

Hyperexacting cardinals

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Joint work in progress with
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Introduction

Following Woodin, seminal results of Jensen can be formulated as the following dichotomy:

Theorem (Jensen)

Exactly one of the following statements holds:

- V is close to L: For every singular cardinal λ , the cardinal λ is singular in L and $(\lambda^+)^L = \lambda^+$ holds.
- V is far from L: Every uncountable cardinal is inaccessible in L .

Woodin proved a surprising result showing that strong large cardinal axioms imply an analog of Jensen's dichotomy for the inner model HOD:

Theorem (Woodin, The HOD Dichotomy)

If δ is an extendible cardinal, then exactly one of the following statements holds:

- **V is close to HOD:** For every singular cardinal $\lambda > \delta$, the cardinal λ is singular in HOD and $(\lambda^+)^{\text{HOD}} = \lambda^+$ holds.
- **V is far from HOD:** Every regular cardinal greater than or equal to δ is measurable in HOD.

The HOD Dichotomy raises various fundamental questions.

In particular, it is not known whether the *far* side of this dichotomy is even possible.

This uncertainty is captured by the following conjecture:

The HOD Conjecture (Woodin)

The theory

$\text{ZFC} + \text{“} \textit{There is an extendible cardinal} \text{”}$

proves that V is close to HOD.

In his work on the HOD Conjecture, Woodin considered the following principle that ensures that V is far from HOD:

Theorem (Woodin)

Assume that $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ is a non-trivial elementary embedding with the property that for every Σ_2 -formula $\varphi(v)$ and every $x \in V_{\lambda+1}$, the statement $\varphi(x)$ holds in V if and only if the statement $\varphi(j(x))$ holds in V . Then λ is a singular cardinal that is regular in HOD.

In particular, if there is an extendible cardinal below λ , then V is far from HOD.

Woodin also proved that the consistency of ZFC with this large cardinal constellation follows from the consistency of ZF with certain choiceless large cardinal axioms.

Ultraextracting cardinals

We now outline joint work with Aguilera and Bagaria that arrives at the above principle from a very different angle.

To motivate central notions, we review the following choiceless large cardinal axiom:

Definition (Goldberg & Schlutzenberg)

A cardinal λ is *rank-Berkeley* if for all $\alpha < \lambda < \zeta$, there is a non-trivial elementary embedding $j : V_\zeta \longrightarrow V_\zeta$ with $\alpha < \text{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

The Kunen Inconsistency ensures that ZFC proves the non-existence of rank-Berkeley cardinals.

Working in ZF^2 , if $j : V \longrightarrow V$ is a non-trivial elementary embedding, then the first non-trivial fixed point of j is a rank-Berkeley cardinal.

The following notion was introduced in the consistency proof of very strong model-theoretic reflection principles:

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *exacting* if for all $\alpha < \lambda < \zeta$, there exists

- an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \rightarrow V_\zeta$ with $\alpha < \text{crit}(j) < \lambda$ and $j(\lambda) = \lambda$.

Theorem (Aguilera–Bagaria–Goldberg–L.)

If there exists an I2-embedding, then there is a transitive, set-sized model of ZFC with an exacting cardinal.

An easy argument shows that there is always an element of $V_{\lambda+1}$ missing from the domain of an embedding $j : X \longrightarrow V_\zeta$ witnessing the exactingness of a cardinal λ .

Therefore, we can strengthen this property by demanding that certain elements of $V_{\lambda+1}$ are contained in X .

Definition (Aguilera–Bagaria–L.)

A cardinal λ is *ultraexacting* if for all $\alpha < \lambda < \zeta$, there exists

- an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow V_\zeta$ with $\alpha < \text{crit}(j) < \lambda$, $j(\lambda) = \lambda$ and $j \upharpoonright V_\lambda \in X$.

This property turns out to be equivalent to Woodin's principle:

Theorem (Aguilera–Bagaria–Goldberg–L.)

A cardinal λ is ultraexacting if and only if there is a non-trivial elementary embedding $j : V_{\lambda+1} \longrightarrow V_{\lambda+1}$ with the property that for every Σ_2 -formula $\varphi(v)$ and every $x \in V_{\lambda+1}$, the statement $\varphi(x)$ holds in V if and only if the statement $\varphi(j(x))$ holds in V .

Theorem (Aguilera–Bagaria–Goldberg–L.)

The following statements are equiconsistent over ZFC:

- There is an ultraexacting cardinal.
- Axiom I0: there is a non-trivial elementary embedding $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ with $\text{crit}(j) < \lambda$.

Hyperexacting cardinals

Key consequences of ultraexactingness are obtained by extending embedding to initial segments of the $L(V_{\lambda+1})$ -hierarchy.

More precisely, given an embedding $j : X \longrightarrow V_\zeta$ witnessing the ultraexactingness of a cardinal λ and an ordinal $\gamma \in X \cap \Theta_{V_{\lambda+1}}$, there exists a function i_γ with domain $L_\gamma(V_{\lambda+1})$ in X satisfying

$$j \upharpoonright (L_\gamma(V_{\lambda+1}) \cap X) = i_\gamma \upharpoonright (L_\gamma(V_{\lambda+1}) \cap X).$$

It can be proven that ultraexactingness alone does not allow an extension of this statement to $\Theta_{V_{\lambda+1}}$.

We now want to strengthen the notion of ultraexactingness demanding the existence of such functions for ordinals above $\Theta_{V_{\lambda+1}}$.

Definition (Aguilera–L.)

A cardinal λ is *hyperexacting* if for all $\alpha < \lambda < \zeta$, there exists

- an elementary submodel X of V_ζ with $V_\lambda \cup \{\lambda\} \subseteq X$, and
- an elementary embedding $j : X \longrightarrow V_\zeta$ with $\alpha < \text{crit}(j) < \lambda$, $j(\lambda) = \lambda$, $j \upharpoonright V_\lambda \in X$ and the property that for all $\gamma \in X \cap \zeta$, there exists a function i_γ with domain γ in X satisfying

$$j \upharpoonright (X \cap \gamma) = i_\gamma \upharpoonright (X \cap \gamma).$$

Theorem (Aguilera–L.)

The consistency of ZF^2 with the existence of a super Reinhardt cardinal implies the consistency of ZFC with an hyperexacting cardinal.

Proposition

If ZFC together with existence of an ultraexacting cardinal is consistent, then this theory does not prove that unboundedly many ordinals are measurable in HOD.

Theorem (Aguilera–L.)

If λ is a hyperexacting cardinal, then every regular cardinal above λ is measurable in HOD.

Proposition

If ZFC together with existence of an ultraexacting cardinal is consistent, then this theory does not prove the existence of an I_0 -embedding.

Theorem (Aguilera–L.)

If λ is a hyperexacting cardinal, then there is an I_0 -embedding $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$.

Theorem (Aguilera–Bagaria–L.)

If λ is an ultraexacting cardinal and $V_{\lambda+1}^\#$ exists, then V_λ is a model of ZFC with a proper class of I0-embeddings.

Theorem (Aguilera–L.)

If λ is a hyperexacting cardinal, then $V_{\lambda+1}^\#$ exists.

Question

Can the consistency of ZFC with the existence of a hyperexacting cardinal be established assuming the consistency of ZFC with some standard large cardinal axiom?

Question

Does the consistency of ZFC with the existence of a hyperexacting cardinal imply the consistency of ZF with a non-trivial elementary embedding $j : V_{\lambda+3} \longrightarrow V_{\lambda+3}$?

Thank you for listening!