EXTREMAL HYPERGRAPH PROBLEMS AND THE REGULARITY METHOD

BRENDAN NAGLE, VOJTĚCH RÖDL, AND MATHIAS SCHACHT

Dedicated to Professor Jaroslav Nešetřil on the occasion of his 60th birthday

Abstract. Szemerédi's regularity lemma asserts that every graph can be decomposed into relatively few random-like subgraphs. This random-like behavior enables one to find and enumerate subgraphs of a given isomorphism type, yielding the so-called counting lemma for graphs. The combined application of these two lemmas is known as the *regularity method for graphs* and has proved useful in graph theory, combinatorial geometry, combinatorial number theory and theoretical computer science.

Recently, the graph regularity method was extended to hypergraphs by Gowers and by Skokan and the authors. The *hypergraph regularity method* has been successfully employed in a handful of combinatorial applications, including alternative proofs to well-known density theorems of Szemerédi and of Furstenberg and Katznelson. In this paper, we apply the hypergraph regularity method to a few extremal hypergraph problems of Ramsey and Turán flavor.

1. Introduction

Szemerédi's regularity lemma asserts that every graph can be decomposed into a bounded number of so-called ε-regular pairs. For a graph $G = (V, E)$ and $\varepsilon > 0$, we say two non-empty disjoint subsets $X, Y \subset V$ are *ε-regular* if for all $X' \subseteq X$, $|X'| > \varepsilon|X|$ and $Y' \subseteq Y$, $|Y'| > \varepsilon|Y|$, we have $|d_G(X, Y) - d_G(X', Y')| < \varepsilon$, where $d_G(X', Y') = |G[X', Y']|/(|X'||Y'|)$ is the *density* of the bipartite subgraph $G[X', Y']$ of G (consisting of all edges $\{x, y\} \in E$ with $x \in X'$ and $y \in Y'$). Szemerédi's lemma is then given as follows.

Theorem 1.1 (Szemerédi's regularity lemma)**.** *For every $\varepsilon > 0$ and integer t_0, there exist integers $T_0 = T_0(\varepsilon, t_0)$ and $N_0 = N_0(\varepsilon, t_0)$ so that for every graph $G = (V, E)$, $|V| \geq N_0$, V admits a partition $V = V_1 \cup \cdots \cup V_t$, $t_0 \leq t \leq T_0$, satisfying*

(i) $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$ and
(ii) all but at most $\varepsilon\binom{t}{2}$ pairs (V_i, V_j), $1 \leq i < j \leq t$, are ε-regular.

Partitions $V = V_1 \cup \cdots \cup V_t$ satisfying (i) and (ii) as above are said to be *t-equitable* and *ε-regular*. Szemerédi's regularity lemma lead to many applications in combinatorial mathematics, particularly in the area of extremal graph theory

Date: February 1, 2006.
2000 *Mathematics Subject Classification.* Primary: 05C35. Secondary: 05C65, 05D10.
Key words and phrases. Turán's theorem, Ramsey theory, removal lemma, regularity lemma for hypergraphs.
The first author was partially supported by NSF grant DMS 0501090.
The second author was partially supported by NSF grant DMS 0300529.
The third author was supported by DFG grant SCHA 1263/1-1.

(see [19, 20] for surveys). Many applications of Szemerédi's lemma depend on the fact that within an appropriately given ε-regular partition $V = V_1 \cup \cdots \cup V_t$, one may enumerate small subgraphs of a fixed isomorphism type. This result is formally due to the following easily proved 'counting lemma' for graphs. In Fact 1.2 below and elsewhere in this paper, we write $x = y \pm \xi$ for reals x and y and some positive $\xi > 0$ for the inequalities $y - \xi < x < y + \xi$.

Fact 1.2 (Graph counting lemma)**.** *For all $d > 0$, $\gamma > 0$ and every positive integer ℓ, there exist $\varepsilon > 0$ and n_0 so that whenever G is an ℓ-partite graph with ℓ-partition $V_1 \cup \cdots \cup V_\ell$, and $|V_1| = \cdots = |V_\ell| = n \geq n_0$, satisfying for all $1 \leq i < j \leq \ell$*

- *(a) $d_G(V_i, V_j) = d \pm \varepsilon$ and*
- *(b) (V_i, V_j) is ε-regular,*

then the number $|\mathcal{K}_\ell(G)|$ of ℓ-cliques in G satisfies $|\mathcal{K}_\ell(G)| = d^{\binom{\ell}{2}} n^\ell (1 \pm \gamma)$.

We refer to a joint application of Theorem 1.1 and Fact 1.2 as the *regularity method* for graphs. Perhaps one of the first applications of this method is due to Ruzsa and Szemerédi [36] who showed it can be used to prove Roth's theorem [34, 35], i.e., Theorem 1.4 below for $d = 1$ and $\ell = 3$. More formally, Ruzsa and Szemerédi used the graph regularity method to prove that every graph G_n on n vertices having $o(n^3)$ triangles contains a triangle-free subgraph G_n' having only $o(n^2)$ fewer edges. Their result can be referred to as the 'triangle removal lemma' (cf. Theorem 1.3 below) and implies, as a corollary, Roth's theorem.

In what follows, a *hypergraph* $\mathcal{H} \subseteq 2^V$ with vertex set V is a collection of subsets from V. We say $\mathcal{H}^{(k)}$ is a *k-uniform hypergraph*, or *k-graph*, for short, if every subset belonging to $\mathcal{H}^{(k)}$ has cardinality k.

An extension of Szemerédi's regularity lemma for 3-graphs has been developed in [10]. More recently, extensions to k-graphs were obtained by Gowers [12, 13] and by Skokan and the current authors [22, 33], and based on that work, subsequently by Tao [41] and the second two authors [29]. Using these techniques, a handful of 3-graph applications appear in [3, 10, 14, 17, 18, 21, 27, 28, 37, 38] and some applications for k-graphs appear in [22, 30, 31, 32] (some of which we discuss momentarily). Tao also obtained some deep number-theoretic applications in [40].

The goal of this paper is to use the hypergraph regularity method established in [22, 33] to investigate some extremal hypergraph problems (see Section 2). The components of the hypergraph regularity method, i.e., the hypergraph regularity lemma of [33] and the hypergraph counting lemma of [22], are technical statements which we will only present later in Section 4 (cf. Remark 2.5). The following so-called *removal lemma*, however, is a direct consequence of the regularity method for hypergraphs.

Theorem 1.3 (Removal lemma, [12, 22, 32])**.** *For fixed k-graph $\mathcal{F}^{(k)}$ on f vertices, suppose $\mathcal{H}_n^{(k)}$ is a k-graph on n vertices containing $o(n^f)$ (not necessarily induced) copies of $\mathcal{F}^{(k)}$. Then, one may remove $o(n^k)$ edges from $\mathcal{H}_n^{(k)}$ to obtain a subhypergraph $\mathcal{G}^{(k)}$ which is $\mathcal{F}^{(k)}$-free, i.e., $\mathcal{G}^{(k)}$ contains no copy of $\mathcal{F}^{(k)}$ at all.*

When $\mathcal{F}^{(k)} = K_{k+1}^{(k)}$, the removal lemma generalizes Ruzsa and Szemerédi's triangle removal lemma (discussed earlier) to k-uniform hypergraphs. Frankl and the second author [10, 26] observed that the the removal lemma (with $\mathcal{F}^{(k)} = K_{k+1}^{(k)}$) implies Szemerédi's theorem (see Theorem 1.4 below with $d = 1$). Subsequently, Solymosi [38, 39] showed that Theorem 1.3 also implies the multidimensional version of Szemerédi's theorem, originally due to Furstenberg and Katznelson [11] (see also [31] for another consequence of Theorem 1.3 of similar flavor).

Theorem 1.4 (multidimensional Szemerédi theorem)**.** *For fixed integers ℓ and d, any set $Z \subseteq \{1, \dots, n\}^d$ containing no homothetic copy of $\{1, \dots, \ell\}^d$ has size $|Z| = o(n^d)$.*

In Theorem 1.4, $\{1, \dots, n\}^d$ denotes, as usual, the d-fold cross product of the set $\{1, \dots, n\}$ with itself. A homothetic copy of $\{1, \dots, \ell\}^d$ is any set of the form $\boldsymbol{a} + c\{1, \dots, \ell\}^d$, where $\boldsymbol{a} \in \{1, \dots, n\}^d$ and c is some positive integer.

2. Results

In this paper, we consider some extremal hypergraph problems of Turán and Ramsey flavor. We begin with some problems of Turán-type.

2.1. A Turán-type problem.

Generalizing Turán's problem for hypergraphs, the following problem was initiated by Brown, Erdős and T. Sós [6]. Let $f^{(r)}(n, v, e)$ denote the maximum number of edges in an r-graph on n vertices in which no v vertices span e (or more) edges. Note that the determination of $f^{(r)}(n, v, \binom{v}{r}) = \mathrm{ex}(n, K_v^{(r)})$ is precisely Turán's problem, on which we shall expand in Section 2.2. It was first proved by Brown, Erdős and T. Sós [6] that $f^{(r)}(n, e(r-k)+k, e) = \Theta(n^k)$. The same authors asked what happens if, instead of on $v = e(r-k)+k$ vertices, one forbids e edges to appear on $v+1 = e(r-k)+k+1$ vertices. In particular, they conjectured $f^{(r)}(n, e(r-k)+k+1, e)$ can be bounded by $o(n^k)$. This conjecture was proved for $e = r = 3$ and $k = 2$ by Ruzsa and Szemerédi [36] and generalized to arbitrary r with $k = 2$ and $e = 3$ by Erdős, Frankl and Rödl [7] and $r > k$ and $e = 3$ by Alon and Shapira [2]. Theorem 1.3 easily implies the upper bound for $r > k \geq 2$ and $e = k+1$. We present the details in Section 3.

Theorem 2.1. *For $r > k \geq 2$, $f^{(r)}(n, (k+1)(r-k+1), k+1) = o(n^k)$.*

Theorem 2.1 was proved for $k = 2$ by Erdős, Frankl and Rödl [7], and for $k = 3$ by Sárközy and Selkow [37].

Remark 2.2. In this paper, the integer notation k is usually reserved for the uniformity of hypergraphs $\mathcal{H}^{(k)}$, while our notation $f^{(r)}(n, v, e)$ appears to break with that tradition (since here, r denotes uniformity). In Theorem 2.1, however, the essential part of proving the assertion $f^{(r)}(n, (k+1)(r-k+1), k+1) = o(n^k)$, in fact, involves appealing to specific auxiliary *k-uniform* hypergraphs $\mathcal{H}^{(k)}$, where the initial uniformity r plays less of a rôle. In this sense, we reserve consistent use of uniformity notation k for later, in the proof, where we feel it is most important.

2.2. Forbidden families.

For an integer k, let $\mathbf{F}^{(k)} = \{\mathcal{F}_i^{(k)}\}_{i \in I}$ be a given (possibly infinite) family of k-graphs. Let $\mathrm{Forb}(n, \mathbf{F}^{(k)})$ denote the family of all k-graphs $\mathcal{H}_n^{(k)}$ on vertex set $\{1, \dots, n\}$ containing no sub-hypergraph isomorphic to $\mathcal{F}_i^{(k)}$ for all $i \in I$. As in the classical Turán problem, set

$$\mathrm{ex}(n, \mathbf{F}^{(k)}) = \max\left\{ \left|\mathcal{H}_n^{(k)}\right| : \mathcal{H}_n^{(k)} \in \mathrm{Forb}(n, \mathbf{F}^{(k)}) \right\}.$$

When $\mathbf{F}^{(k)} = \{K_\ell^{(k)}\}$ consists of the single clique $K_\ell^{(k)}$, determining $\mathrm{ex}(n, K_\ell^{(k)}) = \mathrm{ex}(n, \{K_\ell^{(k)}\})$ is the well-known Turán problem, where even the asymptotic for the case $\ell = 4$ and $k = 3$ remains open today. For $k = 2$, Turán's formula for these numbers is a central result in extremal graph theory. Note that the parameter $\mathrm{ex}(n, K_\ell^{(k)})$ corresponds to $f^{(k)}(n, \ell, \binom{\ell}{k})$ from Section 2.1. In the context of Turán's problem, however, the 'ex' notation appears more commonly than the 'f' notation, and so we shall not break from this tradition here.

Our result in Theorem 2.3 below aims to relate $\mathrm{ex}(n, \mathbf{F}^{(k)})$ with the cardinality $|\mathrm{Forb}(n, \mathbf{F}^{(k)})|$. Observe that since all sub-hypergraphs of a fixed $\mathcal{H}_n^{(k)} \in \mathrm{Forb}(n, \mathbf{F}^{(k)})$ also belong to $\mathrm{Forb}(n, \mathbf{F}^{(k)})$, we have $|\mathrm{Forb}(n, \mathbf{F}^{(k)})| \geq 2^{\mathrm{ex}(n, \mathbf{F}^{(k)})}$. We show that this bound is, in a sense, best possible.

Theorem 2.3. *For every (possibly infinite) family of k-graphs $\mathbf{F}^{(k)} = \{\mathcal{F}_i^{(k)}\}_{i \in I}$, we have*

$$\log_2 |\mathrm{Forb}(n, \mathbf{F}^{(k)})| = \mathrm{ex}(n, \mathbf{F}^{(k)}) + o(n^k)\,.$$

Theorem 2.3 was proved for $k = 2$ and $\mathbf{F}^{(2)} = \{K_\ell^{(2)}\}$ by Erdős, Kleitman and Rothschild [9] and for general $\mathbf{F}^{(2)}$ by Erdős, Frankl and Rödl [7]. Theorem 2.3 was proved for $k = 3$ by the first two authors [21]. Bollobás and Thomason [4] showed that $\lim_{n\to\infty} \log_2 |\mathrm{Forb}(n, \mathbf{F}^{(k)})|/\binom{n}{k}$ exists for any family $\mathbf{F}^{(k)}$ and so Theorem 2.3 provides a combinatorial evaluation of this limit.

We mention that for $k = 2$, an induced version of Theorem 2.3 was established by Prömel and Steger [25] and by Bollobás and Thomason [5]. These results were extended to $k = 3$ by Kohayakawa and the first two authors in [18]. Using the hypergraph regularity method, one may prove an induced version of Theorem 2.3 for general $k \geq 2$, and we hope to address this problem in a forthcoming paper.

2.3. **An induced Ramsey theorem.** For a fixed k-graph $\mathcal{F}^{(k)}$, a k-graph $\mathcal{G}^{(k)}$ is said to be an *induced Ramsey k-graph* for $\mathcal{F}^{(k)}$ if every 2-coloring of $\mathcal{G}^{(k)}$ admits a monochromatic sub-hypergraph isomorphic to $\mathcal{F}^{(k)}$ which appears as an *induced* sub-hypergraph of $\mathcal{G}^{(k)}$. Nešetřil and Rödl [23, 24] and independently Abramson and Harrington [1] proved that every k-graph $\mathcal{F}^{(k)}$ has a Ramsey k-graph $\mathcal{G}^{(k)}$ for $\mathcal{F}^{(k)}$. In this paper, we present another proof of the induced Ramsey theorem (based on the hypergraph regularity method).

Theorem 2.4. *For every integer $k \geq 2$ and every k-graph $\mathcal{F}^{(k)}$, there exists an induced Ramsey k-graph $\mathcal{G}^{(k)}$ for $\mathcal{F}^{(k)}$.*

2.4. **Organization of paper.** In Section 3, we prove Theorem 2.1 using the removal lemma, Theorem 1.3. While Theorem 2.1 is a consequence of the removal lemma, we prove Theorem 2.3 and Theorem 2.4 using the hypergraph regularity method. In Section 4, we present the hypergraph regularity lemma and hypergraph counting lemma. In Section 5, we prove Theorem 2.3. In Section 6, we prove Theorem 2.4.

We conclude the introduction with the following remark.

Remark 2.5. The components of the hypergraph regularity method, the hypergraph regularity lemma and hypergraph counting lemma, take different forms in the versions [12, 13] and [22, 33] and subsequent versions [29] and [41]. While any of these versions would suffice to prove the applications in this paper, we find the recent version of this method due to the second two authors [29] (based on ideas from [22, 33]) most convenient for our purposes. We present these tools in Section 4.

3. Proof of Theorem 2.1

We use Theorem 1.3, the removal lemma, to prove Theorem 2.1. In particular, we use the following corollary of the removal lemma to prove Theorem 2.1.

Corollary 3.1. *For fixed integer $k \geq 2$, let k-graph $\mathcal{H}_n^{(k)}$ on n vertices have the property that each k-tuple $K \in \mathcal{H}_n^{(k)}$ belongs to precisely one copy of the clique $K_{k+1}^{(k)}$. Then, $|\mathcal{H}_n^{(k)}| = o(n^k)$.*

Proof. Corollary 3.1 follows easily from Theorem 1.3 in the case when $\mathcal{F}^{(k)}$ consists of the single k-clique $K_{k+1}^{(k)}$ on $k+1$ vertices.

Let $\mathcal{H}_n^{(k)}$ be given as in the hypothesis of Corollary 3.1. Since each k-tuple $K \in \mathcal{H}_n^{(k)}$ belongs to precisely one copy of $K_{k+1}^{(k)}$, we see that the number of such cliques, $|\mathcal{K}_{k+1}(\mathcal{H}_n^{(k)})|$, satisfies

$$\left|\mathcal{K}_{k+1}(\mathcal{H}_n^{(k)})\right| = \frac{1}{k+1}|\mathcal{H}_n^{(k)}| = o(n^{k+1}). \tag{1}$$

Putting $\mathcal{F}^{(k)} = K_{k+1}^{(k)}$, Theorem 1.3 then asserts that one may delete $o(n^k)$ many k-tuples $K \in \mathcal{H}_n^{(k)}$ to obtain a $K_{k+1}^{(k)}$-free sub-hypergraph $\widetilde{\mathcal{H}}_n^{(k)} \subseteq \mathcal{H}_n^{(k)}$. However, since deleting a k-tuple $K \in \mathcal{H}_n^{(k)}$ destroys exactly one clique $K_{k+1}^{(k)}$, we must have $|\mathcal{K}_{k+1}(\mathcal{H}_n^{(k)})| = o(n^k)$ and Corollary 3.1 follows from (1). □

Proof of Theorem 2.1. Our proof follows the lines of [7, 36, 37], where the earlier established removal lemmas for graphs and 3-graphs were used to prove the special cases $k = 2, 3$. Let $r > k \geq 2$ be given as in Theorem 2.1. Suppose, on the contrary, that there exists $c = c(r,k) > 0$ and positive integer $n_0 = n_0(r,k,c)$ for which

$$f^{(r)}(n, (k+1)(r-k+1), k+1) > cn^k \tag{2}$$

holds for all $n > n_0$. Let $\mathcal{G}^{(r)}$ be an r-graph on $n > n_0(r,k,c)$ vertices with cn^k many r-tuples with the property that no $(k+1)(r-k+1)$ vertices span $(k+1)$ many r-tuples. We shall demonstrate that the existence of such $\mathcal{G}^{(r)}$ contradicts Corollary 3.1.

We begin by reducing the r-graph $\mathcal{G}^{(r)}$ to an r-partite sub-hypergraph $\widetilde{\mathcal{G}}^{(r)}$. A simple averaging argument (see, e.g., [8]) implies that the vertex set $V(\mathcal{G}^{(r)})$ admits an r-partition $V(\mathcal{G}^{(r)}) = V_1 \cup \cdots \cup V_r$ for which

$$\left|\mathcal{G}^{(r)}\left[V_1, \ldots, V_r\right]\right| \geq \frac{r!}{r^r}|\mathcal{G}^{(r)}| > \frac{r!}{r^r}cn^k, \tag{3}$$

where $\mathcal{G}^{(r)}[V_1, \ldots, V_r]$ is the sub-hypergraph of $\mathcal{G}^{(r)}$ consisting of all r-tuples $R \in \mathcal{G}^{(r)}$ with $|R \cap V_i| = 1$ for all $1 \leq i \leq r$. For simplicity, set $\widetilde{\mathcal{G}}^{(r)} = \mathcal{G}^{(r)}[V_1, \ldots, V_r]$.

We now reduce the r-graph $\widetilde{\mathcal{G}}^{(r)}$ to $(k+1)$-graph $\widetilde{\mathcal{G}}^{(k+1)}$ with vertex set $V_1 \cup \cdots \cup V_{k+1}$ as follows: for a $(k+1)$-tuple K^+ satisfying $|K^+ \cap V_i| = 1$, $1 \leq i \leq k+1$, put $K^+ \in \widetilde{\mathcal{G}}^{(k+1)}$ if, and only if, $K^+ \subseteq R$ for some $R \in \widetilde{\mathcal{G}}^{(r)}$. We make the following claim.

Claim 3.2. $|\widetilde{\mathcal{G}}^{(k+1)}| \geq \frac{|\widetilde{\mathcal{G}}^{(r)}|}{k} \overset{(3)}{>} \frac{cr!}{kr^r}n^k$.

Proof. The second inequality immediately follows from (3). To establish the first, we observe that for each $K^+ \in \widetilde{\mathcal{G}}^{(k+1)}$, there are at most k many r-tuples $R \in \widetilde{\mathcal{G}}^{(r)}$ for which $K^+ \subseteq R$ (from which Claim 3.2 then follows). Otherwise, if for some $K^+ \in \widetilde{\mathcal{G}}^{(k+1)}$, there exist $(k+1)$ distinct r-tuples $R_1, \dots, R_{k+1} \in \widetilde{\mathcal{G}}^{(r)}$ each containing K^+, we would have $(k+1)$ many r-tuples spanned on

$$\Big|\bigcup_{i=1}^{k+1} R_i\Big| \le (r-k-1)(k+1) + k + 1 = (r-k)(k+1) < (r-k+1)(k+1)$$

vertices, contradicting our choice of $\mathcal{G}^{(r)}$. □

We proceed with the following claim.

Claim 3.3. *Let $K_0 \subset K_0^+ \in \widetilde{\mathcal{G}}^{(k+1)}$ with $|K_0| = k$. There are at most $k-1$ distinct $(k+1)$-tuples $K_1^+, \dots, K_{k-1}^+ \in \widetilde{\mathcal{G}}^{(k+1)}$ for which $K_0^+ \cap K_i^+ = K_0$, $1 \le i \le k$.*

Proof. Suppose, on the contrary, that some fixed $K_0 \subset K_0^+ \in \widetilde{\mathcal{G}}^{(k+1)}$, $|K_0| = k$, admits k distinct $(k+1)$-tuples $K_1^+, \dots, K_k^+ \in \widetilde{\mathcal{G}}^{(k+1)}$ for which $K_0^+ \cap K_i^+ = K_0$, $1 \le i \le k$. Then, for some $R_0, R_1, \dots, R_k \in \widetilde{\mathcal{G}}^{(r)}$, we would have $(k+1)$-many distinct r-tuples $R_0 \supset K_0^+, R_1 \supset K_1^+, \dots, R_k \supset K_k^+$ spanned on

$$\Big|\bigcup_{i=0}^{k+1} R_i\Big| \le k+k+1+(k+1)(r-k-1) = (k+1)(r-k+1)-1 < (k+1)(r-k+1)$$

vertices, contradicting our choice of $\mathcal{G}^{(r)}$. □

Claim 3.3 immediately implies that for each $K_0^+ \in \widetilde{\mathcal{G}}^{(k+1)}$, at most $(k+1)(k-1) = k^2-1$ distinct $(k+1)$-tuples $K_1^+, \dots, K_{k^2-1}^+ \in \widetilde{\mathcal{G}}^{(k+1)}$ satisfy $|K_0^+ \cap K_i^+| = k$, $1 \le i \le k^2-1$. As such, the $(k+1)$-graph $\widetilde{\mathcal{G}}^{(k+1)}$ contains a sub-hypergraph $\widetilde{\mathcal{G}}_0^{(k+1)}$ of size

$$\Big|\widetilde{\mathcal{G}}_0^{(k+1)}\Big| \ge \frac{|\widetilde{\mathcal{G}}^{(k+1)}|}{k^2-1} \overset{\text{Claim 3.2}}{\ge} \frac{cr!}{k(k^2-1)r^r} n^k \tag{4}$$

consisting of $(k+1)$-tuples $K_0^+ \in \widetilde{\mathcal{G}}^{(k+1)}$, no two of which overlap in k vertices. Indeed, iteratively construct $(k+1)$-graph $\widetilde{\mathcal{G}}_0^{(k+1)}$ by starting with an arbitrary $(k+1)$-tuple $K_0^+ \in \widetilde{\mathcal{G}}^{(k+1)}$, deleting all $(k+1)$-tuples K^+ which overlap with K_0^+ in k vertices, and repeating this procedure until $\widetilde{\mathcal{G}}_0^{(k+1)}$ is produced.

We are now able to conclude the proof of Theorem 2.1. Define $(k+1)$-partite k-graph $\mathcal{H}^{(k)}$ on vertex set $V_1 \cup \dots \cup V_{k+1}$ as follows: for a k-tuple K_0 satisfying $|K_0 \cap V_i| \le 1$, $1 \le i \le k+1$, put $K_0 \in \mathcal{H}^{(k)}$ if, and only if, $K_0 \subset K_0^+$ for some $(k+1)$-tuple $K_0^+ \in \widetilde{\mathcal{G}}_0^{(k+1)}$. We make the following observations.

- $(O1)$ each copy of the clique $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$ corresponds to an edge of $\widetilde{\mathcal{G}}_0^{(k+1)}$, and vice-versa;
- $(O2)$ by construction of $\mathcal{H}^{(k)}$, each edge $K \in \mathcal{H}^{(k)}$ belongs to at least one copy of the clique $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$;
- $(O3)$ by construction of $\widetilde{\mathcal{G}}_0^{(k+1)}$, each edge $K \in \mathcal{H}^{(k)}$ belongs to at most one copy of the clique $K_{k+1}^{(k)}$ in $\mathcal{H}^{(k)}$;
- $(O4)$

$$|\mathcal{H}^{(k)}| = \binom{k+1}{k}\Big|\widetilde{\mathcal{G}}_0^{(k+1)}\Big| \overset{(4)}{\ge} \frac{cr!(k+1)}{k(k^2-1)r^r} n^k = \Omega(n^k).$$

Combining observations $(O2)$, $(O3)$ and $(O4)$, we see that $\mathcal{H}^{(k)}$ is a 'dense' k-graph whose every edge $K \in \mathcal{H}^{(k)}$ belongs to precisely one copy of the clique $K^{(k)}_{k+1}$. This contradicts Corollary 3.1 and hence concludes the proof of Theorem 2.1. □

4. Regularity method for hypergraphs

In this section, we present the hypergraph regularity lemma the hypergraph counting lemma from [29]. We first present all needed definitions and notation in Section 4.1. In Section 4.2, we state both lemmas.

4.1. **Definitions.** We start with some basic concepts and notation.

4.1.1. *Basic concepts.* For integers $\ell \geq j \geq 1$, the notation $[\ell]$ denotes the set of integers $\{1, \dots, \ell\}$ and $[\ell]^j = \binom{[\ell]}{j}$ denotes the set of all unordered j-tuples from $[\ell]$.

In this paper ℓ-partite, j-uniform hypergraphs play a special rôle, where $j \leq \ell$. Given vertex sets $V_1, \dots, V_\ell$, we denote by $K^{(j)}_\ell(V_1, \dots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph (i.e., the family of all j-element subsets $J \subseteq \bigcup_{i \in [\ell]} V_i$ satisfying $|V_i \cap J| \leq 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an (m, ℓ, j)-*cylinder* $\mathcal{H}^{(j)}$ on $V_1 \cup \dots \cup V_\ell$ is any subset of $K^{(j)}_\ell(V_1, \dots, V_\ell)$. The vertex partition $V_1 \cup \dots \cup V_\ell$ is an $(m, \ell, 1)$-cylinder $\mathcal{H}^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $j \leq i \leq \ell$ and set $\Lambda_i \in [\ell]^i$, we denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda \in \Lambda_i} V_\lambda\big]$ the sub-hypergraph of the (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda \in \Lambda_i} V_\lambda$.

For an (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$ and an integer $2 \leq j \leq i \leq \ell$, we denote by $\mathcal{K}_i(\mathcal{H}^{(j)})$ the family of all i-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$ of order i. For $1 \leq i \leq \ell$, we denote by $\mathcal{K}_i(\mathcal{H}^{(1)})$ the family of all i-element subsets of $V(\mathcal{H}^{(1)})$ which 'cross' the partition $V_1 \cup \dots \cup V_\ell$, i.e., $I \in \mathcal{K}_i(\mathcal{H}^{(1)})$ if, and only if, $|I \cap V_s| \leq 1$ for all $1 \leq s \leq \ell$. For $2 \leq j \leq i \leq \ell$, $|\mathcal{K}_i(\mathcal{H}^{(j)})|$ is the number of all copies of $K^{(j)}_i$ in $\mathcal{H}^{(j)}$. Given an $(m, \ell, j-1)$-cylinder $\mathcal{H}^{(j-1)}$ and an (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$, we say $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$. This brings us to one of the main concepts of this paper, the notion of a *complex*.

Definition 4.1 (**(m, ℓ, h)-complex**). *Let $m \geq 1$ and $\ell \geq h \geq 1$ be integers. An* (m, ℓ, h)-complex $\boldsymbol{\mathcal{H}}$ *is a collection of (m, ℓ, j)-cylinders $\{\mathcal{H}^{(j)}\}_{j=1}^h$ such that*

(*a*) *$\mathcal{H}^{(1)}$ is an $(m, \ell, 1)$-cylinder, i.e., $\mathcal{H}^{(1)} = V_1 \cup \dots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$, and*

(*b*) *$\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$ for $2 \leq j \leq h$, i.e., $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.*

We sometimes shorten the terminology (m, ℓ, h)-complex to (ℓ, h)-*complex*, when the cardinality $m = |V_1| = \dots = |V_s|$ isn't of primary concern.

4.1.2. *Relative density and hypergraph regularity.* We begin by defining a relative density of a j-uniform hypergraph w.r.t. $(j-1)$-uniform hypergraph on the same vertex set.

Definition 4.2 (**relative density**). *Let $\mathcal{H}^{(j)}$ be a j-uniform hypergraph and let $\mathcal{H}^{(j-1)}$ be a $(j-1)$-uniform hypergraph on the same vertex set. We define the* density of $\mathcal{H}^{(j)}$ w.r.t. $\mathcal{H}^{(j-1)}$ *as*

$$d\big(\mathcal{H}^{(j)} \big| \mathcal{H}^{(j-1)}\big) = \begin{cases} \frac{\left|\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})\right|}{\left|\mathcal{K}_j(\mathcal{H}^{(j-1)})\right|} & \text{if} \quad \left|\mathcal{K}_j(\mathcal{H}^{(j-1)})\right| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We also define a notion of regularity for (m, j, j)-cylinders w.r.t. some underlying $(m, j, j-1)$-cylinders.

Definition 4.3 (**(ε, d)-regular**). *Let reals $\varepsilon > 0$ and $d \geq 0$ be given along with an (m, j, j)-cylinder $\mathcal{H}^{(j)}$ and underlying $(m, j, j-1)$-cylinder $\mathcal{H}^{(j-1)}$. We say $\mathcal{H}^{(j)}$ is* (ε, d)-regular *w.r.t. $\mathcal{H}^{(j-1)}$ if whenever $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ satisfies*

$$\big|\mathcal{K}_j(\mathcal{Q}^{(j-1)})\big| \geq \varepsilon \big|\mathcal{K}_j(\mathcal{H}^{(j-1)})\big|, \quad \textit{then} \quad d\big(\mathcal{H}^{(j)} \big| \mathcal{Q}^{(j-1)}\big) = d \pm \varepsilon\,.$$

Before continuing, we pause for the following remark.

Remark 4.4. We compare the notion of regularity in Definition 4.3 for $j = 2$ with the traditional definition of an ε-regular pair (given in the beginning of the Introduction). The $(m, 2, 2)$-cylinder $\mathcal{H}^{(2)}$ is, in the traditional terminology, a bipartite graph. The underlying $(m, 2, 1)$-cylinder $\mathcal{H}^{(1)}$ is the bipartition of $\mathcal{H}^{(2)}$, written here as $\mathcal{H}^{(1)} = V_1 \cup V_2$ where $|V_1| = |V_2| = m$. The sub-cylinder $\mathcal{Q}^{(1)} \subseteq V_1 \cup V_2$ is a subset of vertices, which we could write as $\mathcal{Q}^{(1)} = V_1' \cup V_2'$, where $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$. The assumption of Definition 4.3 saying $|\mathcal{K}_2(\mathcal{Q}^{(1)})| \geq \varepsilon |\mathcal{K}_2(\mathcal{H}^{(1)})|$ is identical to saying $|V_1'||V_2'| \geq \varepsilon |V_1||V_2|$. As such, the definition ensures $d(\mathcal{H}^{(2)}|\mathcal{Q}^{(1)}) = d \pm \varepsilon$, or equivalently, $|d(\mathcal{H}^{(2)}|\mathcal{Q}^{(1)}) - d| < \varepsilon$. The quantity $d(\mathcal{H}^{(2)}|\mathcal{Q}^{(1)})$ is the same as $d_{\mathcal{H}^{(2)}}(V_1', V_2')$. The constant d is not necessarily the density $d(\mathcal{H}^{(2)}|\mathcal{H}^{(1)})$, but it is, of course, close to it.

There is only one real difference, therefore, between the notion of graph regularity given in Definition 4.3 when $j = 2$ and the traditional definition of an ε-regular pair. In the traditional definition, we would assume that the subsets $V_1' \subseteq V_1$, $V_2' \subseteq V_2$ individually satisfy the conditions $|V_1'| \geq \varepsilon |V_1|$ and $|V_2'| \geq \varepsilon |V_2|$. In Definition 4.3, we assume the product $|V_1'||V_2'|$ satisfies the single condition $|V_1'||V_2'| \geq \varepsilon |V_1||V_2|$. Quite obviously, however, these two notions are equivalent: *if $\mathcal{H}^{(2)}$ is (ε, d)-regular w.r.t. $\mathcal{H}^{(1)}$, then $\mathcal{H}^{(1)}$ is an ε-regular pair, and if $\mathcal{H}^{(1)}$ is an ε-regular pair, then $\mathcal{H}^{(2)}$ is $(\varepsilon^2, d(\mathcal{H}^{(2)}, \mathcal{H}^{(1)}))$-regular w.r.t. $\mathcal{H}^{(1)}$.*

We now extend the notion of (ε, d)-regularity to (m, ℓ, j)-cylinders $\mathcal{H}^{(j)}$.

Definition 4.5 (**(ε, d)-regular cylinder**). *We say an (m, ℓ, j)-cylinder $\mathcal{H}^{(j)}$ is* (ε, d)-regular *w.r.t. an $(m, \ell, j-1)$-cylinder $\mathcal{H}^{(j-1)}$ if for every $\Lambda_j \in [\ell]^j$, the restriction $\mathcal{H}^{(j)}[\Lambda_j] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda \in \Lambda_j} V_\lambda\big]$ is (ε, d)-regular w.r.t. the restriction $\mathcal{H}^{(j-1)}[\Lambda_j] = \mathcal{H}^{(j-1)}\big[\bigcup_{\lambda \in \Lambda_j} V_\lambda\big]$.*

We now extend the notion of (ε, d)-regularity from cylinders to complexes.

Definition 4.6 (**$(\varepsilon, \boldsymbol{d})$-regular complex**). *Let ε be a positive real and let $\boldsymbol{d} = (d_2, \dots, d_h)$ be a vector of non-negative reals. We say an (m, ℓ, h)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is* $(\varepsilon, \boldsymbol{d})$-regular *if $\mathcal{H}^{(j)}$ is (ε, d_j)-regular w.r.t. $\mathcal{H}^{(j-1)}$ for every $j = 2, \dots, h$.*

4.1.3. *Partitions.* The regularity lemma for k-uniform hypergraphs provides a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ of vertices, pairs, …, and $(k-1)$-tuples of some vertex set. We now discuss the structure of these partitions recursively, following the approach of [33].

Let k be a fixed integer and V be a set of vertices. Let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of V. For every $1 \leq j \leq |\mathscr{P}^{(1)}|$, let $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ be the family of all crossing j-tuples J, i.e., the set of j-tuples which satisfy $|J \cap V_i| \leq 1$ for every $V_i \in \mathscr{P}^{(1)}$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{Cross}_i(\mathscr{P}^{(1)})$ for $1 \le i \le j-1$ have been defined. Then for every $(j-1)$-tuple I in $\mathrm{Cross}_{j-1}(\mathscr{P}^{(1)})$, there exist a unique $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. For every j-tuple J in $\mathrm{Cross}_j(\mathscr{P}^{(1)})$, we define the *polyad of* J

$$\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup \left\{ \mathcal{P}^{(j-1)}(I) \colon \, I \in [J]^{j-1} \right\} .$$

In other words, $\hat{\mathcal{P}}^{(j-1)}(J)$ is the unique set of j partition classes of $\mathscr{P}^{(j-1)}$ each containing a $(j-1)$-subset of J. Observe that $\hat{\mathcal{P}}^{(j-1)}(J)$ can be viewed as a $(j, j-1)$-cylinder, i.e., a j-partite, $(j-1)$-uniform hypergraph. More generally, for $1 \le i < j$, we set

$$\hat{\mathcal{P}}^{(i)}(J) = \bigcup \left\{ \mathcal{P}^{(i)}(I) \colon \, I \in [J]^{i} \right\} \qquad \text{and} \qquad \boldsymbol{\mathcal{P}}(J) = \left\{ \hat{\mathcal{P}}^{(i)}(J) \right\}_{i=1}^{j-1} . \tag{5}$$

Remark 4.7. In this paper, we use $\mathscr{P}^{(j)}$, read "script P", to denote the partition of j-tuples. Partition classes $\mathcal{P}^{(j)} \in \mathscr{P}^{(j)}$ (which are j-uniform hypergraphs on $[n]$) are denoted with "calligraphic P". Unions of special sub-collections of j-graphs $\mathcal{P}^{(j)}$ (which we call polyads) are denoted with "calligraphic P" equipped with a "hat".

Next, we define $\hat{\mathscr{P}}^{(j-1)}$, the family of all polyads

$$\hat{\mathscr{P}}^{(j-1)} = \left\{ \hat{\mathcal{P}}^{(j-1)}(J) \colon \, J \in \mathrm{Cross}_j(\mathscr{P}^{(1)}) \right\} .$$

Note that $\hat{\mathcal{P}}^{(j-1)}(J)$ and $\hat{\mathcal{P}}^{(j-1)}(J')$ are not necessarily distinct for different j-tuples J and J'. We view $\hat{\mathscr{P}}^{(j-1)}$ as a set and, consequently, $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \, \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$.

The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ is

$$\mathscr{P}^{(j)} \prec \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \, \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\} , \tag{6}$$

where '$\prec$' denotes the refinement relation of set partitions. In other words, we require that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$. Note that (6) implies (inductively) that $\boldsymbol{\mathcal{P}}(J)$ defined in (5) is a $(j, j-1)$-complex. On a related note, we shall often drop the argument $J \in \mathrm{Cross}_j(\mathscr{P}^{(1)})$ from the notation $\hat{\mathcal{P}}^{(j-1)}(J)$ (as the families $\mathscr{P}$ with which we work always satisfy $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \neq \emptyset$).

Throughout this paper, we want to control the number of partition classes in $\mathscr{P}^{(j)}$, and more specifically, over the number of classes contained in $\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ for a fixed polyad $\hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}$. We make this precise in the following definition.

Definition 4.8 (**family of partitions**). *Suppose V is a set of vertices, $k \ge 2$ is an integer and $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$ is a* family of partitions on V*, if it satisfies the following:*

(i) $\mathscr{P}^{(1)}$ is a partition of V into a_1 classes,
(ii) $\mathscr{P}^{(j)}$ is a partition of $\mathrm{Cross}_j(\mathscr{P}^{(1)})$ satisfying:

$$\mathscr{P}^{(j)} \quad \textit{refines} \quad \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \, \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$$

$$\textit{and} \quad \left| \left\{ \mathcal{P}^{(j)} \in \mathscr{P}^{(j)} \colon \, \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \right\} \right| = a_j \quad \textit{for every} \quad \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)} .$$

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is t-bounded, if $\max\{a_1, \dots, a_{k-1}\} \le t$.

It is easy to see that for a t-bounded family of partitions $\mathscr{P}$ and an integer $2 \le j \le k-1$, we have

$$|\hat{\mathscr{P}}^{(j-1)}| = \binom{a_1}{j} \prod_{h=2}^{j-1} a_h^{\binom{j}{h}} \le t^{2^t}. \tag{7}$$

We continue with a few final definitions needed to state the hypergraph regularity lemma and corresponding counting lemma.

4.1.4. *Regular partitions.* The following definition describes some of the structure the regularity lemma shall provide.

Definition 4.9 (**$(\eta, \varepsilon, \boldsymbol{a})$-equitable**). *Suppose V is a set of n vertices, η and ε are positive reals, $\boldsymbol{a} = (a_1, \dots, a_{k-1})$ is a vector of positive integers where a_1 divides n.*

We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ on V (as defined in Definition 4.8) is $(\eta, \varepsilon, \boldsymbol{a})$-equitable *if it satisfies the following:*

- (*a*) $\big|[V]^k \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})\big| \le \eta\binom{n}{k}$,
- (*b*) $\mathscr{P}^{(1)} = \{V_i \colon i \in [a_1]\}$ *is an equitable vertex partition, i.e.,* $|V_i| = |V|/a_1$ *for* $i \in [a_1]$, *and*
- (*c*) *for every* $K \in \mathrm{Cross}_k(\mathscr{P}^{(1)})$ *the* $(n/a_1, k, k-1)$*-complex* $\boldsymbol{\mathcal{P}}(K)$ *(see (5)) is* $(\varepsilon, (1/a_2, \dots, 1/a_{k-1}))$*-regular.*

To describe the remaining structure of the regularity lemma, we extend Definition 4.5.

Definition 4.10 (**(δ_k, d_k, r)-regular**). *Let δ_k and d_k be positive reals and r be a positive integer. Suppose $\mathcal{H}^{(k-1)}$ is a $(k-1)$-graph and $\mathcal{H}^{(k)}$ is a k-graph, both of which share the same vertex set. We say $\mathcal{H}^{(k)}$ is* (δ_k, d_k, r)-regular *w.r.t. $\mathcal{H}^{(k-1)}$ if for every collection $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\}$ of not necessarily disjoint sub-hypergraphs of $\mathcal{H}^{(k-1)}$ satisfying*

$$\left|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| > \delta_k \left|\mathcal{K}_k(\mathcal{H}^{(k-1)})\right| ,$$

we have

$$\frac{\big|\mathcal{H}^{(k)} \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\big|}{\big|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\big|} = d_k \pm \delta_k \,.$$

We write $(\delta_k, *, r)$-regular *to mean* $\big(\delta_k, d\big(\mathcal{H}^{(k)}\big|\mathcal{H}^{(k-1)}\big), r\big)$*-regular.*

We need one last definition to state the regularity lemma.

Definition 4.11 (**(δ_k, r)-regular w.r.t. $\mathscr{P}$**). *Suppose δ_k is a positive real and r is a positive integer. Let $\mathcal{H}^{(k)}$ be a k-uniform hypergraph with vertex set V and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on V. We say $\mathcal{H}^{(k)}$* is (δ_k, r)-regular w.r.t. $\mathscr{P}$, *if*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } \mathcal{H}^{(k)} \text{ is not } (\delta_k, *, r)\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}\Big\}\Big| \le \delta_k \binom{|V|}{k}.$$

4.2. **Hypergraph regularity lemma and counting lemma.** The regularity lemma of [29] is given as follows.

Theorem 4.12 (Regularity lemma)**.** *Let $k \geq 2$ be a fixed integer. For all positive constants η and δ_k and functions $r\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0,1]$ there are integers $t_{\mathrm{Thm.4.12}}$ and $n_{\mathrm{Thm.4.12}}$ so that the following holds.*

For every k-uniform hypergraph $\mathcal{H}^{(k)}$ satisfying $|V(\mathcal{H}^{(k)})| = n \geq n_{\mathrm{Thm.4.12}}$ and $t_{\mathrm{Thm.4.12}}!$ dividing n, there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that

(i) $\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and $t_{\mathrm{Thm.4.12}}$-bounded;
(ii) $\mathcal{H}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$.

The following hypergraph counting lemma corresponds to Theorem 4.12.

Theorem 4.13 (Counting lemma)**.** *For all integers $\ell \geq k \geq 2$ and positive constants $\gamma > 0$ and $d_k > 0$, there exists $\delta_k > 0$ such that for all integers $a_{k-1}, \dots, a_2$, there are a constant $\delta > 0$ and positive integers r and m_0 so that the following holds. Suppose*

(i) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta, (1/a_2, \dots, 1/a_{k-1}))$-regular $(m, \ell, k-1)$-complex with $m \geq m_0$, and
(ii) for every $\Lambda_k \in [\ell]^k$, the k-graph $\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $(\delta_k, d_{\Lambda_k}, r)$-regular w.r.t. $\mathcal{R}^{(k-1)}[\Lambda_k]$ for some $d_{\Lambda_k} \geq d_k$.

Then

$$\left|\mathcal{K}_\ell(\mathcal{H}^{(k)})\right| \geq (1-\gamma) d_k^{\binom{\ell}{k}} \prod_{j=2}^{k-1} \left(\frac{1}{a_j}\right)^{\binom{\ell}{j}} \times m^\ell .$$

5. Proof of Theorem 2.3

The main idea in proving Theorem 2.3 is not difficult, but since it involves appealing to the regularity lemma and counting lemma for hypergraphs, its appearance is technical. We therefore begin this section by sketching this main idea in the (more transparent) case of graphs, following the work of [7]. In the following outline, we restrict our attention to the special case when $\mathbf{F}^{(2)} = \{K_3^{(2)}\}$ consists of the (single) triangle $K_3 = K_3^{(2)}$. We mention that, if we focus our attention to when $\mathbf{F}^{(2)}$ consists of a single graph, our choice here of K_3 makes little difference in the argument. However, restricting our attention to when $\mathbf{F}^{(2)}$ consists of only *finitely* many graphs frees us from one detail which is similarly technical for graphs as it is for hypergraphs.

5.1. **The graph case with $\mathbf{F}^{(2)} = K_3$.** Fix $\nu > 0$. We sketch the proof that

$$\left|\mathrm{Forb}(n, K_3)\right| \leq 2^{\mathrm{ex}(n,K_3) + \nu n^3}$$

holds for all large integers n. The main components of the proof are the Szemerédi regularity lemma, Theorem 1.1, and the counting lemma (for graphs), Fact 1.2.

We begin by 'regularizing' every graph $G = \mathcal{G}^{(2)}$ in the collection $\mathrm{Forb}(n, K_3)$. To that end, we pick some 'small' $0 < \varepsilon = \varepsilon(\nu) \ll \nu$ (we won't determine a formula for ε at this time since we plan to bypass, in this outline, the calculations using this formula) and 'large' integer $t_0 = t_0(\nu) \gg 1/\nu$. As we make these choices, we also pick an auxiliary constant $\varepsilon \ll d_0 \ll \nu$ which is 'small' w.r.t. ν but 'large' w.r.t. ε. Theorem 1.1 guarantees an integer $T_0 = T_0(\varepsilon, t_0)$ so that, with n large, every graph $G \in \mathrm{Forb}(n, K_3)$ admits an ε-regular, t_G-equitable partition $V(G) = V_1^G \cup \cdots \cup V_{t_G}^G$ where $t_0 \le t_G \le T_0$. For $G \in \mathrm{Forb}(n, K_3)$, we shall write $\mathcal{P}_G$ for the ε-regular, t_G-equitable partition $V(G) = V_1^G \cup \cdots \cup V_{t_G}^G$, $t_0 \le t_G \le T_0$, obtained above. We fix, for each $G \in \mathrm{Forb}(n, K_3)$, the partition $\mathcal{P}_G$ now obtained (and if G admits multiple such, we simply pick one, arbitrarily). In all that follows, $n = n(\nu, d_0, \varepsilon, t_0, T_0)$ is sufficiently large w.r.t. all the constants mentioned above.

We first decompose $\mathrm{Forb}(n, K_3)$ into equivalence classes. We say two graphs G_1 and $G_2 \in \mathrm{Forb}(n, K_3)$ are equivalent if, and only if, $\mathcal{P}_{G_1} = \mathcal{P}_{G_2}$. (In other words, the ε-regular partitions $\mathcal{P}_{G_1}$ and $\mathcal{P}_{G_2}$ fixed above split the vertices $\{1, \ldots, n\}$ in precisely the same way.) Let $\mathrm{Forb}(n, K_3) = \Pi_1 \cup \cdots \cup \Pi_N$ be the partition of $\mathrm{Forb}(n, K_3)$ associated with this equivalence relation. Then $|\mathrm{Forb}(n, K_3)| = \sum_{a=1}^{N} |\Pi_a|$, and clearly, there are at most $N \le T_0^n = 2^{o(n^2)}$ partitions of the vertices $\{1, \ldots, n\}$. Thus, it suffices to estimate $|\Pi_a|$ for an arbitrary index $1 \le a \le N$.

For the remainder of this outline, fix $1 \le a \le N$. There is a common partition $\mathcal{P}_a$ of $\{1, \ldots, n\}$ that every graph $G \in \Pi_a$ admits as its fixed ε-regular partition $\mathcal{P}_G$. We write $\mathcal{P}_a$ as $V_1 \cup \cdots \cup V_t$, where $t_0 \le t \le T_0$. Now, for a fixed $G \in \Pi_a$, we shall record for which pairs (V_i, V_j) of the partition $\mathcal{P}_a$ the graph G is 'dense' and 'regular'. More formally, for $1 \le i < j \le t$, write $\boldsymbol{x}_G = (x_{ij}^G : 1 \le i < j \le t)$, where

$$x_{ij}^G = \begin{cases} 1 & \text{if} \quad d_G(V_i, V_j) \ge d_0 \text{ and } V_i, V_j \text{ is } \varepsilon\text{-regular w.r.t. } G, \\ 0 & \text{otherwise.} \end{cases}$$

For fixed $\boldsymbol{x} \in \{0,1\}^{\binom{t}{2}}$, we set $\Pi_a(\boldsymbol{x}) = \{G \in \Pi_a : \boldsymbol{x}_G = \boldsymbol{x}\}$ and observe

$$|\Pi_a| = \sum \left\{ |\Pi_a(\boldsymbol{x})| \colon \ \boldsymbol{x} \in \{0,1\}^{\binom{t}{2}} \right\}.$$

Since there are only $2^{\binom{t}{2}} \le 2^{T_0^2} = 2^{O(1)} = 2^{o(n^2)}$ vectors $\boldsymbol{x} \in \{0,1\}^{\binom{t}{2}}$, it suffices to estimate $|\Pi_a(\boldsymbol{x})|$ for a fixed but arbitrary $\boldsymbol{x} \in \{0,1\}^{\binom{t}{2}}$.

With $\boldsymbol{x}$ fixed, and a fixed before, we now define $D_a(\boldsymbol{x})$ as the graph with vertex set $\{1, \ldots, t\}$ and edges $\{i, j\}$, $1 \le i < j \le t$, corresponding to when the pair (V_i, V_j) is 'dense' and 'regular' w.r.t. *every* graph $G \in \Pi_a(\boldsymbol{x})$, i.e., when $x_{ij} = 1$. If we can show

$$|D_a(\boldsymbol{x})| \le \mathrm{ex}(t, K_3) \tag{8}$$

then it will be easy to show

$$|\Pi_a(\boldsymbol{x})| \le 2^{\mathrm{ex}(n, K_3) + \frac{\nu}{2} n^2}. \tag{9}$$

Establishing the implication (8) $\implies$ (9) is standard, and so we only highlight it here. Indeed, using standard considerations of ε-regular partitions, one may easily show that for any $G \in \Pi_a(\boldsymbol{x})$

$$\begin{aligned} \Big| \big\{ \{v_i, v_j\} \in E(G) \colon \ v_i \in V_i, \, v_j \in V_j, \text{ either } i = j \text{ or } x_{ij} = 0 \big\} \Big| \\ < \left(\frac{1}{t_0} + \varepsilon + d_0 \right) n^2 \ll \frac{\nu}{2} n^2 \end{aligned} \tag{10}$$

where the last 'inequality' holds by virtue of the fact that we chose $1/t_0$, ε, and d_0 much smaller than ν. Hence there are essentially $2^{\frac{\nu}{2} n^2}$ choices for the subgraphs of graphs $G \in \Pi_a(\boldsymbol{x})$ induced on vertex classes V_i $(i = 1, \ldots, t)$ and on pairs (V_i, V_j)

with $x_{ij} = 0$. The number of subgraphs on pairs (V_i, V_j) with $x_{ij} = 1$ is (ignoring precise error calculations) approximately

$$2^{\sum_{\{i,j\}\in D_a(\boldsymbol{x})} |V_i||V_j|} \sim 2^{\sum_{\{i,j\}\in D_a(\boldsymbol{x})} \frac{n^2}{t^2}} \overset{(8)}{\sim} 2^{\mathrm{ex}(n,K_3)} \tag{11}$$

where the last asymptotic employs (8) and makes use of the fact that $\mathrm{ex}(t, K_3)/\binom{t}{2} \sim \mathrm{ex}(n, K_3)/\binom{n}{2}$ whenever t and n are large (recall $t \geq t_0$, where we picked t_0 'large'). Since every graph $G \in \Pi_a(\boldsymbol{x})$ behaves 'identically' on the common partition $\mathcal{P}_a$, every graph $G \in \Pi_a(\boldsymbol{x})$ must consist of one of the (essentially) $2^{\frac{\nu}{2}n^2}$ many subgraphs counted in (10), and one of the (essentially) $2^{\mathrm{ex}(n,K_3)}$ subgraphs counted in (11). This completes the sketch of (8) $\implies$ (9).

We finish the present outline by proving (8), and to that end, we use the counting lemma, Fact 1.2. Indeed, if $|D_a(\boldsymbol{x})| > \mathrm{ex}(t, K_3)$, then $D_a(\boldsymbol{x})$ contains a copy of the triangle K_3. Let i, j, k denote the vertices of this triangle (which correspond to the vertex classes V_i, V_j, V_k of the partition $\mathcal{P}_a$) and fix any graph $G_0 \in \Pi_a(\boldsymbol{x})$. By definition of $D_a(\boldsymbol{x})$, each of the pairs $\{V_i, V_j\}$, $\{V_j, V_k\}$ and $\{V_i, V_k\}$ are ε-regular w.r.t. G_0 and also satisfy

$$d_{G_0}(V_i, V_j),\ d_{G_0}(V_j, V_k),\ d_{G_0}(V_i, V_k) \geq d_0.$$

By the counting lemma, Fact 1.2, the graph G_0 contains at least $\sim d_0^3(n/t)^3 > 0$ many triangles K_3, which contradicts that $G_0 \in \mathrm{Forb}(n, K_3)$. This completes the outline.

Before proceeding to the actual proof of Theorem 2.3, we make the following remark.

Remark 5.1. As we mentioned before, one has to work a little harder, whether for graphs or hypergraphs, when the set $\mathbf{F}^{(k)}$ consists of infinitely many elements rather than finitely many. These details were not addressed in our outline, but are addressed in our proof of Theorem 2.3. As well, in our proof of Theorem 2.3, we shall define a k-graph $\mathcal{D}_\alpha(\boldsymbol{x})$ in (26) which is an analogue to the graph $D_a(\boldsymbol{x})$ (cf. (8)). For reasons we do not mention here, we define $\mathcal{D}_\alpha(\boldsymbol{x})$ in a slightly different way than we defined $D_a(\boldsymbol{x})$. In the end, however, the invocation of the counting lemma will be precisely the same as in the outline above.

5.2. **Setting up the proof of Theorem 2.3.** In our proof of Theorem 2.3, we use the following notation. For an integer n and a family of k-graphs $\mathbf{F}^{(k)}$, set

$$\widetilde{\mathrm{ex}}(n, \mathbf{F}^{(k)}) = \frac{\mathrm{ex}(n, \mathbf{F}^{(k)})}{\binom{n}{k}}\,.$$

It is well known (see [16]) that the sequence $(\widetilde{\mathrm{ex}}(n, \mathbf{F}^{(k)}))_{n=1}^{\infty}$ is non-increasing, and hence,

$$\pi(\mathbf{F}^{(k)}) = \lim_{n\to\infty} \widetilde{\mathrm{ex}}(n, \mathbf{F}^{(k)}) \tag{12}$$

exists. Note that when $\pi(\mathbf{F}^{(k)}) = 0$ the assertion of Theorem 2.3 is trivial. Indeed,

$$|\mathrm{Forb}(n, \mathbf{F}^{(k)})| \leq \sum_{s=0}^{o(n^k)} \binom{\binom{n}{k}}{s} = 2^{o(n^k)}\,.$$

Henceforth, we shall assume $\pi(\mathbf{F}^{(k)}) > 0$.

It suffices to prove Theorem 2.3 for n divisible by a fixed but arbitrary integer T. In particular, suppose that, for fixed $\nu > 0$ and fixed integer T, for every integer $m > m_0(k,\nu,T)$, we have

$$|\mathrm{Forb}(mT, \mathbf{F}^{(k)})| \le 2^{\mathrm{ex}(mT,\mathbf{F}^{(k)})+\nu(mT)^k}.$$

Then it easily follows that for all integers $n > n_0(k,\nu,T)$,

$$|\mathrm{Forb}(n, \mathbf{F}^{(k)})| \le 2^{\mathrm{ex}(n,\mathbf{F}^{(k)})+2\nu n^k}.$$

Indeed, for an integer n, write $(m-1)T \le n < mT$ for some integer m. Then, with m and n sufficiently large, we have

$$\begin{aligned}
\log_2|\mathrm{Forb}(n,\mathbf{F}^{(k)})| &\le \log_2|\mathrm{Forb}(mT,\mathbf{F}^{(k)})| \le \mathrm{ex}(mT,\mathbf{F}^{(k)}) + \nu(mT)^k \\
&= \widetilde{\mathrm{ex}}(mT,\mathbf{F}^{(k)})\binom{mT}{k} + \nu(mT)^k \le \pi(\mathbf{F}^{(k)})\binom{mT}{k} + \nu(mT)^k + o((mT)^k) \\
&\le \pi(\mathbf{F}^{(k)})\binom{n+T}{k} + \nu(n+T)^k + o((n+T)^k) = \pi(\mathbf{F}^{(k)})\binom{n}{k} + \nu n^k + o(n^k) \\
&\le \widetilde{\mathrm{ex}}(n,\mathbf{F}^{(k)})\binom{n}{k} + \nu n^k + o(n^k) \le \widetilde{\mathrm{ex}}(n,\mathbf{F}^{(k)})\binom{n}{k} + 2\nu n^k \\
&= \mathrm{ex}(n,\mathbf{F}^{(k)}) + 2\nu n^k,
\end{aligned}$$

where the next to last inequality follows from the sequence $(\widetilde{\mathrm{ex}}(s,\mathbf{F}^{(k)}))_{s=1}^{\infty})$ being non-increasing with limit $\pi(\mathbf{F}^{(k)})$.

We now prove that for every $\nu > 0$, there exist integers $T = T(\nu)$ and $n_0 = n_0(\nu, T)$ so that for every $n \ge n_0$ divisible by T,

$$\log_2|\mathrm{Forb}(n,\mathbf{F}^{(k)})| \le \mathrm{ex}(n,\mathbf{F}^{(k)}) + \nu\binom{n}{k}. \tag{13}$$

As our proof depends on Theorems 4.12 and 4.13, we first discuss a sequence of auxiliary constants.

5.3. **Constants.** Let $\nu > 0$ be given. Let $f_0 \in \mathbb{N}$ be sufficiently large so that

$$\widetilde{\mathrm{ex}}(f_0,\mathbf{F}^{(k)}) < \pi(\mathbf{F}^{(k)}) + \frac{\nu}{8}\,. \tag{14}$$

Choose $0 < \eta = d_0 < 1/9$ so that

$$(1-\eta)^{1/(k-1)} \ge 1 - \frac{1}{f_0} \qquad \text{and} \qquad 4d_0\log_2\frac{\mathrm{e}}{3d_0} \le \frac{\nu}{4} \tag{15}$$

(note that the last inequality uses $x\log_2 x \to 0$ as $x \to 0^+$). For fixed integers f_0 and k and constants $\gamma = 1/2$ and $d_k = d_0$, let

$$\delta_k = \delta_k^{(4.13)}(f_0, k, 1/2, d_0) \tag{16}$$

be the constant guaranteed by Theorem 4.13. We may assume, without loss of generality, that

$$\delta_k \le d_0\,. \tag{17}$$

For positive integer variables $y_{k-1},\dots,y_2$, let

$$\delta(y_{k-1},\dots,y_2) = \delta^{(4.13)}(f_0,k,1/2,d_0,\delta_k,y_{k-1},\dots,y_2) \tag{18}$$

$$r(y_{k-1},\dots,y_2) = r^{(4.13)}(f_0,k,1/2,d_0,\delta_k,y_{k-1},\dots,y_2) \tag{19}$$

be the functions guaranteed by Theorem 4.13.

We now define further constants in terms of the regularity lemma, Theorem 4.12. With input parameters η and δ_k and functions [1] $\delta(y_{k-1},\dots,y_2)$ and $r(y_{k-1},\dots,y_2)$ defined above, Theorem 4.12 guarantees integer constants

$$t = t^{(4.12)}(\eta,\delta_k,\delta,r) \quad \text{and} \quad n_0 = n^{(4.12)}(\eta,\delta_k,\delta,r). \tag{20}$$

The constant T mentioned in (13) is set to be

$$T = t!\,.$$

Now, for $n > n_0$ divisible by T and sufficiently large, we verify (13).

5.4. **Proof of (13).** According to Theorem 4.12, every k-graph $\mathcal{G}^{(k)}$ on n vertices (n defined above) admits an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable t-bounded family of partitions $\mathscr{P}$ with respect to which $\mathcal{G}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular. As such, for each $\mathcal{G}^{(k)} \in \mathrm{Forb}(n, \mathbf{F}^{(k)})$, we may associate a family of partitions $\mathscr{P}_{\mathcal{G}^{(k)}}$ (if $\mathcal{G}^{(k)}$ admits multiple such partitions, we simply choose one of them). Accordingly, we may impose an equivalence relation $\sim$ on $\mathrm{Forb}(n, \mathbf{F}^{(k)})$ according to the following rule: for $\mathcal{G}^{(k)}, \widetilde{\mathcal{G}}^{(k)} \in \mathrm{Forb}(n, \mathbf{F}^{(k)})$,

$$\mathcal{G}^{(k)} \sim \widetilde{\mathcal{G}}^{(k)} \quad \Longleftrightarrow \quad \mathscr{P}_{\mathcal{G}^{(k)}} = \mathscr{P}_{\widetilde{\mathcal{G}}^{(k)}}\,. \tag{21}$$

Let $\mathrm{Forb}(n, \mathbf{F}^{(k)}) = \Pi_1 \cup \dots \cup \Pi_N$ be the partition of $\mathrm{Forb}(n, \mathbf{F}^{(k)})$ induced by $\sim$. To prove (13), we first seek to bound the parameter $N = N(n)$.

Clearly, N is at most the number of t-bounded families of partitions on the vertex set $[n]$. For a fixed vector $\boldsymbol{a} = (a_1,\dots,a_{k-1})$, there are at most $\prod_{j=1}^{k-1} a_j^{\binom{n}{j}}$ families of partitions $\mathscr{P}(k-1,\boldsymbol{a})$ on the vertex set $[n]$. Consequently,

$$N \le \sum_{\boldsymbol{a}} \Big\{ \prod_{j=1}^{k-1} a_j^{\binom{n}{j}} \colon\ 1 \le a_j \le t \text{ for } j = 1,\dots,k-1 \Big\} \le t^{k-1} \times t^{\sum_{j=1}^{k-1}\binom{n}{j}} = 2^{O(n^{k-1})}\,. \tag{22}$$

We now seek to bound $|\Pi_\alpha|$ for every $\alpha = 1,\dots,N$. Fix $1 \le \alpha \le N$ and, correspondingly, family of partitions $\mathscr{P}_\alpha = \{\mathscr{P}_\alpha^{(1)},\dots,\mathscr{P}_\alpha^{(k-1)}\}$, i.e., the family associated to every $\mathcal{G}^{(k)} \in \Pi_\alpha$. With each $\mathcal{G}^{(k)} \in \Pi_\alpha$, we associate the vector

$$\boldsymbol{x}_{\mathcal{G}^{(k)}} = \Big(x_{\hat{\mathcal{P}}^{(k-1)}} \colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)} \Big) \in \{0,1\}^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|}, \tag{23}$$

where, for fixed $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)}$,

$$x_{\hat{\mathcal{P}}^{(k-1)}} = \begin{cases} 1 & \text{if } d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \ge d_0 \text{ and} \\ & \qquad \mathcal{G}^{(k)} \text{ is } (\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}_\alpha}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

From (7) and the t-boundedness of the family $\mathscr{P}_\alpha$,

$$|\{\boldsymbol{x}_{\mathcal{G}^{(k)}} \colon\ \mathcal{G}^{(k)} \in \Pi_\alpha\}| \le 2^{t^{2^k}} = O(1)\,. \tag{25}$$

---

[1] Note that the input functions functions $\delta(y_{k-1},\dots,y_2)$ and $r(y_{k-1},\dots,y_2)$ have $k-2$ variables while Theorem 4.12 would allow us to consider $k-1$ variables. In particular, Theorem 4.12 would allow us to include a variable y_1 corresponding to the number of vertex classes the output family of partitions $\mathscr{P}$ will have. We have no need for this feature in our argument here, so we hold the variable y_1 constant.

With $\alpha \in [N]$ fixed, fix vector $\boldsymbol{x} \in \{0,1\}^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|}$ and define

$$\Pi_\alpha(\boldsymbol{x}) = \left\{\mathcal{G}^{(k)} \in \Pi_\alpha \colon \boldsymbol{x}_{\mathcal{G}^{(k)}} = \boldsymbol{x}\right\}.$$

We prove the following lemma.

Lemma 5.2. $\log_2 |\Pi_\alpha(\boldsymbol{x})| \le \mathrm{ex}(n, \mathbf{F}^{(k)}) + \frac{\nu}{2}\binom{n}{k}$.

Lemma 5.2, combined with (22) and (25), easily implies (13) (and hence, Theorem 2.3). Indeed

$$\begin{aligned}|\mathrm{Forb}(n, \mathbf{F}^{(k)})| = \sum_{\alpha=1}^{N} |\Pi_\alpha| &= \sum_{\alpha=1}^{N} \sum_{\boldsymbol{x}} |\Pi_\alpha(\boldsymbol{x})| \\ &\le 2^{O(n^{k-1})} \times O(1) \times 2^{\mathrm{ex}(n,\mathbf{F}^{(k)}) + \frac{\nu}{2}\binom{n}{k}} \le 2^{\mathrm{ex}(n,\mathbf{F}^{(k)}) + \nu\binom{n}{k}}\end{aligned}$$

where the last inequality holds for sufficiently large n.

We now proceed to prove Lemma 5.2.

5.5. **Proof of Lemma 5.2.** Fix $\alpha \in [N]$ and, correspondingly, $\mathscr{P}_\alpha = \mathscr{P}_\alpha(k-1, \boldsymbol{a}^{\mathscr{P}_\alpha})$ with $\boldsymbol{a}^{\mathscr{P}_\alpha} = (a_1, \dots, a_{k-1})$ and fix $\boldsymbol{x} = (x_{\hat{\mathcal{P}}^{(k-1)}} \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}_\alpha^{(k-1)})$. Define $\mathcal{D}_\alpha(\boldsymbol{x})$ to be the set of k-tuples $K \in \mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})$ for which each $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$ is 'dense and regular' w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$:

$$\mathcal{D}_\alpha(\boldsymbol{x}) = \bigcup\left\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon x_{\hat{\mathcal{P}}^{(k-1)}} = 1 \text{ (cf. (24))}\right\}. \tag{26}$$

We make the following claim.

Claim 5.3. $|\mathcal{D}_\alpha(\boldsymbol{x})| \le (\widetilde{\mathrm{ex}}(n, \mathbf{F}^{(k)}) + \frac{\nu}{4})\binom{n}{k}$.

Our proof of Claim 5.3 is based on the counting lemma, Theorem 4.13. On the other hand, Lemma 5.2 is a simple consequence of Claim 5.3. As such, we go ahead and assume Claim 5.3, for the moment, and finish the proof of Lemma 5.2, before we verify Claim 5.3.

Finishing the proof of Lemma 5.2, note that every edge $K \in \binom{[n]}{k} \setminus \mathcal{D}_\alpha(\boldsymbol{x})$ satisfies that either

(I) K is non-crossing in $\mathscr{P}^{(1)}$,
(II) or $x_{\hat{\mathcal{P}}^{(k-1)}(K)} = 0$, i.e., by (24), polyad $\hat{\mathcal{P}}^{(k-1)}(K)$ is either 'sparse' or 'irregular' (for every $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$).

However, since every $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}_\alpha}), \boldsymbol{a}^{\mathscr{P}_\alpha})$-equitable family $\mathscr{P}_\alpha$, the number of edges $K \in \binom{[n]}{k}$ satisfying (I) or (II) is at most

$$(\eta + \delta_k + d_0)\binom{n}{k} \overset{(15),(17)}{\le} 3d_0\binom{n}{k}.$$

(Indeed, the equitability of family $\mathscr{P}_\alpha$ ensures that there are at most $\eta\binom{n}{k}$ non-crossing edges. The fact that every $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. family $\mathscr{P}_\alpha$ ensures that at most $\delta_k\binom{n}{k}$ many k-tuples belong to irregular polyads. Finally, sparse polyads (with density smaller than d_0), in total, can only give rise to at most $d_0\binom{n}{k}$ many k-tuples.)

Now, every $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$ can be written as a disjoint union $\mathcal{G}^{(k)} = \mathcal{G}_1^{(k)} \cup \mathcal{G}_2^{(k)}$ where $\mathcal{G}_1^{(k)} \subseteq \mathcal{D}_\alpha(\boldsymbol{x})$ and $|\mathcal{G}_2^{(k)}| \leq 3d_0\binom{n}{k}$. As such,

$$|\Pi_\alpha(\boldsymbol{x})| \leq 2^{|\mathcal{D}_\alpha(\boldsymbol{x})|} \times \sum_{j=0}^{3d_0\binom{n}{k}} \binom{\binom{n}{k}}{j} \overset{\text{Claim 5.3}}{\leq} 2^{(\widetilde{\mathrm{ex}}(n,\mathbf{F}^{(k)})+\frac{\nu}{4})\binom{n}{k}} \times n^k \left(\frac{\mathrm{e}}{3d_0}\right)^{3d_0\binom{n}{k}},$$

which implies (with n large)

$$\log_2 \big|\Pi_\alpha(\boldsymbol{x})\big| \leq \left(\widetilde{\mathrm{ex}}(n,\mathbf{F}^{(k)}) + \frac{\nu}{4} + 4d_0 \log\frac{\mathrm{e}}{3d_0}\right)\binom{n}{k} \overset{(15)}{\leq} \mathrm{ex}(n,\mathbf{F}^{(k)}) + \frac{\nu}{2}\binom{n}{k},$$

as promised by Lemma 5.2.

It now only remains to prove Claim 5.3.

5.6. **Proof of Claim 5.3.** Let $\alpha \in [N]$ and $\boldsymbol{x} \in \{0,1\}^{|\hat{\mathscr{P}}_\alpha^{(k-1)}|}$ be fixed. For crossing set $A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})$, define auxiliary k-graph

$$\mathrm{Dense}(A) = \left\{K \in \tbinom{A}{k} :\ x_{\hat{\mathcal{P}}^{(k-1)}(K)} = 1 \text{ (cf. (24))}\right\}.$$

Double-counting pairs (A,K) where $K \in \mathrm{Dense}(A)$ and $A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})$ yields

$$\big|\mathcal{D}_\alpha(\boldsymbol{x})\big| \left(\frac{n}{a_1}\right)^{a_1-k} = \sum_{A\in\mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})} |\,\mathrm{Dense}(A)|. \tag{27}$$

As such, we may infer Claim 5.3 from the the following assertion:

$$\max\{|\,\mathrm{Dense}(A)| :\ A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})\} < \left(\widetilde{\mathrm{ex}}(a_1,\mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{a_1}{k}. \tag{28}$$

Indeed, since $|\mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})| = (n/a_1)^{a_1}$, we combine (27) and (28) to say

$$\big|\mathcal{D}_\alpha(\boldsymbol{x})\big| < \left(\widetilde{\mathrm{ex}}(a_1,\mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{a_1}{k}\left(\frac{n}{a_1}\right)^k \leq \left(\widetilde{\mathrm{ex}}(a_1,\mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{n}{k}.$$

Since[2] $a_1 \geq f_0$ (where f_0 is given in (14)) and the sequence $(\widetilde{\mathrm{ex}}(s,\mathbf{F}^{(k)}))_{s=1}^\infty$ is non-increasing with limit $\pi(\mathbf{F}^{(k)})$ (see (12)), we have

$$\big|\mathcal{D}_\alpha(\boldsymbol{x})\big| < \left(\widetilde{\mathrm{ex}}(f_0,\mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{n}{k} \overset{(14)}{<} \left(\pi(\mathbf{F}^{(k)}) + \frac{\nu}{4}\right)\binom{n}{k} \leq \left(\widetilde{\mathrm{ex}}(n,\mathbf{F}^{(k)}) + \frac{\nu}{4}\right)\binom{n}{k}.$$

Thus, it remains to prove the assertion in (28).

Proof of (28). On the contrary, suppose there exists $A \in \mathrm{Cross}_{a_1}(\mathscr{P}_\alpha^{(1)})$ so that

$$|\,\mathrm{Dense}(A)| \geq \left(\widetilde{\mathrm{ex}}(a_1,\mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{a_1}{k}. \tag{29}$$

As such, we claim there must also exist $B \in \binom{A}{f_0}$ (see (14)) such that the sub-hypergraph $\mathrm{Dense}_B(A)$ of $\mathrm{Dense}(A)$ induced on B contains at least $\mathrm{ex}(f_0,\mathbf{F}^{(k)}) + 1$

[2]It is easy to see $a_1 \geq f_0$. Indeed, since $\mathscr{P}_\alpha$ is an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}_\alpha}), \boldsymbol{a}^{\mathscr{P}_\alpha})$-equitable family of partitions and since $|\mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})| = \binom{a_1}{k}(\frac{n}{a_1})^k$, we have

$$1 - \eta \leq |\mathrm{Cross}_k(\mathscr{P}_\alpha^{(1)})|\binom{n}{k}^{-1} \leq \left(1 - \frac{1}{a_1}\right)^{k-1}$$

where the last inequality holds with n sufficiently large. The assertion $a_1 \geq f_0$ then follows from our choice of η in (15), i.e., $1 - \eta \geq (1 - f_0^{-1})^{k-1}$.

edges. Indeed, supposing otherwise, the number M of pairs (K,B), $K \in \binom{B}{k}$, $B \in \binom{A}{f_0}$, would, on the one hand, satisfy

$$M \leq \binom{a_1}{f_0} \mathrm{ex}(f_0, \mathbf{F}^{(k)}) = \widetilde{\mathrm{ex}}(f_0, \mathbf{F}^{(k)}) \binom{f_0}{k}\binom{a_1}{f_0}. \tag{30}$$

On the other hand, by the choice of A in (29),

$$M \geq \left(\widetilde{\mathrm{ex}}(a_1, \mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{a_1}{k}\binom{a_1-k}{f_0-k}.$$

The monotonicity of the sequence $(\widetilde{\mathrm{ex}}(s, \mathbf{F}^{(k)}) : s \geq 1)$ then gives

$$\begin{aligned} M &\geq \left(\widetilde{\mathrm{ex}}(a_1, \mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{a_1}{k}\binom{a_1-k}{f_0-k} \\ &\geq \left(\pi(\mathbf{F}^{(k)}) + \frac{\nu}{8}\right)\binom{a_1}{k}\binom{a_1-k}{f_0-k} \overset{(14)}{>} \widetilde{\mathrm{ex}}(f_0, \mathbf{F}^{(k)})\binom{a_1}{k}\binom{a_1-k}{f_0-k}, \end{aligned}$$

contradicting (30).

Fix $B \in \binom{A}{f_0}$ for which the f_0-vertex sub-hypergraph $\mathrm{Dense}_B(A)$ of $\mathrm{Dense}(A)$ induced on B contains at least $\mathrm{ex}(f_0, \mathbf{F}^{(k)})+1$ edges. Then, there exists $\mathcal{F}^{(k)} \in \mathbf{F}^{(k)}$ so that its copy $\mathcal{F}_0^{(k)}$ appears as a sub-hypergraph of $\mathrm{Dense}_B(A)$. In order to derive a contradiction from our assumption in (29), we use the counting lemma, Theorem 4.13, to find a copy of the same $\mathcal{F}^{(k)} \in \mathbf{F}^{(k)}$ in any (and every) $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$. Since $\Pi_\alpha(\boldsymbol{x}) \subseteq \mathrm{Forb}(\mathbf{F}^{(k)})$, we have an immediate contradiction.

To that end, fix $\mathcal{G}^{(k)} \in \Pi_\alpha(\boldsymbol{x})$ and let $F = V(\mathcal{F}_0^{(k)}) \subseteq B$. For each $K \in \binom{F}{k}$, set

$$\mathcal{H}_K^{(k)} = \begin{cases} \mathcal{G}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) & \text{if} \quad K \in \mathcal{F}_0^{(k)}, \\ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) & \text{otherwise.} \end{cases}$$

Set

$$\mathcal{H}^{(k)} = \bigcup \left\{ \mathcal{H}_K^{(k)} \colon K \in \tbinom{F}{k} \right\}.$$

With $\mathcal{H}^{(k)}$ defined above, observe that every element of $\mathcal{K}_f(\mathcal{H}^{(k)})$, $f = |F|$ corresponds to a copy of $\mathcal{F}^{(k)}$ appearing as a sub-hypergraph of $\mathcal{G}^{(k)}$. If we show $|\mathcal{K}_f(\mathcal{H}^{(k)})| > 0$, then we derive a contradiction, and hence, (28) follows.

To show $|\mathcal{K}_f(\mathcal{H}^{(k)})| > 0$, we apply the counting lemma, Theorem 4.13, to $\mathcal{H}^{(k)}$ and $\boldsymbol{\mathcal{Q}} = \{\mathcal{Q}^{(j)}\}_{j=1}^{k-1}$ where $\mathcal{Q}^{(j)} = \bigcup \left\{ \mathcal{P}^{(j)}(J) \colon J \in \tbinom{F}{j} \right\}$ for $j = 1, \dots, k-1$. We first check that the assumptions of Theorem 4.13 are met by $\mathcal{H}^{(k)}$ and $\boldsymbol{\mathcal{Q}}$:

(1) Since $\mathscr{P}_\alpha$ is an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable family, the $(n/a_1, f, k-1)$-complex $\boldsymbol{\mathcal{Q}}$ is $(\delta(\boldsymbol{a}^{\mathscr{P}}), (1/a_2, \dots, 1/a_{k-1}))$-regular. Moreover, we chose the function δ in (18) appropriately for an application of Theorem 4.13;

(2) For each $K \in \mathcal{F}_0^{(k)} \subseteq \mathrm{Dense}_B(A) \subseteq \mathrm{Dense}(A)$, the definition of $\boldsymbol{x}$ in (23) guarantees that $\mathcal{H}_K^{(k)} = \mathcal{G}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K))$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K) \subseteq \mathcal{Q}^{(k-1)}$ and that $d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(K)) \geq d_0$. We note that δ_k and r were chosen in (16) and (19) appropriately for an application of Theorem 4.13;

(3) For each $K \in \binom{F}{k} \setminus \mathcal{F}_0^{(k)}$, the k-graph $\mathcal{H}_K^{(k)} = \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K))$ is easily seen to be $(\varepsilon, 1, s)$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ for every $\varepsilon > 0$ and $s \in \mathbb{N}$. As such, $\mathcal{H}_K$ is $(\delta_k, 1, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$.

Hence, we can apply the hypergraph counting lemma to $\mathcal{H}^{(k)}$ and $\boldsymbol{\mathcal{Q}}$. We conclude

$$\left|\mathcal{K}_f(\mathcal{H}^{(k)})\right| \geq \frac{1}{2} d_0^{\binom{f}{k}} \prod_{j=2}^{k-1} \left(\frac{1}{a_j}\right)^{\binom{f}{j}} \left(\frac{n}{a_1}\right)^f \geq \frac{1}{2} d_0^{\binom{f_0}{k}} \prod_{j=2}^{k-1} \left(\frac{1}{a_j}\right)^{\binom{f_0}{j}} \left(\frac{n}{a_1}\right)^{f_0} > 0.$$

This proves (28). □

6. Proof of Theorem 2.4

Theorem 2.4 is a simple consequence of the following lemma.

Lemma 6.1. *Let k-graph $\mathcal{F}^{(k)}$ on f vertices be given. For every $c > 0$, there exist $\varepsilon > 0$ and integers $\widetilde{r}$, T and n_0 so that a given k-graph $\mathcal{G}^{(k)}$ on vertex set $[n] = \{1, \dots, n\}$, with $n \geq n_0$ and n divisible by T, is an induced Ramsey k-graph for $\mathcal{F}^{(k)}$ whenever the following conditions are met:*

- *(i) $|\mathcal{K}_s(\mathcal{G}^{(k)})| \geq c\binom{n}{s}$ where $s = R^{(k)}(f, f)$ is the Ramsey number for $K_f^{(k)}$;*
- *(ii) $\mathcal{G}^{(k)}$ is $(\varepsilon, d(\mathcal{G}^{(k)}|\mathcal{P}^{(k-1)}), \widetilde{r})$-regular, $d(\mathcal{G}^{(k)}|\mathcal{P}^{(k-1)}) \in [\frac{1}{4}, \frac{3}{4}]$, w.r.t. every $(k-1)$-graph $\mathcal{P}^{(k-1)} \subseteq \binom{[n]}{k-1}$ which satisfies $|\mathcal{K}_k(\mathcal{P}^{(k-1)})| \geq n^k/\log n$.*

Lemma 6.1 implies Theorem 2.4. Indeed, it is easy to verify that, with probability tending to 1 as $n \to \infty$, i.e., *asymptotically almost surely (a.a.s.)*, the binomial random k-graph $\mathcal{G}^{(k)}(n, 1/2)$ satisfies the hypothesis of Lemma 6.1 with $c = (1/2)^{\binom{s}{k}-1}$ and with arbitrary choices of $\varepsilon > 0$ and integers $\widetilde{r}$ and T. In particular, Chebyshev's inequality verifies that $\mathcal{G}^{(k)}(n, 1/2)$ satisfies (*i*), a.a.s. For completeness, we verify in the Appendix (see Fact A.1) that $\mathcal{G}^{(k)}(n, 1/2)$ satisfies a.a.s. (*ii*).

The goal of this section is, therefore, to prove Lemma 6.1. As our proof depends on Theorems 4.12 and 4.13, we again first discuss a sequence of auxiliary constants.

6.1. Constants.

Let k-graph $\mathcal{F}^{(k)}$ on f vertices be given. Set, as in the hypothesis of Lemma 6.1,

$$s = R^{(k)}(f, f). \tag{31}$$

As in the hypothesis of Lemma 6.1, let $c > 0$ be given. We define $\varepsilon > 0$ and integers $\widetilde{r}$ and t in terms of Theorem 4.12 and 4.13.

As in Theorem 4.13, put $\ell = f$, $\gamma = 1/2$ and $d_k = 1/8$ and let

$$\delta_k^{(4.13)} = \delta_k^{(4.13)}(f, k, 1/2, d_k)$$

be the constant guaranteed by Theorem 4.13. Set

$$\eta = \delta_k = \min\left\{\frac{1}{2}\delta_k^{(4.13)}, \frac{c}{4}\binom{s}{k}^{-1}\right\} \tag{32}$$

For positive integer variables $y_{k-1}, \dots, y_2$, let

$$\delta(y_{k-1}, \dots, y_2) = \delta^{(4.13)}(f, k, 1/2, d_k, y_{k-1}, \dots, y_2), \tag{33}$$

$$r(y_{k-1}, \dots, y_2) = r^{(4.13)}(f, k, 1/2, d_k, y_{k-1}, \dots, y_2) \tag{34}$$

be the functions guaranteed by Theorem 4.13. Without loss of generality, we assume that $r(y_{k-1}, \dots, y_2)$ is monotone increasing in every coordinate.

We now define more auxiliary constants. In Theorem 4.12, let constants η and δ_k and functions r and δ be the parameters chosen in (32)–(34). Theorem 4.12 guarantees integer *constants*

$$t = t^{(4.12)}(\eta, \delta_k, r, \delta) \quad \text{and} \quad n_0 = n_0^{(4.12)}(\eta, \delta_k, r, \delta). \tag{35}$$

We set

$$\varepsilon = \delta_k, \quad T = t! \quad \text{and} \quad \widetilde{r} = r(t, \dots, t). \tag{36}$$

Let $n > n_0$ be divisible by T and be sufficiently large whenever needed. This concludes our discussion of the constants.

6.2. **Proof of Lemma 6.1.** With the constants above, let $\mathcal{G}^{(k)}$ be a k-graph on n vertices satisfying the hypothesis of Lemma 6.1. Let $\mathcal{G}^{(k)} = \mathcal{R}^{(k)} \cup \mathcal{B}^{(k)}$ be any two-coloring with colors 'red' and 'blue'. We prove that one of $\mathcal{R}^{(k)}$ or $\mathcal{B}^{(k)}$ contains a copy of $\mathcal{F}^{(k)}$ as a sub-hypergraph which is induced in $\mathcal{G}^{(k)}$.

With constants η, and δ_k and functions r and δ defined above, we apply Theorem 4.12 to the k-graph $\mathcal{R}^{(k)}$ to obtain $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t-bounded family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ with respect to which $\mathcal{R}^{(k)}$ is $(\delta_k, r(\boldsymbol{a}^{\mathscr{P}}))$-regular. Observe, that due to our choice of $\widetilde{r}$ in (36) and the monotonicity thereof,

$$r(\boldsymbol{a}^{\mathscr{P}}) \leq \widetilde{r}. \tag{37}$$

We now consider the polyads of $\mathscr{P}$.

Set[3]

$$\hat{\mathscr{P}}^{(k-1)}_{\text{bad}} = \left\{ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \colon\ |\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| < n^k / \log n \right\}.$$

Note that the t-boundedness of $\mathscr{P}$ gives for sufficiently large n

$$\left| \bigcup \left\{ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\text{bad}} \right\} \right| \leq \binom{a_1}{k} \prod_{j=2}^{k-1} a_j^{\binom{k}{j}} \times \frac{n^k}{\log n} \leq \frac{c}{4\binom{s}{k}} \binom{n}{k}. \tag{38}$$

Set

$$\begin{aligned} \hat{\mathscr{P}}^{(k-1)}_{\text{reg}} = \Big\{ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \setminus \hat{\mathscr{P}}^{(k-1)}_{\text{bad}} \colon & \\ \mathcal{R}^{(k)} \text{ is } (\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))\text{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)} \Big\}. & \end{aligned} \tag{39}$$

While $\hat{\mathscr{P}}^{(k-1)}_{\text{reg}}$ is defined in terms of the k-graph $\mathcal{R}^{(k)}$ only, the following fact observes that both $\mathcal{R}^{(k)}$ and $\mathcal{B}^{(k)}$ are 'regular' w.r.t. every polyad $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\text{reg}}$.

Fact 6.2.

$$\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\text{reg}} \quad \Longrightarrow \quad \mathcal{B}^{(k)} \textit{ is } (2\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))\textit{-regular w.r.t. } \hat{\mathcal{P}}^{(k-1)}.$$

Proof of Fact 6.2. Indeed, for fixed $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\text{reg}}$, we know

(1) $\mathcal{R}^{(k)}$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ (by definition of $\hat{\mathscr{P}}^{(k-1)}_{\text{reg}}$);
(2) $\mathcal{G}^{(k)}$ is $(\varepsilon, d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}), \widetilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$, where $d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \in [\frac{1}{4}, \frac{3}{4}]$ (see (ii) of Lemma 6.1).

[3] We note that one could, in fact, show that $\hat{\mathscr{P}}^{(k-1)}_{\text{bad}} = \varnothing$. This would follow from the fact that there are only a bounded number (independent of n) of polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}$ and each of them corresponds to a $(\delta, (1/a_2, \dots, 1/a_{k-1}))$-regular $(n/a_1, k, k-1)$-complex. In this situation, one can argue that with $\delta \ll \min\{1/a_1, \dots, 1/a_{k-1}\}$ we have $|\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})| = (1 \pm f(\delta)) \prod_{h=2}^{k-1} (1/a_h)^{\binom{k}{h}} \times (n/a_1)^k$ where $f(\delta) \to 0$ as $\delta \to 0$. Rather than making this precise, however, we chose in our current proof to use the fact that (sparse) polyads $\hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\text{bad}}$ can have only little influence.

As such, it may be directly verified from Definition 4.10 that the difference $\mathcal{B}^{(k)} = \mathcal{G}^{(k)} \setminus \mathcal{R}^{(k)}$ is $(\varepsilon + \delta_k, *, \min\{r(\boldsymbol{a}^{\mathscr{P}}), \widetilde{r}\})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}$ (with complementary density $d(\mathcal{B}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) = d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) - d(\mathcal{R}^{(k)}|\hat{\mathcal{P}}^{(k-1)})$). Recalling $\varepsilon = \delta_k$ from (36) and $r(\boldsymbol{a}^{\mathscr{P}}) \le \widetilde{r}$ from (37), Fact 6.2 follows. $\square$

We proceed with the first of two easy claims that will prove Lemma 6.1.

Claim 6.3. *For $s = R^{(k)}(f,f)$ fixed in (31), there exists $S \in \mathrm{Cross}_s(\mathscr{P}^{(1)})$ so that every $K \in \binom{S}{k}$ has $\hat{\mathcal{P}}^{(k-1)}(K) \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{reg}}$.*

Proof of Claim 6.3. Set

$$\widetilde{\mathcal{G}}^{(k)} = \mathcal{G}^{(k)} \cap \mathrm{Cross}_k(\mathscr{P}^{(1)}) \cap \Big\{ \bigcup \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{reg}} \Big\}. \tag{40}$$

Observe that every $S \in \mathcal{K}_s(\widetilde{\mathcal{G}}^{(k)})$ satisfies the properties required by the claim. As such, it suffices to prove $|\mathcal{K}_s(\widetilde{\mathcal{G}}^{(k)})| > 0$. Recall that our hypothesis in Lemma 6.1 assumes that $|\mathcal{K}_s(\mathcal{G}^{(k)})| > c\binom{n}{s}$. We show that, in deleting the few edges of $\mathcal{G}^{(k)}$ to obtain $\widetilde{\mathcal{G}}^{(k)}$, we don't destroy all of these cliques.

First, we check that $|\mathcal{G}^{(k)} \setminus \widetilde{\mathcal{G}}^{(k)}|$ is small. Indeed, since $\mathscr{P}$ is an $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable family of partitions,

$$\big|\mathcal{G}^{(k)} \setminus \mathrm{Cross}_k(\mathscr{P}^{(1)})\big| \le \eta \binom{n}{k}. \tag{41}$$

Combining (38) with the fact that $\mathcal{R}^{(k)}$ is $(\delta_k, r(a^{\mathscr{P}}))$-regular w.r.t. $\mathscr{P}$ we have (in view of (39)) that

$$\Big|\mathcal{G}^{(k)} \setminus \Big\{ \bigcup \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}) \colon \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{reg}} \Big\}\Big| \le \left(\delta_k + \frac{c}{4\binom{s}{k}}\right)\binom{n}{k}. \tag{42}$$

Consequently, we infer from (40), (41), and (42) that

$$|\mathcal{G}^{(k)} \setminus \widetilde{\mathcal{G}}^{(k)}| \le \left(\eta + \delta_k + \frac{c}{4\binom{s}{k}}\right)\binom{n}{k} \overset{(32)}{\le} \frac{3c}{4}\binom{s}{k}^{-1}\binom{n}{k}. \tag{43}$$

Now, since each k-tuple of $\mathcal{G}^{(k)} \setminus \widetilde{\mathcal{G}}^{(k)}$ can belong to at most $\binom{n-k}{s-k}$ cliques $K_s^{(k)}$, we see that (43) implies

$$\begin{aligned} \big|\mathcal{K}_s(\widetilde{\mathcal{G}}^{(k)})\big| &\ge \big|\mathcal{K}_s(\mathcal{G}^{(k)})\big| - \frac{3c}{4}\binom{s}{k}^{-1}\binom{n}{k}\binom{n-k}{s-k} \\ &= \big|\mathcal{K}_s(\mathcal{G}^{(k)})\big| - \frac{3c}{4}\binom{n}{s} \overset{(i)}{\ge} \frac{c}{4}\binom{n}{s} > 0 \end{aligned}$$

where we used property (i) from the hypothesis of Lemma 6.1. $\square$

As guaranteed by Claim 6.3, fix $S \in \mathrm{Cross}_s(\mathscr{P}^{(1)})$ of size $s = R^{(k)}(f,f)$ whose every $K \in \binom{S}{k}$ has $\hat{\mathcal{P}}^{(k-1)}(K) \in \hat{\mathscr{P}}^{(k-1)}_{\mathrm{reg}}$. We continue with the second of two easy claims that will prove Lemma 6.1.

Claim 6.4. *There exists a set $F \in \binom{S}{f}$ such that either*

$$d(\mathcal{R}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(K)) \ge \tfrac{1}{8} \quad \textit{for every } K \in \tbinom{F}{k}, \tag{44}$$

or

$$d(\mathcal{B}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(K)) \ge \tfrac{1}{8} \quad \textit{for every } K \in \tbinom{F}{k}. \tag{45}$$

Proof of Claim 6.4. For each $K \in \binom{S}{k}$, define an auxiliary two-coloring

$$\chi(K) = \begin{cases} \text{`red'} & \text{if} \quad d(\mathcal{R}^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) \geq \frac{1}{8}\,, \\ \text{`blue'} & \text{otherwise.} \end{cases}$$

Note that, by the definition of $\hat{\mathscr{P}}^{(k-1)}_{\rm bad}$ and assumption (ii) of Lemma 6.1, we have $d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \geq 1/4$ for every $\hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\rm bad}$. Consequently, for every $\hat{\mathcal{P}}^{(k-1)} \notin \hat{\mathscr{P}}^{(k-1)}_{\rm bad}$ either $d(\mathcal{R}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \geq 1/8$ or $d(\mathcal{B}^{(k)}|\hat{\mathcal{P}}^{(k-1)}) \geq 1/8$. (In this way, $\chi(K)$ =‘blue’ implies $d(\mathcal{B}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(K)) \geq 1/8$.) Now, it follows from $s = R^{(k)}(f,f)$ in (31) that there exists a set $F \in \binom{S}{f}$ such that χ is constant on $\binom{F}{k}$. Claim 6.4 then follows. □

We now deduce Lemma 6.1 from Claim 6.3 and 6.4. Let $F \in \binom{S}{f}$ be the set with the properties guaranteed by Claim 6.4. Then, the polyads $\hat{\mathcal{P}}^{(k-1)}(K)$ across $K \in \binom{F}{k}$ are all ‘dense’ in the same color $\mathcal{R}^{(k)}$ or $\mathcal{B}^{(k)}$. Recall Fact 6.2 ensures these same polyads are also all ‘regular’ across $K \in \binom{F}{k}$. It therefore doesn't matter which of (44) or (45) holds, and so we assume, without loss of generality, that the former does.

Fix a copy $\mathcal{F}_0^{(k)}$ of $\mathcal{F}^{(k)}$ on the set F, i.e., $V(\mathcal{F}_0^{(k)}) = F$. We construct a sub-hypergraph $\mathcal{H}^{(k)} \subseteq \mathcal{G}^{(k)}$ as follows. For $K \in \binom{F}{k} = \binom{V(\mathcal{F}_0^{(k)})}{k}$, set

$$\mathcal{H}^{(k)}_K = \begin{cases} \mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) & \text{if} \quad K \in \mathcal{F}_0^{(k)}\,, \\ \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) \setminus \mathcal{G}^{(k)} & \text{otherwise.} \end{cases}$$

Define

$$\mathcal{H}^{(k)} = \bigcup \left\{ \mathcal{H}^{(k)}_K \colon\ K \in \tbinom{F}{k} \right\}.$$

With $\mathcal{H}^{(k)}$ defined above, observe that every element of $\mathcal{K}_f(\mathcal{H}^{(k)})$ corresponds to a copy of $\mathcal{F}^{(k)} \subset \mathcal{R}^{(k)}$ which is induced in $\mathcal{G}^{(k)}$. To conclude the proof of Lemma 6.1, therefore, it suffices to show $|\mathcal{K}_f(\mathcal{H}^{(k)})| > 0$. To this end, we use the counting lemma, Theorem 4.13, and first check that it is appropriate to do so.

Indeed, for $j = 1, \dots, k-1$, set

$$\mathcal{Q}^{(j)} = \bigcup \left\{ \mathcal{P}^{(j)}(J) \colon\ J \in \tbinom{F}{j} \right\}$$

and $\boldsymbol{\mathcal{Q}} = \{\mathcal{Q}^{(j)}\}_{j=1}^{k-1}$. We observe the following.

(1) $\boldsymbol{\mathcal{Q}}$ is a $(\delta(\boldsymbol{a}^{\mathscr{P}}), r(1/a_2, \dots, 1/a_{k-1}))$-regular $(n/a_1, f, k-1)$-complex, where the function δ was chosen in (33) appropriately for an application of Theorem 4.13;

(2) For $K \in \mathcal{F}_0^{(k)}$, we combine Claim 6.3 and Claim 6.4 to see that $\mathcal{H}^{(k)}_K = \mathcal{R}^{(k)} \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K))$ is $(\delta_k, *, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ with density $d(\mathcal{R}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(K)) \geq \frac{1}{8}$. We note that $\delta_k \leq \delta_k^{(4.13)}(f, k, 1/2, d_k)$ and $r = r(\boldsymbol{a}^{\mathscr{P}})$ were chosen in (32) and (34), resp., appropriately for an application of Theorem 4.13;

(3) For each $K \in \binom{F}{k} \setminus \mathcal{F}_0^{(k)}$, we have, by (ii) of Lemma 6.1, that $\mathcal{G}^{(k)}$ is $(\varepsilon, d_K, \widetilde{r})$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ with $d_K = d(\mathcal{G}^{(k)}|\hat{\mathcal{P}}^{(k-1)}(K)) \in [\frac{1}{4}, \frac{3}{4}]$.

Since $\varepsilon = \delta_k$ and $\widetilde{r} \geq r(\boldsymbol{a}^{\mathscr{P}})$ (cf. (36) and (37)), the k-graph $\mathcal{G}^{(k)}$ is therefore also $(\delta_k, d_K, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$. As such, the complement $\mathcal{H}_K^{(k)} = \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(K)) \setminus \mathcal{G}^{(k)}$ is then also $(\delta_k, d_K, r(\boldsymbol{a}^{\mathscr{P}}))$-regular w.r.t. $\hat{\mathcal{P}}^{(k-1)}(K)$ with density $\bar{d}_K = d(\mathcal{H}_K^{(k)} | \hat{\mathcal{P}}^{(k-1)}(K)) = 1 - d_K \in [\frac{1}{4}, \frac{3}{4}]$.

Hence, we can apply the counting lemma to $\mathcal{H}^{(k)}$ and $\boldsymbol{\mathcal{Q}}$. As such, we conclude

$$\left|\mathcal{K}_f(\mathcal{H}^{(k)})\right| \geq \frac{1}{2}\left(\frac{1}{8}\right)^{\binom{f}{k}} \prod_{j=2}^{k-1}\left(\frac{1}{a_j}\right)^{\binom{f}{j}}\left(\frac{n}{a_1}\right)^{f} > 0,$$

and Lemma 6.1 is proved.

Appendix A.

Fact A.1. *With probability at least $(1-\exp(-n^k/\log^6 n))$ the binomial random hypergraph $\mathcal{G}^{(k)}(n,1/2)$ is $(1/\log n, 1/2, \log n)$-regular w.r.t. to every $(k-1)$-uniform hypergraph $\mathcal{P}^{(k-1)} \subseteq \binom{[n]}{k-1}$ for which $|\mathcal{K}_k(\mathcal{P}^{(k-1)})| \geq n^k/\log n$.*

Proof. The proof of Fact A.1 follows standard lines. For simplicity of notation, set $r = \log n$. Fix any $(k-1)$-graph $\mathcal{P}^{(k-1)} \subseteq \binom{[n]}{k-1}$ for which

$$\left|\mathcal{K}_k(\mathcal{P}^{(k-1)})\right| > \frac{n^k}{\log n}. \tag{46}$$

Let $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\}$ be a family of r sub-hypergraphs of $\mathcal{P}^{(k-1)}$ for which

$$\left|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| \geq \frac{1}{\log n}\left|\mathcal{K}_k(\mathcal{P}^{(k-1)})\right| \overset{(46)}{>} \frac{n^k}{\log^2 n}. \tag{47}$$

Set $X(\boldsymbol{\mathcal{Q}}^{(k-1)}) = |\mathcal{G}^{(k)}(n,1/2) \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})|$. Then, $X(\boldsymbol{\mathcal{Q}}^{(k-1)})$ is binomially distributed random variable with expectation

$$\begin{aligned}\mathbb{E}\big[X(\boldsymbol{\mathcal{Q}}^{(k-1)})\big] &= \mathbb{E}\left[\left|\mathcal{G}^{(k)}(n,1/2) \cap \bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right|\right] \\ &= \frac{1}{2}\left|\bigcup_{i\in[r]} \mathcal{K}_k(\mathcal{Q}_i^{(k-1)})\right| \overset{(47)}{>} \frac{n^k}{2\log^2 n}.\end{aligned} \tag{48}$$

We apply the Chernoff inequality (cf. [15]) to conclude

$$\begin{aligned}\mathbb{P}\left(\left|X(\boldsymbol{\mathcal{Q}}^{(k-1)}) - \mathbb{E}\big[X(\boldsymbol{\mathcal{Q}}^{(k-1)})\big]\right| > \frac{1}{\log n}\mathbb{E}\big[X(\boldsymbol{\mathcal{Q}}^{(k-1)})\big]\right) \\ \leq 2\exp\left\{-\frac{\mathbb{E}\big[X(\boldsymbol{\mathcal{Q}}^{(k-1)})\big]}{3\log^2 n}\right\} \overset{(48)}{<} 2\exp\left\{-\frac{n^k}{6\log^4 n}\right\}.\end{aligned} \tag{49}$$

For a given $\mathcal{P}^{(k-1)}$ satisfying (46), let $B(\mathcal{P}^{(k-1)})$ be the event that there exist a family $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{\mathcal{Q}_1^{(k-1)}, \dots, \mathcal{Q}_r^{(k-1)}\}$ of r sub-hypergraphs of $\mathcal{P}^{(k-1)}$ such that (47) and $|X(\boldsymbol{\mathcal{Q}}^{(k-1)}) - \mathbb{E}[X(\boldsymbol{\mathcal{Q}}^{(k-1)})]| > \mathbb{E}[X(\boldsymbol{\mathcal{Q}}^{(k-1)})]/\log n$. As there are at most $2^{r|\mathcal{P}^{(k-1)}|} \leq 2^{n^{k-1}\log n}$ families $\boldsymbol{\mathcal{Q}}^{(k-1)}$ of sub-hypergraphs of $\mathcal{P}^{(k-1)}$, we see

$$\mathbb{P}\big(B(\mathcal{P}^{(k-1)})\big) \overset{(49)}{<} 2\cdot 2^{n^{k-1}\log n}\exp\left\{-\frac{n^k}{6\log^4 n}\right\} < \exp\left\{-\frac{n^k}{\log^5 n}\right\}. \tag{50}$$

We now conclude the proof of Fact A.1. Note that (50) almost proves what we want. Namely, we have fixed an appropriate $(k-1)$-graph $\mathcal{P}^{(k-1)}$ (i.e., which satisfies (46)) and have proved that it is very unlikely that $\mathcal{G}^{(k)}(n,1/2)$ fails to be $(1/\log n, 1/2, \log n)$-regular w.r.t. $\mathcal{P}^{(k-1)}$. We simply want the same assertion for every appropriate $(k-1)$-graph $\mathcal{P}^{(k-1)}$, Since there are at most $2^{\binom{n}{k-1}}$ many $(k-1)$-graphs $\mathcal{P}^{(k-1)}$ satisfying (46), we see

$$\mathbb{P}\left(\bigcup\left\{B(\mathcal{P}^{(k-1)})\colon\ \mathcal{P}^{(k-1)} \subseteq \binom{[n]}{k-1} \text{ satisfying (46)}\right\}\right)$$
$$\overset{(50)}{<} 2^{\binom{n}{k-1}} \exp\left\{-\frac{n^k}{\log^5 n}\right\} < \exp\left\{-\frac{n^k}{\log^6 n}\right\}.$$

This conclude our proof of Fact A.1. $\square$

References

[1] F. G. Abramson and L. A. Harrington, *Models without indiscernibles*, J. Symbolic Logic **43** (1978), no. 3, 572–600. 2.3

[2] N. Alon and A. Shapira, *On an extremal problem of Brown, Erdős and Sós*, Combinatorica, to appear. 2.1

[3] C. Avart, V. Rödl, and M. Schacht *Every monotone 3-graph property is testable*, submitted. 1

[4] B. Bollobás and A. Thomason, *Projections of bodies and hereditary properties of hypergraphs*, Bull. London Math. Soc. **27** (1995), no. 5, 417–424. 2.2

[5] ———, *Hereditary and monotone properties of graphs*, The mathematics of Paul Erdős, II, Algorithms Combin., vol. 14, Springer, Berlin, 1997, pp. 70–78. 2.2

[6] W. G. Brown, P. Erdős, and V. T. Sós, *Some extremal problems on r-graphs*, New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich, 1971), Academic Press, New York, 1973, pp. 53–63. 2.1

[7] P. Erdős, P. Frankl, and V. Rödl, *The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent*, Graphs Combin. **2** (1986), no. 2, 113–121. 2.1, 2.1, 2.2, 3, 5

[8] P. Erdős and D. J. Kleitman, *On coloring graphs to maximize the proportion of multicolored k-edges*, J. Combinatorial Theory **5** (1968), 164–169. 3

[9] P. Erdős, D. J. Kleitman, and B. L. Rothschild, *Asymptotic enumeration of K_n-free graphs*, Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973), Tomo II, Accad. Naz. Lincei, Rome, 1976, pp. 19–27. Atti dei Convegni Lincei, No. 17. 2.2

[10] P. Frankl and V. Rödl, *Extremal problems on set systems*, Random Structures Algorithms **20** (2002), no. 2, 131–164. 1, 1

[11] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for commuting transformations*, J. Analyse Math. **34** (1978), 275–291. 1

[12] W. T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem*, submitted. 1, 1.3, 2.5

[13] ———, *Quasirandomness, counting and regularity for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1–2, 143–184. 1, 2.5

[14] P. E. Haxell, B. Nagle, and V. Rödl, *Integer and fractional packings in dense 3-uniform hypergraphs*, Random Structures Algorithms **22** (2003), no. 3, 248–310. 1

[15] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience, New York, 2000. A

[16] G. Katona and M. Nemetz, T.and Simonovits, *On a problem of Turán in the theory of graphs*, Mat. Lapok **15** (1964), 228–238. 5.2

[17] Y. Kohayakawa, B. Nagle, and V. Rödl, *Efficient testing of hypergraphs (extended abstract)*, Automata, languages and programming, Lecture Notes in Comput. Sci., vol. 2380, Springer, Berlin, 2002, pp. 1017–1028. 1

[18] ______, *Hereditary properties of triple systems*, Combin. Probab. Comput. **12** (2003), no. 2, 155–189. 1, 2.2

[19] J. Komlós and M. Simonovits, *Szemerédi's regularity lemma and its applications in graph theory*, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352. 1

[20] János Komlós, Ali Shokoufandeh, Miklós Simonovits, and Endre Szemerédi, *The regularity lemma and its applications in graph theory*, Theoretical aspects of computer science (Tehran, 2000), Lecture Notes in Comput. Sci., vol. 2292, Springer, Berlin, 2002, pp. 84–112. 1

[21] B. Nagle and V. Rödl, *The asymptotic number of triple systems not containing a fixed one*, Discrete Math. **235** (2001), no. 1-3, 271–290, Combinatorics (Prague, 1998). 1, 2.2

[22] B. Nagle, V. Rödl, and M. Schacht, *The counting lemma for regular k-uniform hypergraphs*, Random Structures Algorithms, to appear. 1, 1.3, 2.5

[23] J. Nešetřil and V. Rödl, *Partitions of finite relational and set systems*, J. Combinatorial Theory Ser. A **22** (1977), no. 3, 289–312. 2.3

[24] ______, *Two proofs of the Ramsey property of the class of finite hypergraphs*, European J. Combin. **3** (1982), no. 4, 347–352. 2.3

[25] H. J. Prömel and A. Steger, *Excluding induced subgraphs. III. A general asymptotic*, Random Structures Algorithms **3** (1992), no. 1, 19–31. 2.2

[26] V. Rödl, *Some developments in Ramsey theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) (Tokyo), Math. Soc. Japan, 1991, pp. 1455–1466. 1

[27] V. Rödl and A. Ruciński, *Ramsey properties of random hypergraphs*, J. Combin. Theory Ser. A **81** (1998), no. 1, 1–33. 1

[28] V. Rödl, A. Ruciński, and E. Szemerédi, *Dirac's theorem for 3-uniform hypergraphs*, Combin. Probab. Comput. **15** (2006), no. 1–2, 229–251. 1

[29] V. Rödl and M. Schacht, *Regular partitions of hypergraphs*, submitted. 1, 2.5, 4, 4.2

[30] V. Rödl, M. Schacht, M. Siggers, and N. Tokushige, *Integer and fractional packings of hypergraphs*, submitted. 1

[31] V. Rödl, M. Schacht, E. Tengan, and N. Tokushige, *Density theorems and extremal hypergraph problems*, Israel J. Math., to appear. 1, 1

[32] V. Rödl and J. Skokan, *Applications of the regularity lemma for uniform hypergraphs*, Random Structures Algorithms, to appear. 1, 1.3

[33] ______, *Regularity lemma for k-uniform hypergraphs*, Random Structures Algorithms **25** (2004), no. 1, 1–42. 1, 2.5, 4.1.3

[34] K. F. Roth, *Sur quelques ensembles d'entiers*, C. R. Acad. Sci. Paris **234** (1952), 388–390. 1

[35] ______, *On certain sets of integers*, J. London Math. Soc. **28** (1953), 104–109. 1

[36] I. Z. Ruzsa and E. Szemerédi, *Triple systems with no six points carrying three triangles*, Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II, North-Holland, Amsterdam, 1978, pp. 939–945. 1, 2.1, 3

[37] G. N. Sárközy and S. Selkow, *On a Turán-type hypergraph problem of Brown, Erdős and T. Sós*, Discrete Math. **297** (2005), no. 1-3, 190–195. 1, 2.1, 3

[38] J. Solymosi, *Note on a generalization of Roth's theorem*, Discrete and computational geometry, Algorithms Combin., vol. 25, Springer, Berlin, 2003, pp. 825–827. 1, 1

[39] ______, *A note on a question of Erdős and Graham*, Combin. Probab. Comput. **13** (2004), no. 2, 263–267. 1

[40] T. Tao, *The Gaussian primes contain arbitrarily shaped constellations*, submitted. 1

[41] ______, *A variant of the hypergraph removal lemma*, J. Combinatorial Theory Ser. A, to appear. 1, 2.5

Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557, USA.

E-mail address: `nagle@unr.edu.`

Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30322, USA.

E-mail address: `rodl@mathcs.emory.edu.`

Institut für Informatik, Humboldt-Universität zu Berlin, Unter den Linden 6, D-10099, Berlin, Germany

E-mail address: `schacht@informatik.hu-berlin.de`