Homotopical algebra and homotopy colimits

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New interactions between homotopical algebra and quantum field theory
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Homotopical algebra – What for?

Often, we are interested in (co)homology groups.
$H^2(X, \mathbb{Z}) = [X, \mathbb{C}P^\infty] = [X, BU(1)]$ classifies line bundles on a space X.
Many geometric invariants of a manifold M can be understood via its de Rham cohomology groups.
In order to calculate or understand such (co)homology groups, we often have to perform constructions on the level of (co)chain complexes: quotients, direct sums,...
For these constructions one needs models.
Homotopical algebra: Study of homological/homotopical questions via model categories.
Definition given by Quillen in 1967 [Q].
Flexible framework, can be used for chain complexes, topological spaces, algebras over operads, and many more – allows us to do homotopy theory.

Chain complexes, I

Let R be an associative ring and let Ch_R denote the category of non-negatively graded chain complexes of R-modules.
The objects are families of R-modules $C_n, n \geq 0$, together with R-linear maps, the differentials, $d = d_n \colon C_n \to C_{n-1}$ for all $n \geq 1$ such that $d_{n-1} \circ d_n = 0$ for all n.
Morphisms are chain maps $f_* \colon C_* \to D_*$. These are families of R-linear maps $f_n \colon C_n \to D_n$ such that $d_n \circ f_n = f_{n-1} \circ d_n$ for all n.
The nth homology group of a chain complex C_* is

$$H_n(C_*) = ker(d_n \colon C_n \to C_{n-1})/im(d_{n+1} \colon C_{n+1} \to C_n).$$

$ker(d_n \colon C_n \to C_{n-1})$ are the n-cycles of C_*, Z_nC_*, and $im(d_{n+1} \colon C_{n+1} \to C_n)$ are the n-boundaries of C_*, B_nC_*.
Here, we use the convention that $Z_0C_* = C_0$. Chain maps f_* induce well-defined maps on homology groups $H_n(f)$:
$H_n(f) \colon H_n(C_*) \to H_n(D_*)$, $H_n(f)[c] := [f_n(c)]$.

The homotopy category

A chain map f_* is called a quasi-isomorphism if the induced map

$$H_n f \colon H_n(C_*) \to H_n(D_*)$$

is an isomorphism for all $n \geq 0$.
For understanding homology groups of chain complexes we would like to have a category $Ch_R[qi^{-1}]$ where we invert the quasi-isomorphisms.
Such a category is usually hard to construct. (How can you compose morphisms? How can you make this well-defined?...)
Model categories give such a construction.

Model categories, I

A model category is a category $\mathcal{C}$ together with three classes of maps

- the weak equivalences, (we)
- the cofibrations (cof) and
- the fibrations (fib).

These classes are closed under compositions and every identity map is in each of the classes.
An $f \in fib \cap we$ is called an acyclic fibration and a $g \in cof \cap we$ is called an acyclic cofibration.
We indicate weak equivalences by $\xrightarrow{\sim}$, cofibrations by $\rightarrowtail$ and fibrations by $\twoheadrightarrow$.
These classes of maps have to satisfy a lot of compatibility conditions...

Model categories, II

- M1 The category $\mathcal{C}$ has all limits and colimits.
- M2 (2-out-of-3): If f, g are morphisms in $\mathcal{C}$ such that $g \circ f$ is defined, then if two of the maps f, g, $g \circ f$ are weak equivalences, then so is the third.
- M3 If f is a retract of g and g is in *we*, *cof* or *fib*, then so is f.
- M4 For every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle q} \\ B & \xrightarrow{\beta} & Y \end{array}$$

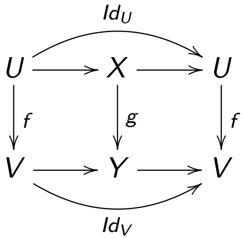
  in $\mathcal{C}$ where i is a cofibration and q is an acyclic fibration or where i is an acyclic cofibration and q is a fibration, a lift ξ exists with $q \circ \xi = \beta$ and $\xi \circ i = \alpha$.
- M5 Every morphism f in $\mathcal{C}$ can be factored as $f = p \circ j$ and $q \circ i$, where j is an acyclic cofibration and p is a fibration, q is an acyclic fibration and i is a cofibration.

Model categories, II

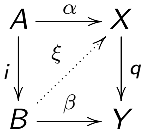
M1 allows us to make constructions.

M2: think of maps that induce isomorphisms on homology or homotopy groups. These will automatically satisfy 2-out-of-3.

M3: f is a retract of g if it fits into a commutative diagram



M4: The lift ξ in



is not required to be unique!

M5: Can be used for constructing projective/injective resolutions, CW-approximations etc.

Chain complexes, II

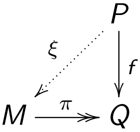
The category Ch_R has several model category structures. The one we will use is: A chain map $f\colon C_* \to D_*$ is a

- weak equivalence, if f_* is a quasi-isomorphism, i.e., $H_* f_*$ is an isomorphism for all $n \geq 0$,
- fibration, if $f_n\colon C_n \to D_n$ is an epimorphism for all $n \geq 1$,
- cofibration, if $f_n\colon C_n \to D_n$ is a monomorphism with projective cokernel for all $n \geq 0$.

This *does* define a model category structure on Ch_R.
What are projective modules?

Projective modules

Let R be a ring. A left R-module P is projective if for every epimorphism $\pi\colon M \to Q$ of R-modules and every morphism $f\colon P \to Q$ of R-modules there is an R-linear morphism $\xi\colon P \to M$ that lifts f to M:



If $R = \mathbb{Z}$ then the projective modules are exactly the free ones, that is, $P = \bigoplus_I \mathbb{Z}$.
If R is a field, then every module is projective.

A light exposure to a typical argument

There are spheres and disks in Ch_R!

$$(\mathbb{S}^n)_m = \begin{cases} R, & m = n, \\ 0, & \text{otherwise.} \end{cases}$$

The sphere complex has $d = 0$ for all m.

$$(\mathbb{D}^n)_m = \begin{cases} R, & m = n, n-1, \\ 0, & \text{otherwise.} \end{cases}$$

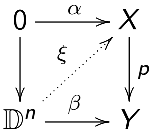
Here $d\colon (\mathbb{D}^n)_n = R \to R = (\mathbb{D}^n)_{n-1}$ is the identity map.
Exercise: Calculate the homology groups of spheres and disks. Show that every chain map from $\mathbb{S}^n$ to a chain complex C_* picks out an n-cycle $c \in Z_n(C_*)$ and that every chain map from $\mathbb{D}^n$ to a chain complex C_* picks out an element $x \in C_n$. Therefore there is a canonical map $i_n\colon \mathbb{S}^{n-1} \to \mathbb{D}^n$.

Lemma

1) A morphism in Ch_R is a fibration if and only if it has the lifting property with respect to all maps $0 \to \mathbb{D}^n$ with $n \geq 1$.

2) A morphism in Ch_R is an acyclic fibration if and only if it has the lifting property with respect to all maps $i_n \colon \mathbb{S}^{n-1} \to \mathbb{D}^n$ for $n \geq 0$.

Proof: of 1): We assume that there is a lift ξ in the diagram

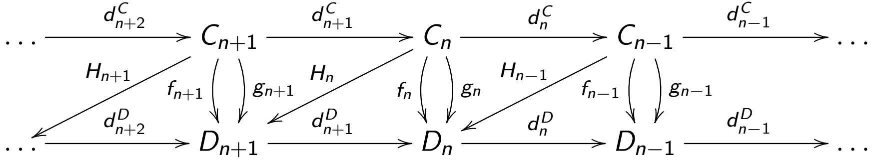


for all $n \geq 1$ and we have to show that p_n is surjective for all $n \geq 1$. Any $y \in Y_n$ corresponds to $\beta \colon \mathbb{D}^n \to Y$, sending $1_R \in \mathbb{D}^n_n$ to y. A lift ξ picks an element $x \in X_n$ and the property $p_n \circ \xi_n = \beta_n$ ensures that x is a preimage of y under p_n, hence p_n is surjective.

Towards the homotopy category

When are two chain maps $f_*, g_* \colon C_* \to D_*$ homotopic?
A chain homotopy H between f_* and g_* is a sequence of R-linear maps $(H_n)_{n \in \mathbb{N}_0}$ with $H_n \colon C_n \to D_{n+1}$ such that for all n

$$d^D_{n+1} \circ H_n + H_{n-1} \circ d^C_n = f_n - g_n.$$



If f_* is chain homotopic to g_*, then $H_* f = H_* g$.

We can express this in a more "geometric" way.
The cylinder on C_* is the chain complex $cyl(C)_*$ with $cyl(C)_n = C_n \oplus C_{n-1} \oplus C_n$ and with $d\colon cyl(C)_n \to cyl(C)_{n-1}$ given by the matrix

$$d = \begin{pmatrix} d_n & id & 0 \\ 0 & -d_{n-1} & 0 \\ 0 & -id & d_n \end{pmatrix}$$

The "top" and the "bottom" of the cylinder embed as

$$C_n \to cyl(C)_n, \quad c \mapsto (c, 0, 0)$$

and

$$C_n \to cyl(C)_n, \quad c \mapsto (0, 0, c).$$

There is also a map $q\colon cyl(C)_* \to C_*$ sending (c_1, c_2, c_3) to $c_1 + c_3$. These maps are chain maps.
Exercise: Two chain maps $f_*, g_*\colon C_* \to D_*$ are chain homotopic if and only if they extend to a chain map

$$f_* + H_* + g_*\colon cyl(C)_* \to D_*.$$

Cylinder objects in a model category

Let C be an object in a model category $\mathcal{C}$. We call an object cyl_C a cylinder object for C, if there are morphisms

$$C \sqcup C \xrightarrow{i} cyl_C \xrightarrow[\sim]{q} C$$

that factor the fold map $\nabla \colon C \sqcup C \to C$.
For $\mathcal{C} = Ch_R$ the categorical sum $C_* \sqcup C_*$ is the direct sum $C_* \oplus C_*$ and the fold map ∇ sends (c_1, c_2) to $c_1 + c_2$.
$cyl(C)_*$ as above is a cylinder object: we can take $i(c_1, c_2) = (c_1, 0, c_2)$ and $q \colon cyl(C)_* \to C_*$ as above.

A cylinder object cyl_C is good, if i is a cofibration and it is very good if in addition q is an acyclic fibration.
Warning: In general, cyl_C won't be functorial in C!
In Ch_R our cylinder object $cyl(C)_*$ won't be good in general: i is not a cofibration in general, because the cokernel of i_n is C_{n-1} which won't be projective in general. However, q is always surjective in all degrees, hence a fibration.
Good and very good cylinder objects exist thanks to M5.
The map $i \colon C \sqcup C \to cyl_C$ has components $i_0 \colon C \to cyl_C$ and $i_1 \colon C \to cyl_C$ given by the two maps $C \to C \sqcup C$.

Left homotopies

Two morphisms in a model category $f, g\colon C \to D$ are called left homotopic, if there is a cylinder object cyl_C of C and a morphism $H\colon cyl_C \to D$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$.
Problems:

- Being left homotopic is no equivalence relation in general.
- There is a dual notion of being right homotopic (using "path objects" instead of cylinder objects) and these notions don't agree in general.

We need to restrict to nice objects!

Digression: initial and terminal objects

Every chain complex C_* receives a unique chain map f from the trivial chain complex 0 with $0_n = 0$ for all $n \geq 0$, the trivial abelian group,
$f_n = 0 \colon 0 \to C_n$, and it also has a unique chain map $g \colon C_* \to 0$, sending everything to zero.
In the category of topological spaces every topological space X receives a unique map from the empty topological space $\varnothing$ (by convention) and for every one-point topological space $\{*\}$ there is a unique continuous map $p \colon X \to \{*\}$.
Definition: An object i in a category $\mathcal{C}$ is called initial, if every object C of $\mathcal{C}$ has a unique morphism $f \in \mathcal{C}(i, C)$. Dually, an object t of $\mathcal{C}$ is called terminal, if for every object C of $\mathcal{C}$ there is a unique morphism $g \in \mathcal{C}(C, t)$.
So, 0 is initial and terminal in the category Ch_R and $\varnothing$ is inital in *Top* whereas any one-point space is terminal in *Top*.

Cofibrant and fibrant objects

Initial objects and terminal objects exist in every model category.
Definition: An object C in a model category is cofibrant, if the unique morphism $i \to C$ is a cofibration. Dually, an object P in a model category is fibrant, if the unique morphism $P \to t$ is a fibration.
In Ch_R every object is fibrant, but only those chain complexes C_* with C_n projective for all $n \geq 0$ are cofibrant.
For every object X in a model category, we can factor the unique map $i \to X$ as

$$i \rightarrowtail^{f} QX \overset{q}{\underset{\sim}{\twoheadrightarrow}} X$$

with $f \in cof$ and $q \in fib \cap we$. We call this a cofibrant replacement of X. (This can be made functorial in X.)
In Ch_R this gives projective resolutions of any R-module M viewed as $\mathbb{S}^0(M)$.

The homotopy category of a model category

For a cofibrant object QX we can factor the unique map $QX \to t$ as

$$QX \underset{\sim}{\overset{j}{\rightarrowtail}} RQX \overset{p}{\twoheadrightarrow} t$$

with $j \in cof \cap we$ and $p \in fib$. Then we have an object RQX that is both fibrant and cofibrant and has a zig-zag of weak equivalences

$$X \underset{\sim}{\overset{q}{\twoheadleftarrow}} QX \underset{\sim}{\overset{j}{\rightarrowtail}} RQX$$

Definition: The homotopy category, $Ho(\mathcal{C})$, of a model category $\mathcal{C}$ has as objects the objects of $\mathcal{C}$ and $Ho(\mathcal{C})(X, Y)$ is the set of (left) homotopy classes of maps from RQX to RQY.

This is the right thing: There is a functor $\gamma\colon \mathcal{C} \to Ho(\mathcal{C})$ with $\gamma(X) = X$ and $\gamma(f\colon X \to Y) = [RQf\colon RQX \to RQY]$.

Theorem: For any f in $\mathcal{C}$ we have: $\gamma(f)$ is an isomorphism in $Ho(\mathcal{C})$ if and only if f is a weak equivalence.

So $Ho(\mathcal{C})$ is a model for $\mathcal{C}[we^{-1}]$!

What are diagrams?

Take any small category $\mathcal{D}$. That is a category whose objects constitute an actual set and not a proper class. Let $\mathcal{C}$ be an arbitrary category.

A $\mathcal{D}$-diagram in $\mathcal{C}$ is a functor $F \colon \mathcal{D} \to \mathcal{C}$: So for every object D of $\mathcal{D}$ you have an object $F(D)$ of $\mathcal{C}$ and for every morphism $f \in \mathcal{D}(D_1, D_2)$ you get a morphism $F(f) \colon F(D_1) \to F(D_2)$. This has to be consistent: for $g \in \mathcal{D}(D_2, D_3)$ we have $F(g) \circ F(f) = F(g \circ f)$ and $F(id_D) = id_{F(D)}$ for all objects D of $\mathcal{D}$.

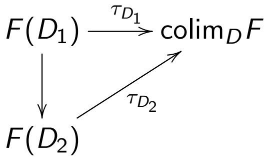
Examples:

- $\mathcal{D} = (2 \leftarrow 0 \rightarrow 1)$ and $\mathcal{C} = Ch_R$ gives a diagram $F(2) \leftarrow F(0) \rightarrow F(1)$ of chain complexes and chain maps.
- For $\mathcal{D} = (0 \rightarrow 1 \rightarrow 2 \rightarrow \ldots)$ and $\mathcal{C} = Top$ we get a sequence $F(0) \rightarrow F(1) \rightarrow F(2) \rightarrow \ldots$ of topological spaces and continuous maps.
- If S is any set, then we can consider it as a category whose only morphisms are identity maps. A functor $F \colon S \to \mathcal{C}$ for any $\mathcal{C}$ is just an S-indexed family of objects.

What are colimits?

Let $F\colon \mathcal{D} \to \mathcal{C}$ be a functor as above. Then a colimit of F is an object $\mathrm{colim}_{\mathcal{D}}F$ of $\mathcal{C}$ that is "as close to the diagram that F defines as it can be".

Definition: A colimit of F over $\mathcal{D}$ is an object $\mathrm{colim}_D F$ of $\mathcal{C}$ together with morphisms $\tau_D\colon F(D) \to \mathrm{colim}_D F$ in $\mathcal{C}$ such that for all $f \in \mathcal{D}(D_1, D_2)$



commutes. Furthermore, if C is any other object of $\mathcal{C}$ with morphisms $\eta_D\colon F(D) \to C$ such that

$$\eta_{D_2} \circ F(f) = \eta_{D_1} \quad \forall\, f \in \mathcal{D}(D_1, D_2)$$

then there is a unique morphism $\xi\colon \mathrm{colim}_{\mathcal{D}}F \to C$ with $\xi \circ \tau_D = \eta_D$ for all objects D of $\mathcal{D}$.

Examples of colimits

- Colimits for $\mathcal{D} = (2 \leftarrow 0 \rightarrow 1)$ are called pushouts. In Ch_R the pushout of $F(2) \leftarrow F(0) \rightarrow F(1)$ is the chain complex

$$(F(2) \oplus F(1))/\sim$$

where $\sim$ identifies the image of $F(0)$ in $F(1)$ and $F(2)$. This fit into a diagram

$$\begin{array}{ccc} F(0) & \longrightarrow & F(2) \\ \big\downarrow & & \big\downarrow \tau_2 \\ F(1) & \xrightarrow{\tau_1} & (F(2) \oplus F(1))/\sim \end{array}$$

For $\tau_0 \colon F(0) \rightarrow (F(2) \oplus F(1))/\sim$ you take the map from $F(0)$ to $(F(2) \oplus F(1))/\sim$ in the diagram (they are both the same).

Examples of colimits – continued

- For a diagram of the form $F(0) \to F(1) \to F(2) \to \dots$ in *Top* the colimit is given by $\bigsqcup_{n \geq 0} F(n)/\sim$ where $\sim$ identifies $x \in F(m)$ with the image of x in $F(n)$ under the maps in the sequence for $m \leq n$. Such colimits are called sequential colimits.
- A colimit over a diagram indexed on a set S viewed as a category is the coproduct of the objects $F(s)$, $s \in S$ and is denoted by $\bigsqcup_S F(s)$. For sets or topological spaces you get the disjoint union of the $F(s)$, for chain complexes you get $\bigoplus_S F(s)$.

Homotopy invariance

Slogan: Homotopy colimits are homotopy invariant colimits

What does that mean?

Usual colimits are *not* homotopy invariant:

Take the pushout of

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & * \\ \downarrow & & \\ * & & \end{array}$$

Here $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}, |x| = 1\}$ is the unit sphere in $\mathbb{R}^{n+1}$.

An explicit formula for the pushout is $* \sqcup */\sim$ where the two points are glued together, so

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

is a pushout diagram.

But the unit $(n+1)$-disk $\mathbb{D}^{n+1} = \{y \in \mathbb{R}^{n+1}, |y| \leq 1\}$ is contractible, so homotopy equivalent to a point $*$.
The pushout of

$$\begin{array}{ccc} \mathbb{S}^n & \longrightarrow & \mathbb{D}^{n+1} \\ \downarrow & & \\ \mathbb{D}^{n+1} & & \end{array}$$

is $\mathbb{S}^{n+1}$.
Thus replacing $*$ by the homotopy equivalent $\mathbb{D}^{n+1}$ changed the homotopy type of the pushout.
That's bad, if you want to work up to homotopy...

What should a homotopy colimit do for us?

In a model category all colimits exist by assumption. We can actually view the colimit as a functor

$$\mathrm{colim}_{\mathcal{D}} \colon \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$$

where $\mathcal{C}^{\mathcal{D}}$ denotes the category of functors from $\mathcal{D}$ to $\mathcal{C}$. It is left adjoint to the constant functor

$$\Delta \colon \mathcal{C} \to \mathcal{C}^{\mathcal{D}}, \quad \Delta(C)(D) = C \quad \forall\, D$$

and Δ sends any morphism in $\mathcal{D}$ to the identity map on C.
We want to transform this into a functor

$$\mathrm{hocolim}_{\mathcal{D}} \colon Ho(\mathcal{C}^{\mathcal{D}}) \to Ho(\mathcal{C})$$

...at least, if $\mathcal{C}^{\mathcal{D}}$ possesses a model category structure and thus a homotopy category, $Ho(\mathcal{C}^{\mathcal{D}})$. (Warning: $Ho(\mathcal{C}^{\mathcal{D}}) \neq Ho(\mathcal{C})^{\mathcal{D}}$!)

Model category definition of hocolims

Assume that $\mathcal{C}^{\mathcal{D}}$ possesses a model category structure. Then if the colimit functor $\mathrm{colim}_{\mathcal{D}}$ preserves cofibrations and if the functor Δ preserves fibrations, then there is an adjoint pair of functors

$$Ho(\mathcal{C}^{\mathcal{D}}) \underset{R\Delta}{\overset{\mathrm{hocolim}_{\mathcal{D}}}{\rightleftarrows}} Ho(\mathcal{C})$$

Recipe for $\mathrm{hocolim}_{\mathcal{D}} F$:

1. Take your diagram F and its cofibrant replacement $i \rightarrowtail Q(F) \overset{\sim}{\twoheadrightarrow} F$ in $\mathcal{C}^{\mathcal{D}}$.
2. The colimit $\mathrm{colim}_D Q(F)$ models $\mathrm{hocolim}_{\mathcal{D}} F$.

Why are we not happy with that?

Usually, model structures on diagram categories $\mathcal{C}^{\mathcal{D}}$ are complicated.
The cofibrant replacement of a diagram in $\mathcal{C}^{\mathcal{D}}$ is *not* just given by the cofibrant replacement of each $F(D)$, but is way more involved.
How do we get explicit models?

Bousfield-Kan, Hirschhorn, Rodríguez-González

- 1972: Bousfield and Kan constructed models for homotopy colimits for diagrams in simplicial sets; those are combinatorial models of topological spaces.
- People observed that the Bousfield-Kan construction transfers to many other settings “with a simplicial structure” (see Hirschhorn's book [H]).
- Rodríguez-González [RG] gave a systematic account on the question, when there is a Bousfield-Kan model of a homotopy colimit.

Examples of homotopy colimits, I

The double mapping cylinder. We saw that ordinary pushouts in topological spaces are *not* homotopy invariant.
Consider a diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_2 \\ {\scriptstyle g}\downarrow & & \\ X_1 & & \end{array}$$

of topological spaces and continuous maps. (I.e. $F(i) = X_i$).
Replace X_0, the space you use for gluing, by the cylinder $X_0 \times [0, 1]$.
The homotopy colimit of the diagram can be expressed as

$$(X_1 \sqcup X_0 \times [0, 1] \sqcup X_2)/\sim$$

where you glue points $(x_0, 0) \in X_0 \times [0, 1]$ to $g(x_0)$ and $(x_0, 1)$ to $f(x_0)$.

Examples of homotopy colimits, II

For a sequential diagram of topological spaces $X_0 \to X_1 \to X_2 \to \dots$ the telescope is an explicit model of $\text{hocolim}_{\mathbb{N}_0} X$:

1. Replace every X_n by the cylinder $X_n \times [n, n+1]$.
2. Glue the points $(x_n, n+1) \in X_n \times [n, n+1]$ to the points $(f_n(x_n), n+1) \in X_{n+1} \times [n+1, n+2]$.
3. This gives a telescope

$$\left(\bigsqcup_{n \geq 0} X_n \times [n, n+1] \right) / \sim .$$

Example: hocolim in non-negative chain complexes

Let $\mathcal{D}$ be any small category and let $F\colon \mathcal{D} \to Ch_R$ be any functor. Rodríguez-González describes an explicit model of $\mathrm{hocolim}_{\mathcal{D}} F$:

1. We consider morphisms in the category $\mathcal{D}$. Let $N(\mathcal{D})_n$ be the set of morphisms $D_0 \xrightarrow{f_1} D_1 \xrightarrow{f_2} \ldots \xrightarrow{f_n} D_n$. Here, by convention $N(\mathcal{D})_0$ is the set of objects of $\mathcal{D}$.
2. If we denote an element of $N(\mathcal{D})_n$ as above as $\underline{f} = (f_n, \ldots, f_1)$, then we can define

$$d_i(f_n, \ldots, f_1) := \begin{cases} (f_n, \ldots, f_2), & i = 0, \\ (f_n, \ldots, f_{i+2}, f_{i+1} \circ f_i, f_{i-1}, \ldots, f_1), & 0 < i < n, \\ (f_{n-1}, \ldots, f_1), & i = n. \end{cases}$$

3. Thus d_i erases the object D_i, so in d_0 f_1 is omitted because its source is gone, in d_n f_n is omitted because it lost its target, and all the inner d_i force a composition because the intermediate object disappeared.
4. We call D_0 the source of $\underline{f} = (f_n, \ldots, f_1)$ and denote it by $s\underline{f}$.

We can build a double chain complex out of our diagram and out of the above construction:
Each $F(D)$ is a chain complex with a differential $d\colon F(D)_n \to F(D)_{n-1}$. We can build

$$\delta\colon \bigoplus_{\underline{f}\in N(\mathcal{D})_n} F(s\underline{f}) \to \bigoplus_{\underline{g}\in N(\mathcal{D})_{n-1}} F(s\underline{g})$$

by using the alternating sum $\sum_{i=0}^{n}(-1)^i d_i$ of the d_i's above.
The resulting double complex looks as follows:

$$\begin{array}{ccccc}
\cdots & & \cdots & & \cdots \\
{\scriptstyle\delta}\big\downarrow & & {\scriptstyle-\delta}\big\downarrow & & \\
\bigoplus_{(f_1)\in N(\mathcal{D})_1} F(s(f_1))_0 & \xleftarrow{\;d\;} & \bigoplus_{(f_1)\in N(\mathcal{D})_1} F(s(f_1))_1 & \xleftarrow{\;d\;} & \cdots \\
{\scriptstyle\delta}\big\downarrow & & {\scriptstyle-\delta}\big\downarrow & & \\
\bigoplus_{D\in\mathcal{D}} F(D)_0 & \xleftarrow{\quad d\quad} & \bigoplus_{D\in\mathcal{D}} F(D)_1 & \xleftarrow{\quad d\quad} & \cdots
\end{array}$$

The associated total complex is a model for the homotopy colimit. This is rather involved, but explicit and useful for constructions.

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