = \min_{2\le i<k} \frac{1}{a_i}$ to get that

$$|\mathcal{K}_k(\hat{\mathcal{P}})| \ge \left(1-\frac{1}{4}\right)\left(\frac{n}{a_1}\right)^k \cdot \prod_{i=2}^{k-1}\left(\frac{1}{a_i}\right)^{\binom{k}{i}} \ge \frac{3}{4}\frac{m^k}{L^{2^k}} \ge \frac{m^k}{\ln m}\,,$$

where the last inequality is from the choice of m_{Dec}.

- $\mathcal{G}$ is $(\varepsilon(\boldsymbol{a}), d(\mathcal{G}|\hat{\mathcal{P}}))$-regular w.r.t. $\hat{\mathcal{P}}$ since by assumption of Lemma 3.8 the hypergraph $\mathcal{G}$ satisfies property (ii) of Lemma 3.6.
- By definition of $p_{(\boldsymbol{\mathcal{F}},\hat{\mathcal{P}})}$ and equation (4) we get

$$\sum_{\boldsymbol{\mathcal{F}}\in\mathscr{C}^{>0}_{\hat{\mathcal{P}}}} p_{(\boldsymbol{\mathcal{F}},\hat{\mathcal{P}})} \le 1.$$

- Since $m > m_{\mathrm{Dec}}$ and ε is monotone in every coordinate, we have

$$\frac{k\ln m}{m} \le \frac{\varepsilon(L,\dots,L))^3}{5} \le \frac{\varepsilon(\boldsymbol{a})^3}{5}\,.$$

- From property (vi) of Lemma 3.6 (which holds by the assumption of Lemma 3.8) and the choice of function ε, we have for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}^{>0}_{\hat{\mathcal{P}}}$

$$p_{(\boldsymbol{\mathcal{F}},\hat{\mathcal{P}})} d(\mathcal{G}|\hat{\mathcal{P}}) = \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) > \frac{\mu}{5}\prod_{i=1}^{k-1}\left(\frac{1}{a_i}\right)^{|\Delta_i(F_0)|-\binom{k}{i}} > 3\varepsilon(\boldsymbol{a}).$$

Thus for each $\hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)}$ we can apply Lemma 4.1 to $\mathcal{G}\cap\mathcal{K}_k(\hat{\mathcal{P}})$ with $d = d(\mathcal{G}|\hat{\mathcal{P}})$, $\varepsilon = \varepsilon(\boldsymbol{a})$, and $u = |\mathscr{C}^{>0}_{\hat{\mathcal{P}}}|$, to get partition $\mathcal{G}\cap\mathcal{K}_k(\hat{\mathcal{P}}) = \mathcal{T}_{\hat{\mathcal{P}}} \cup \bigcup_{\boldsymbol{\mathcal{F}}\in\mathscr{C}^{>0}_{\hat{\mathcal{P}}}} \mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})}$ such that $\mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})}$ is $(3\varepsilon(\boldsymbol{a}), p_{(\boldsymbol{\mathcal{F}},\hat{\mathcal{P}})} d(\mathcal{G}|\hat{\mathcal{P}}))$-regular w.r.t. $\hat{\mathcal{P}}$. We define the promised $\mathscr{C}$-decomposition of $\mathcal{G}$ by setting

$$\mathcal{G}_{\boldsymbol{\mathcal{F}}} = \begin{cases} \bigcup\left\{\mathcal{G}_{(\hat{\mathcal{P}},\boldsymbol{\mathcal{F}})} \colon \hat{\mathcal{P}} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } \hat{\mathcal{P}} \subseteq \mathcal{F}^{(k-1)}\right\} & \text{if } \ \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) > 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, if $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) = 0$, then $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ is $(3\varepsilon(\boldsymbol{a}), 0, F_0)$-regular w.r.t. $\mathcal{F}^{(k-1)}$. Moreover, since for every $\boldsymbol{\mathcal{F}} \in \mathscr{C}$ with $\overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}) > 0$ we have that $p_{(\boldsymbol{\mathcal{F}},\hat{\mathcal{P}})} d(\mathcal{G}|\hat{\mathcal{P}}) = \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}})$ independent of $\hat{\mathcal{P}}$, the hypergraph $\mathcal{G}_{\boldsymbol{\mathcal{F}}}$ defined above is also $(3\varepsilon(\boldsymbol{a}), \overline{\varphi}^*_{\mathcal{G}}(\boldsymbol{\mathcal{F}}), F_0)$-regular w.r.t. $\mathcal{F}^{(k-1)}$, which concludes the proof. □

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Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA
E-mail address: `rodl@mathcs.emory.edu`

Humboldt Universität zu Berlin, Institut für Informatik, Unter den Linden 6, 10099 Berlin, Germany
E-mail address: `schacht@informatik.hu-berlin.de`

Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA
E-mail address: `msigger@mathcs.emory.edu`

College of Education, Ryukyu University, Nishihara, Okinawa, 903-0213, Japan
E-mail address: `hide@edu.u-ryukyu.ac.jp`