WEAK COMPACTNESS CARDINALS FOR STRONG LOGICS AND SUBTLETY PROPERTIES OF THE CLASS OF ORDINALS

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Abstract. Motivated by recent work of Boney, Dimopoulos, Gitman and Magidor, we characterize the existence of weak compactness cardinals for all abstract logics through combinatorial properties of the class of ordinals. This analysis is then used to show that, in contrast to the existence of strong compactness cardinals, the existence of weak compactness cardinals for abstract logics does not imply the existence of a strongly inaccessible cardinal. More precisely, it is proven that the existence of a proper class of subtle cardinals is consistent with the axioms of ZFC if and only if it is not possible to derive the existence of strongly inaccessible cardinals from the existence of weak compactness cardinals for all abstract logics. Complementing this result, it is shown that the existence of weak compactness cardinals for all abstract logics implies that unboundedly many ordinals are strongly inaccessible in the inner model HOD of all hereditarily ordinal definable sets.

1. Introduction

The theory of strong axioms of infinity and the study of extensions of first-order logic are deeply connected through results showing that the ability to generalize fundamental structural features of first-order logic to stronger logics is equivalent to the existence of large cardinals.[1] A classical example of such a result is a theorem of Magidor in [14] that shows that the existence of a *strong compactness cardinal* for second-order logic $\mathcal{L}^2$ (i.e., the existence of a cardinal κ with the property that every unsatisfiable $\mathcal{L}^2$-theory contains an unsatisfiable subtheory of cardinality less than κ) is equivalent to the existence of an extendible cardinal. In [16], Makowsky proved an analog of this result that provides a large cardinal characterization of the existence of strong compactness cardinals for all abstract logics.[2] Recall that *Vopěnka's Principle* is the scheme of axioms stating that for every proper class[3] of graphs, there are two distinct members of the class with a homomorphism between them. The validity of this combinatorial principle is known to be characterizable through

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[1]Throughout this paper, we refer to properties of cardinal numbers that imply weak inaccessibility as *large cardinal properties*.

[2]The definition of an abstract logic and all related model-theoretic notions can be found in Section 5.

[3]Throughout this paper, we work in ZFC. Therefore, all classes are definable by first-order formulas in the language of set theory, possibly using sets as parameters.

the existence of certain large cardinals (see [1, Section 4] and [21, Section 6]). Makowsky's result now states the following:

Theorem 1.1 ([16, Theorem 2])**.** *The following schemes are equivalent over* ZFC*:*

(1) *Vopěnka's Principle.*
(2) *Every abstract logic has a strong compactness cardinal.*

The work presented in this paper is motivated by recent results of Boney, Dimopoulos, Gitman and Magidor in [5] that deal with the existence of *weak compactness cardinals* for abstract logics, i.e., the existence of cardinals κ with the property that every unsatisfiable theory of cardinality κ contains an unsatisfiable subtheory of cardinality less than κ. These results show that the existence of weak compactness cardinals for strong logics is connected to the combinatorial concept of *subtleness*, introduced by Jensen and Kunen in [11]. Remember that an infinite cardinal κ is *subtle* if for every sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ with $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$ and every closed unbounded subset C of κ, there are $\alpha, \beta \in C$ with $\alpha < \beta$ and $A_\beta \cap \alpha = A_\alpha$. A short argument shows that subtle cardinals are inaccessible limits of totally indescribable cardinals. This large cardinal notion has a canonical class-version, i.e., one defines "Ord *is subtle*" to be the scheme of sentences stating that for every class sequence $\langle A_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with $A_\alpha \subseteq \alpha$ for all $\alpha \in \mathrm{Ord}$ and every closed unbounded class C of ordinals, there exist $\alpha, \beta \in C$ with $\alpha < \beta$ and $A_\beta \cap \alpha = A_\alpha$. The results of [5] now show that, in the presence of a definable well-ordering of V,[4] the validity of this principle is connected to the existence of weak compactness cardinals for abstract logics:

Theorem 1.2 ([5, Theorem 5.3])**.** *The following schemes are equivalent over* ZFC *together with the existence of a definable well-ordering of* V*:*

(1) *Ord is subtle.*
(2) *Every abstract logic has a stationary class of weak compactness cardinals.*

This result immediately raises the following questions:

- Is it also possible to characterize the mere existence of weak compactness cardinals for every abstract logic (instead of the existence of stationary classes of such cardinals) through combinatorial properties of the class of ordinals?
- Is it possible to establish such characterizations in the absence of definable well-orderings of V?

The results of this paper will provide affirmative answers to both of these questions. Moreover, this analysis will be based on arguments showing that the validity of the involved combinatorial principles for the class of ordinals is equivalent to the existence of certain large cardinals.

First, in order to prove results that do not rely on the existence of definable well-orderings, we will make use of the following variation of the above class-principle that was introduced by Bagaria and the author in their work on *principles of structural reflection*:

Definition 1.3 ([2])**.** We let "Ord *is essentially subtle*" denote the scheme of sentences stating that for every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$ and every closed unbounded class C of ordinals, there exist $\alpha, \beta \in C$ and $A \in E_\beta$ with $\alpha < \beta$ and $A \cap \alpha \in E_\alpha$.

Obviously, if Ord is essentially subtle, then Ord is subtle. Moreover, in the presence of a definable well-ordering of V, these principles are equivalent. In particular, since the assumption that Ord is

[4]Note that the existence of such a well-ordering is equivalent to the statement that $\mathrm{V} = \mathrm{HOD}_x$ for some set x and can therefore be formulated as a single sentence in the language of set theory.

subtle is easily seen to be downwards absolute from V to the constructible universe L, it follows that these two principles are equiconsistent over ZFC. In contrast, by combining a result in [5] with Theorem 1.4 below, it is possible to show that, over ZFC, the assumption that Ord is subtle does not imply that Ord is essentially subtle (see Corollary 6.3 below). The following direct strengthening of Theorem 1.2, proven in Section 6 below, now provides the desired ZFC-characterization of the existence of stationary classes of weak compactness cardinals for all abstract logics:

Theorem 1.4. *The following schemes of sentences are equivalent over* ZFC*:*

(1) Ord *is essentially subtle.*
(2) *Every abstract logic has a stationary class of weak compactness cardinals.*

Next, we present a result showing that the statement that every abstract logic has at least one weak compactness cardinal is also equivalent to a combinatorial property of the class Ord. This property generalizes the notion of *faintness*, introduced by Matet in [17], from cardinals to the class Ord. Matet defined a cardinal κ to be *faint* if for every sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ with $A_\alpha \subseteq \alpha$ for all $\alpha < \kappa$ and every $\xi < \kappa$, there are $\xi < \alpha < \beta < \kappa$ with $A_\beta \cap \alpha = A_\alpha$. The results of [17] then show that a cardinal is faint if and only if it is either subtle or a limit of subtle cardinals. Following earlier patterns, we introduce the following class-version of faintness:

Definition 1.5. We let "Ord *is essentially faint*" denote the scheme of sentences stating that for every class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$ and every ordinal ξ, there are ordinals $\xi < \alpha < \beta$ and $A \in E_\beta$ with $A \cap \alpha \in E_\alpha$.

By definition, if Ord is essentially subtle, then Ord is essentially faint. Moreover, it is easy to see that the existence of a proper class of subtle cardinals implies that Ord is essentially faint: Given a class sequence $\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle$ with $\emptyset \neq E_\alpha \subseteq \mathcal{P}(\alpha)$ for all $\alpha \in \mathrm{Ord}$ and an ordinal ξ, the given assumption allows us to pick a subtle cardinal $\kappa > \xi$ and we can then use the Axiom of Choice to find a sequence $\langle A_\alpha \mid \alpha < \kappa\rangle$ with $A_\alpha \in E_\alpha$ for all $\alpha < \kappa$. The subtleness of κ then ensures that the closed unbounded subset (ξ, κ) of κ contains ordinals $\alpha < \beta$ with $A_\alpha = A_\beta \cap \alpha$. In particular, it follows that there are ordinals $\xi < \alpha < \beta$ and $A \in E_\beta$ with $A \cap \alpha \in E_\alpha$.

The following result, again proven in Section 6, shows that this principle exactly determines the combinatorial property of Ord needed for the desired characterization:

Theorem 1.6. *The following schemes of sentences are equivalent over* ZFC*:*

(1) Ord *is essentially faint.*
(2) *Every abstract logic has a weak compactness cardinal.*

As mentioned earlier, the proofs of the above theorems rely heavily on characterizations of the involved combinatorial class-principles through large cardinal assumptions. This analysis, carried out in Sections 2 and 3 below, can also be used to show that the ability to generalize other structural properties of first-order logic to all abstract logics can also be characterized through the validity of the studied combinatorial principles for Ord. More specifically, motivated by an unpublished result of Stavi that provides an analog to Theorem 1.1 for the existence of *Löwenheim–Skolem–Tarski numbers* for abstract logics (see Theorem 5.4 below), we will prove analogs of Theorems 1.4 and 1.6 that characterize the existence of *strict Löwenheim–Skolem–Tarski numbers*, studied by Bagaria and Väänänen in [3], through subtlety properties of Ord (see Theorem 5.5).

The results presented above naturally raise questions about the relationship between the principle "Ord *is essentially subtle*" and the principle "Ord *is essentially faint*". Trivially, the former principle implies the latter. However, it is not obvious whether these principles are equivalent.

Somewhat surprisingly, the next result shows that the inequivalence of the two principles has consistency strength strictly greater than the consistency strength of the individual principles. Moreover, this result uses the inequivalence of the two principles to derive the existence of certain large cardinals not only in the constructible universe L, but also in the class HOD of all hereditarily ordinal definable sets.

Theorem 1.7. *If* Ord *is essentially faint and only boundedly many cardinals are subtle in* HOD*, then* Ord *is essentially subtle.*

In particular, in the unlikely case that the axioms of ZFC prove that there are only set-many subtle cardinals, the above theorem implies that these axioms also prove that the principle "Ord *is essentially subtle*" is equivalent to the principle "Ord *is essentially faint*". Our later results will show that the converse of this implication also holds, i.e., if the axioms of ZFC are consistent with the existence of a proper class of subtle cardinals, then the given principles are not provably equivalent (see Theorem 1.10 below).

Since the existence of a subtle cardinal clearly implies the existence of a set-sized model of ZFC in which Ord is essentially subtle, Theorem 1.7 directly yields the following equiconsistency:

Corollary 1.8. *The following theories are equiconsistent:*

(1) ZFC + "Ord *is essentially subtle*".
(2) ZFC + "Ord *is essentially faint*". □

Motivated by Theorem 1.7, we now consider the possibility of separating the principle "Ord *is essentially faint*" from the principle "Ord *is essentially subtle*". Our results will show that this question is closely related to questions about the existence of sentences in the language of set theory that imply all sentences appearing in the given schemes of sentences. To motivate these results, we will first observe that, over ZFC, the principle "Ord *is essentially subtle*" cannot be derived from a single consistent sentence. In Section 2, we will show that this observation directly follows from the results of [2, Section 9] that show that for every natural number n, the assumption that Ord is essentially subtle implies the existence of class-many strongly inaccessible cardinals κ with the property that V_κ is a Σ_n-elementary submodel of V.

Proposition 1.9. *If ϕ is a sentence in the language of set theory with the property that* ZFC + ϕ *is consistent, then*

$$\text{ZFC} + \phi \nvdash \text{"Ord } \textit{is essentially subtle}\text{"}.$$

In contrast, assuming the consistency of sufficiently strong large cardinal assumptions, it is easy to find consistent sentences that provably imply that Ord is essentially faint. As observed above, the statement that there is a proper class of subtle cardinals is an example of a sentence with this property. Our further analysis will yield additional examples of such sentences. In particular, we will produce two incompatible sentences with the given property (see Theorem 1.14 below). The following theorem now shows that the existence of such a consistent sentence is actually equivalent to the possibility of separating the two principles. Moreover, it shows that Theorem 1.7 already provides the correct consistency strength for the inequality of these principles.

Theorem 1.10. *The following statements are equivalent:*

(1) ZFC + "Ord *is essentially faint*" $\nvdash$ "Ord *is essentially subtle*".
(2) *There exists a sentence ϕ in the language of set theory such that the theory* ZFC + ϕ *is consistent and*

$$\text{ZFC} + \phi \vdash \text{"Ord } \textit{is essentially faint}\text{"}.$$

(3) *The theory*

$$\mathrm{ZFC} + \text{“}\textit{There is a proper class of subtle cardinals}\text{”}$$

is consistent.

Next, we consider the possibility of axiomatizing the validity of the given principles by single sentences. Proposition 1.9 already shows that this is not possible for the principle stating that Ord is essentially subtle. A short argument in Section 4 will prove the following result that provides the same conclusion for the principle “Ord *is essentially faint*”.

Corollary 1.11. *If the theory*

$$\mathrm{ZFC} + \text{“Ord }\textit{is essentially faint}\text{”}$$

is consistent, then no sentence ϕ in the language of set theory satisfies the following statements:

(1) $\mathrm{ZFC} + \text{“Ord }\textit{is essentially faint}\text{”} \vdash \phi$.
(2) $\mathrm{ZFC} + \phi \vdash \text{“Ord }\textit{is essentially faint}\text{”}$.

We now consider the question whether the validity of the scheme “Ord *is essentially subtle*” can be finitely axiomatized over the theory ZFC+“Ord *is essentially faint*”. The following observation, proven in Section 4, motivates the formulation of the subsequent theorem:

Proposition 1.12. *There exists a theory* T *in the language of set theory such that the following statements hold:*

(1) $\mathrm{ZFC} + \text{“Ord }\textit{is essentially subtle}\text{”} \vdash \mathrm{T}$.
(2) $\mathrm{ZFC} + \text{“Ord }\textit{is essentially faint}\text{”} + \mathrm{T} \vdash \text{“Ord }\textit{is essentially subtle}\text{”}$.
(3) *For every sentence ϕ in* T*, we have*

$$\mathrm{ZFC} + \neg\phi \vdash \text{“Ord }\textit{is essentially faint}\text{”}.$$

It is now natural to ask whether the theory T appearing in the above proposition can be replaced by a single sentence. The next result, also proven in Section 4, shows that the answer to this question depends on the consistency of the class-version of the large cardinal property of being a subtle limit of subtle cardinals:

Theorem 1.13. *The following statements are equivalent:*

(1) *There is no sentence ϕ in the language of set theory satisfying the following statements:*
   (a) $\mathrm{ZFC} + \text{“Ord }\textit{is essentially subtle}\text{”} \vdash \phi$.
   (b) $\mathrm{ZFC} + \text{“Ord }\textit{is essentially faint}\text{”} + \phi \vdash \text{“Ord }\textit{is essentially subtle}\text{”}$.
   (c) $\mathrm{ZFC} + \neg\phi \vdash \text{“Ord }\textit{is essentially faint}\text{”}$.
(2) *There is no sentence ϕ in the language of set theory satisfying the above statements* (1a) *and* (1b).
(3) *The theory*

$$\mathrm{ZFC} + \text{“}\textit{There is a proper class of subtle cardinals}\text{”} + \text{“Ord }\textit{is essentially subtle}\text{”}$$

is consistent.

The final topic that will be treated in this paper is the relationship between the existence of weak compactness cardinals for abstract logics and the existence of strongly inaccessible cardinals. By combining Theorem 1.1 with the results of [1, Section 4], it is easy to see that the existence of strong compactness cardinals for all abstract logics implies the existence of a proper class of strongly inaccessible cardinals. Moreover, a combination of Theorem 1.4 with results in [2, Section

9] (see Theorem 2.4 below) shows that the same conclusion follows from the assumption that every abstract logic has a stationary class of weak compactness cardinals. In contrast, it turns out that the existence of weak compactness cardinals for abstract logics is compatible with the non-existence of strongly inaccessible cardinals. The corresponding arguments will also allow us to show that there can be incompatible sentences witnessing (2) in Theorem 1.10.

Theorem 1.14. *The following statements are equivalent:*

(1) *The theory*

$$\text{ZFC} + \text{“}\textit{There is a proper class of subtle cardinals}\text{”}$$

is consistent.

(2) *The theory*

$$\text{ZFC} + \text{“Ord } \textit{is essentially faint}\text{”} + \text{“}\textit{There are no strongly inaccessible cardinals}\text{”}$$

is consistent.

(3) *There is a sentence ϕ in the language of set theory such that* $\text{ZFC} + \phi$ *is consistent and*

$$\text{ZFC} + \phi \vdash \text{“Ord } \textit{is essentially faint}\text{”} + \text{“}\textit{There are no strongly inaccessible cardinals}\text{”}.$$

While Theorem 1.14 shows that the existence of weak compactness cardinals for all abstract logics does not imply the existence of strongly inaccessible cardinals, our analysis allows us to conclude that the given assumption at least implies the existence of strongly inaccessible cardinals in the inner model HOD:

Theorem 1.15. *If* Ord *is essentially faint, then unboundedly many ordinals are strongly inaccessible cardinals in* HOD.

Note that the proof of this result will actually show that strongly inaccessible cardinals with certain strong combinatorial properties exist in HOD (see Corollary 4.5 below).

2. $C^{(n)}$-strongly unfoldable cardinals

In this section, we review the large cardinal characterization of the principle “Ord *is essentially subtle*” derived in [2, Section 9]. The starting point of these results is the notion of *strongly unfoldable cardinal* introduced by Villaveces in [22]. Villaveces defined an inaccessible cardinal κ to be *strongly unfoldable* if for every ordinal λ and every transitive ZF^--model M of cardinality κ with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $\text{V}_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\text{crit}\,(j) = \kappa$ and $j(\kappa) \geq \lambda$. In [2], Bagaria and the author introduced natural strengthenings of this definition that demand the existence of embeddings into models with stronger correctness properties. Following [1], for every natural number n, we let $C^{(n)}$ denote the class of all Σ_n-*correct* ordinals, i.e., the class of all ordinals α with the property that V_α is a Σ_n-elementary submodel of the set-theoretic universe V, denoted by $\text{V}_\alpha \prec_{\Sigma_n} \text{V}$. It is then easy to show that for each natural number n, the class $C^{(n)}$ is a closed unbounded class of ordinals that is definable by a Π_n-formula without parameters. Moreover, short arguments show that $C^{(0)}$ is the class of all ordinals and $C^{(1)}$ is the class of all cardinals κ satisfying $\text{H}(\kappa) = \text{V}_\kappa$. In particular, there is a sentence φ in the language of set theory with the property that $C^{(1)}$ is equal to the class of all ordinals α with the property that φ holds in V_α.

Definition 2.1 ([2])**.** Given a natural number n, an inaccessible cardinal κ is $C^{(n)}$-*strongly unfoldable* if for every ordinal $\lambda \in C^{(n)}$ greater than κ and every transitive ZF^--model M of cardinality κ

with $\kappa \in M$ and ${}^{<\kappa}M \subseteq M$, there is a transitive set N with $\mathrm{V}_\lambda \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \lambda$ and $\mathrm{V}_\lambda \prec_{\Sigma_n} \mathrm{V}^N_{j(\kappa)}$.

The following basic properties of these large cardinal notions were established in [2, Section 9]:

Proposition 2.2 ([2])**.** (1) *A cardinal is strongly unfoldable if and only if it is $C^{(n)}$-strongly unfoldable for some natural number $n < 2$.*
(2) *Given natural numbers $m < n$, every $C^{(n)}$-strongly unfoldable cardinal is $C^{(m)}$-strongly unfoldable.*
(3) *For every natural number n, all $C^{(n)}$-strongly unfoldable cardinals are elements of $C^{(n+1)}$.*

In [19], Rathjen defined a cardinal κ to be *shrewd* if for every formula $\Phi(v_0, v_1)$ in the language of set theory, every ordinal $\gamma > \kappa$ and every $A \subseteq \mathrm{V}_\kappa$ with the property that $\Phi(A, \kappa)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ such that $\Phi(A \cap \mathrm{V}_\alpha, \alpha)$ holds in V_β. The results of [12] then show that the notions of shrewdness and strong unfoldability coincide. The proof of this equivalence relies on an embedding characterization of shrewdness in [13] that closely resembles Magidor's classical characterization of supercompactness in [14]. The following result establishes analogous equivalences for the notion of $C^{(n)}$-strong unfoldability (see [2, Theorem 9.4]):

Lemma 2.3 ([2])**.** *Given a natural number $n > 0$, the following statements are equivalent for every cardinal κ:*
(1) *The cardinal κ is $C^{(n)}$-strongly unfoldable.*
(2) *For every $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$, every ordinal $\gamma \in C^{(n)}$ greater than κ, and every subset A of V_κ with the property that $\varphi(\kappa, A)$ holds in V_γ, there exist ordinals $\alpha < \beta < \kappa$ with $\beta \in C^{(n)}$ and the property that $\varphi(\alpha, A \cap V_\alpha)$ holds in V_β.*
(3) *For every ordinal $\gamma \in C^{(n)}$ greater than κ and every $z \in V_\gamma$ there exists an ordinal $\bar{\gamma} \in C^{(n)} \cap \kappa$, a cardinal $\bar{\kappa} < \bar{\gamma}$, an elementary submodel X of $V_{\bar{\gamma}}$ with $V_{\bar{\kappa}} \cup \{\bar{\kappa}\} \subseteq X$, and an elementary embedding $j : X \longrightarrow V_\gamma$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$.*

Analogous to the characterization of Vopěnka's Principle through the existence of $C^{(n)}$-extendible cardinals provided by the results of [1, Section 4], it turns out that the existence $C^{(n)}$-strongly unfoldable cardinals for all natural numbers n characterizes the validity of the principle "Ord *is essentially subtle*" (see [2, Theorem 1.17]).

Theorem 2.4 ([2])**.** *The following schemes of axioms are equivalent over* ZFC*:*
(1) Ord *is essentially subtle.*
(2) *For every natural number n, there exists a $C^{(n)}$-strongly unfoldable cardinal.*
(3) *For every natural number n, there exists a proper class of $C^{(n)}$-strongly unfoldable cardinals.*

The above results now allow us to show that no consistent sentence can imply that Ord is essentially subtle. Note that an almost identical argument shows that Vopěnka's Principle is not the consequence of a single consistent sentence.

Proof of Proposition 1.9. Assume, towards a contradiction, that there is a sentence ϕ in the language of set theory such that the theory $\mathrm{ZFC} + \phi$ is consistent and proves that Ord is essentially subtle. Work in a model of $\mathrm{ZFC} + \phi$. By combining Theorem 2.4 with Proposition 2.2.(3), we can now find an inaccessible cardinal κ with the property that ϕ holds in V_κ. Let κ be minimal with this property. Since V_κ is a model of $\mathrm{ZFC} + \phi$, we can repeat this argument in V_κ and find an inaccessible cardinal $\mu < \kappa$ such that ϕ holds in V_μ, contradicting the minimality of κ. □

3. $C^{(n)}$-Weakly shrewd cardinals

In this section, we will formulate and prove an analog of Theorem 2.4 for the principle "Ord *is essentially faint*". The starting point of this analysis is the notion of *weak shrewdness* introduced in [13] as a natural weakening of shrewdness. This large cardinal property still entails many of the interesting combinatorial properties of strongly unfoldable cardinals, but it no longer implies that the given cardinal is strongly inaccessible or an element of $C^{(2)}$. In [13], a cardinal κ is defined to be *weakly shrewd* if for every formula $\varphi(v_0, v_1)$ in the language of set theory, every cardinal $\theta > \kappa$ and every subset z of κ with the property that $\varphi(z, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist cardinals $\bar{\kappa} < \bar{\theta}$ such that $\bar{\kappa} < \kappa$ and $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$. The results of [13] show that when we modify this definition to also demand that we can find cardinals $\bar{\theta}$ with the given property below κ (as it is the case in the definition of shrewdness), we obtain a strictly stronger large cardinal property. In particular, the canonical argument showing that shrewd cardinal are strongly inaccessible cannot be adapted to weakly shrewd cardinals.

The results of [13, Section 3] show that every shrewd cardinal is weakly shrewd and every weakly shrewd cardinal is weakly Mahlo. Moreover, these results show that, starting from a subtle cardinal, it is possible to construct a model of set theory that contains a weakly shrewd cardinal that is not strongly inaccessible and therefore also not shrewd. Moreover, it is shown that weakly shrewd cardinals can even exist below the cardinality of the continuum. The results of this paper will show that these consistency results make use of the correct large cardinal hypothesis (see Lemma 4.3 below). Motivated by Lemma 2.3, we strengthen the notion of weak shrewdness in the same way as the transition to $C^{(n)}$-strong unfoldability strengthens the notions of shrewdness.

Definition 3.1. Given a natural number n, a cardinal κ is *$C^{(n)}$-weakly shrewd* if for every formula $\varphi(v_0, v_1)$ in the language of set theory, every cardinal $\kappa < \theta \in C^{(n)}$ and every subset z of κ with the property that $\varphi(z, \kappa)$ holds in $\mathrm{H}(\theta)$, there exist a cardinal $\bar{\theta} \in C^{(n)}$ and a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$ such that $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$.

We now derive some basic properties of this newly defined notion:

Proposition 3.2. (1) *Given natural numbers $m < n$, every $C^{(n)}$-weakly shrewd cardinal is $C^{(m)}$-weakly shrewd.*

(2) *A cardinal is weakly shrewd if and only if it is $C^{(n)}$-weakly shrewd for some $n < 2$.*

Proof. (1) Let κ be a $C^{(n)}$-weakly shrewd cardinal. Fix a formula $\varphi(v_0, v_1)$ in the language of set theory, a cardinal $\kappa < \theta \in C^{(m)}$ and $z \subseteq \kappa$ such that $\varphi(z, \kappa)$ holds in $\mathrm{H}(\theta)$. Pick a cardinal $\theta < \vartheta \in C^{(n)}$. Then, in $\mathrm{H}(\vartheta)$, there exists a cardinal $\kappa < \nu \in C^{(m)}$ with the property that $\varphi(z, \kappa)$ holds in $\mathrm{H}(\nu)$. By our assumptions on κ, we can now find a cardinal $\bar{\vartheta} \in C^{(n)}$ and a cardinal $\bar{\kappa} < \min(\kappa, \bar{\vartheta})$ such that, in $\mathrm{H}(\bar{\vartheta})$, there exists a cardinal $\bar{\kappa} < \bar{\nu} \in C^{(m)}$ with the property that $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\nu})$. Since $\bar{\vartheta} \in C^{(n)}$ and $n > 0$, we then know that, in V, the ordinal $\bar{\nu}$ is a cardinal in $C^{(m)}$ with the property that $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\nu})$.

(2) By definition and (1), it suffices to show that every weakly shrewd cardinal is $C^{(1)}$-weakly shrewd. Let κ be a weakly shrewd cardinal, let $\varphi(v_0, v_1)$ be a formula in the language of set theory, let $\kappa < \theta \in C^{(1)}$ be a cardinal and let z be a subset of κ with the property that $\varphi(z, \kappa)$ holds in $\mathrm{H}(\theta)$. Since θ is an element of $C^{(1)}$, we then know that, in $\mathrm{H}(\theta)$, for every ordinal α, the class V_α is a set. Our assumption then allows us to find a cardinal $\bar{\theta}$ and a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$ with the property that, in $\mathrm{H}(\bar{\theta})$, the statement $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds and for every ordinal α, the class V_α is a set. This completes the proof, because this conclusion ensures that $\bar{\theta}$ is an element of $C^{(1)}$. □

In combination with [13, Proposition 3.3], the above proposition shows that all $C^{(n)}$-weakly shrewd cardinals are weakly Mahlo. In particular, if a $C^{(n)}$-weakly shrewd cardinal is an element of $C^{(1)}$, then it is strongly inaccessible. Motivated by [13, Lemma 3.1], we now prove an analog of Lemma 2.3 for $C^{(n)}$-weak shrewdness.

Lemma 3.3. *Given a natural number $n > 0$, the following statements are equivalent for every cardinal κ:*

(1) *The cardinal κ is $C^{(n)}$-weakly shrewd.*
(2) *For every cardinal $\kappa < \theta \in C^{(n)}$ and every $y \in \mathrm{H}(\theta)$, there exists a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $y \in \mathrm{ran}(j)$.*

Proof. First, assume that (1) holds, $\kappa < \theta \in C^{(n)}$ is a cardinal and y is an element of $\mathrm{H}(\theta)$. Pick a cardinal $\theta < \vartheta \in C^{(n)}$ and an elementary submodel Y of $\mathrm{H}(\theta)$ of cardinality κ with the property that $\kappa \cup \{\kappa, y\} \subseteq Y$. In addition, fix a bijection $b : \kappa \longrightarrow Y$ satisfying $b(0) = \kappa$, $b(1) = \langle \kappa, y \rangle$ and $b(\omega \cdot (1 + \alpha)) = \alpha$ for all $\alpha < \kappa$. Finally, define z to be the set of all elements of κ of the form $\prec \ell, \alpha_0, \ldots, \alpha_{n-1} \succ$[5] with the property that $\ell < \omega$ is the Gödel number of a set-theoretic formula with n free variables, $\alpha_0, \ldots, \alpha_{n-1} < \kappa$ and $\mathsf{Sat}(Y, \ell, \langle b(\alpha_0), \ldots, b(\alpha_{n-1}) \rangle)$ holds, where Sat denotes the formalized satisfaction relation (see, for example, [7, Section I.9]).

By our assumption and the fact that Sat is uniformly Δ_1-definable over models of ZF^-, we can now find a cardinal $\bar{\vartheta} \in C^{(n)}$ and a cardinal $\bar{\kappa} < \min(\kappa, \bar{\vartheta})$ with the property that, in $\mathrm{H}(\bar{\vartheta})$, there exists a cardinal $\bar{\kappa} < \bar{\theta} \in C^{(n)}$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ of cardinality $\bar{\kappa}$ with $\bar{\kappa} + 1 \subseteq X$ and a bijection $a : \bar{\kappa} \longrightarrow X$ with $a(0) = \bar{\kappa}$, $a(\omega \cdot (1 + \alpha)) = \alpha$ for all $\alpha < \bar{\kappa}$ and the property that the set $z \cap \bar{\kappa}$ consists of all elements of κ of the form $\prec \ell, \alpha_0, \ldots, \alpha_{n-1} \succ$ such that $\ell < \omega$ is the Gödel number of a set-theoretic formula with n free variables, $\alpha_0, \ldots, \alpha_{n-1} < \bar{\kappa}$ and $\mathsf{Sat}(X, \ell, \langle a(\alpha_0), \ldots, a(\alpha_{n-1}) \rangle)$ holds. Since $\bar{\vartheta} \in C^{(n)}$, we now know that, in V, the ordinal $\bar{\theta}$ is a cardinal in $C^{(n)}$ and the set X is an elementary submodel of $\mathrm{H}(\bar{\theta})$. If we now define

$$j \;=\; b \circ a^{-1} : X \longrightarrow \mathrm{H}(\theta),$$

then it is easy to check that j is an elementary embedding with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $y \in \mathrm{ran}(j)$.

Now, assume that (2) holds, $\varphi(v_0, v_1)$ is a formula in the language of set theory, $\kappa < \theta \in C^{(n)}$ is a cardinal and z is a subset of κ with the property that $\varphi(z, \kappa)$ holds in $\mathrm{H}(\theta)$. Our assumptions then allow us to find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$. We then know that $z \cap \bar{\kappa} \in X$ and $j(z \cap \bar{\kappa}) = z$. But, this shows that $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in both X and $\mathrm{H}(\bar{\theta})$. □

Next, we observe that the existence of subtle cardinals entails the existence of many $C^{(n)}$-weakly shrewd cardinals:

Lemma 3.4. *Given a natural number $n > 0$, every subtle cardinal is a stationary limit of strongly inaccessible $C^{(n)}$-weakly shrewd cardinals.*

Proof. Assume, towards a contradiction, that there is a subtle cardinal δ and a closed unbounded subset C of δ that consists of uncountable cardinals that are not $C^{(n)}$-weakly shrewd. Given $\gamma \in C$, we can use Lemma 3.3 to find $\ell_\gamma < \omega$ and $z_\gamma \subseteq \gamma$ such that ℓ_γ is the Gödel number of a set-theoretic formula $\varphi(v_0, v_1)$ with the property that there exists a cardinal $\gamma < \theta \in C^{(n)}$ such that $\varphi(z, \gamma)$ holds

[5] We use $\prec \cdot, \ldots, \cdot \succ$ to denote (iterated applications of) the *Gödel pairing function*.

in $\mathrm{H}(\theta)$ and for every cardinal $\bar{\theta} \in C^{(n)}$, there is no cardinal $\bar{\gamma} < \min(\gamma, \bar{\theta})$ with the property that $\varphi(z \cap \bar{\gamma}, \bar{\gamma})$ holds in $\mathrm{H}(\bar{\theta})$. Now, let $\langle E_\gamma \mid \gamma \in C \rangle$ denote the unique sequence with

$$E_\gamma \;=\; \prec 0, \ell_\gamma \succ \;\cup\; \{\prec 1, \beta \succ \mid \beta \in z_\gamma\}$$

for all $\gamma \in C$. Using the subtlety of δ, we can now find $\beta < \gamma$ in C with $E_\gamma \cap \beta = E_\beta$. We then know that $\ell_\beta = \ell_\gamma$ and $z_\gamma \cap \beta = z_\beta$. But, this contradicts the choice of ℓ_γ and z_γ, because, if $\varphi(v_0, v_1)$ is the formula coded by ℓ_γ, then β is a cardinal below γ with the property that there exists $\beta < \theta \in C^{(n)}$ such that $\varphi(z_\gamma \cap \beta, \beta)$ holds in $\mathrm{H}(\theta)$.

Since subtle cardinals are regular limit points of the class $C^{(1)}$, the above computations show that subtle cardinals are stationary limits of $C^{(n)}$-weakly shrewd cardinals that are elements of $C^{(1)}$. This yields the statement of the lemma, because, as observed earlier, $C^{(n)}$-weakly shrewd cardinals are inaccessible if and only if they are elements of $C^{(1)}$. □

We close this section by proving the desired characterization of the principle "Ord *is essentially faint*" through the existence of $C^{(n)}$-weakly shrewd cardinals:

Theorem 3.5. *The following schemes of axioms are equivalent over* ZFC*:*

(1) Ord *is essentially faint.*
(2) *For every natural number n, there exists a proper class of $C^{(n)}$-weakly shrewd cardinals.*

Proof. First, assume, towards a contradiction, that (1) holds, $n > 0$ is a natural number and there exists an ordinal ξ with the property that there are no $C^{(n)}$-weakly shrewd cardinals above ξ. Then there is a unique class function E with domain Ord such that the following statements hold:

- $E(\alpha) = \mathcal{P}(\alpha)$ for all ordinal $\alpha \leq \xi$.
- $E(\alpha + 1) = \{\{\prec 0, \alpha \succ\}\}$ for all ordinals $\alpha \geq \xi$.
- If $\alpha > \xi$ is a singular limit ordinal, then $E(\alpha)$ consists of all subsets of α of the form

  $$\{\prec 1, \mathrm{cof}(\alpha) \succ\} \;\cup\; \{\prec 2, \eta, c(\eta) \succ \mid \eta < \mathrm{cof}(\alpha)\},$$

  where $c : \mathrm{cof}(\alpha) \longrightarrow \alpha$ is a strictly increasing cofinal function.
- If $\alpha > \xi$ is a regular cardinal, then $E(\alpha)$ consists of all subsets of α of the form

  $$\{\prec 3, \ell \succ\} \;\cup\; \{\prec 4, \eta \succ \mid \eta \in z\},$$

  where ℓ is the Gödel number of a set-theoretic formula $\varphi(v_0, v_1)$ and z is a subset of α with the property that there exists a cardinal $\alpha < \theta \in C^{(n)}$ such that $\varphi(z, \alpha)$ holds in $\mathrm{H}(\theta)$ and for every cardinal $\bar{\theta} \in C^{(n)}$, there is no cardinal $\bar{\alpha} < \min(\alpha, \bar{\theta})$ with the property that $\varphi(z \cap \bar{\alpha}, \bar{\alpha})$ holds in $\mathrm{H}(\bar{\theta})$.

Our assumption then ensures that $E(\alpha) \neq \emptyset$ holds for all $\alpha \in \mathrm{Ord}$. Therefore, we can find ordinals $\xi < \alpha < \beta$ and $A \in E(\beta)$ with $A \cap \alpha \in E(\alpha)$. The definition of E then ensures that α and β are both regular cardinals. Fix a set-theoretic formula $\varphi(v_0, v_1)$ and a subset z of β such that $A = \{\prec 3, \ell \succ\} \cup \{\prec 4, \eta \succ \mid \eta \in z\}$, where ℓ is the Gödel number of $\varphi(v_0, v_1)$. Since

$$A \cap \alpha \;=\; \{\prec 3, \ell \succ\} \cup \{\prec 4, \eta \succ \mid \eta \in z \cap \alpha\} \;\in\; E(\alpha),$$

we now know that there exists a cardinal $\alpha < \theta \in C^{(n)}$ with the property that $\varphi(z \cap \alpha, \alpha)$ holds in $\mathrm{H}(\theta)$. Since $\alpha < \min(\beta, \theta)$, this contradicts the fact that A is an element of $E(\beta)$.

Now, assume that (2) holds and E is a class function with domain Ord and the property that $\emptyset \neq E(\alpha) \subseteq \mathcal{P}(\alpha)$ holds for all $\alpha \in \mathrm{Ord}$. Fix an ordinal ξ. Pick a natural number $n > 0$ such that there exists a Σ_n-formula $\varphi(v_0, v_1, v_2)$ and a parameter y defining E. By our assumption, there is a $C^{(n)}$-weakly shrewd cardinal $\kappa > \xi$ with $y \in \mathrm{H}(\kappa)$. Pick $A \in E(\kappa)$ and a cardinal $\kappa < \theta \in C^{(n)}$.

Then Lemma 3.3 yields a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $A, y, \xi \in \mathrm{ran}(j)$. We then know that $\xi < \bar{\kappa}$, $y \in \mathrm{H}(\bar{\kappa}) \cap X$ with $j(y) = y$ and $A \cap \bar{\kappa} \in X$ with $j(A \cap \bar{\kappa}) = A$. Moreover, the fact that $\theta \in C^{(n)}$ ensures that, in $\mathrm{H}(\theta)$, there exists a set $\mathcal{A}$ such that $A \in \mathcal{A}$ and $\varphi(\kappa, \mathcal{A}, y)$ holds. This implies that, in $\mathrm{H}(\bar{\theta})$, there exists a set $\bar{\mathcal{A}}$ such that $A \cap \bar{\kappa} \in \bar{\mathcal{A}}$ and $\varphi(\bar{\kappa}, \bar{\mathcal{A}}, y)$ holds. Since $\bar{\theta} \in C^{(n)}$, we then know that $\varphi(\bar{\kappa}, \bar{\mathcal{A}}, y)$ also holds in V and we can conclude that $A \cap \bar{\kappa} \in \bar{\mathcal{A}} = E(\bar{\kappa})$. □

4. Separating the schemes

The goal of this section is to identify scenarios in which the principle "Ord *is essentially faint*" holds while the principle "Ord *is essentially subtle*" fails. In the light of Theorems 2.4 and 3.5, this task is closely connected to the analysis of $C^{(n)}$-weakly shrewd cardinals that are not $C^{(n)}$-strongly unfoldable. For the case $n = 1$, this analysis is provided by [13, Lemma 3.5], which shows that a weakly shrewd cardinal is strongly unfoldable if and only if it is an element of $C^{(2)}$. The following lemma provides a direct analog of this result for $C^{(n)}$-weakly shrewd cardinals:

Lemma 4.1. *Given a natural number $n > 0$, the following statements are equivalent for every $C^{(n)}$-weakly shrewd cardinal:*

1. *The cardinal κ is $C^{(n)}$-strongly unfoldable.*
2. *The cardinal κ is an element of $C^{(n+1)}$.*
3. *There is no ordinal $\varepsilon \geq \kappa$ with the property that the set $\{\varepsilon\}$ is definable by a Σ_{n+1}-formula with parameters in $\mathrm{H}(\kappa)$.*

Proof. First, assume that (1) fails and (3) holds. Given $\alpha < \kappa$, the fact that the set $\{|\mathrm{V}_\alpha|\}$ is definable by a Σ_2-formula with parameter α allows us to use our assumption to conclude that $|\mathrm{V}_\alpha| < \kappa$ holds. In particular, we know that the set V_κ has cardinality κ in this setting. Since (1) fails, Lemma 2.3 shows that there is a set-theoretic formula $\varphi(v_0, v_1)$, a cardinal $\kappa < \theta \in C^{(n)}$ and a subset A of V_κ such that $\varphi(A, \kappa)$ holds in $\mathrm{H}(\theta)$ and there is no cardinal $\bar{\theta} \in C^{(n)} \cap \kappa$ such that $\varphi(A \cap \mathrm{V}_{\bar{\kappa}}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$ for some cardinal $\bar{\kappa} < \bar{\theta}$. Using the fact that κ is weakly inaccessible and the set V_κ has cardinality κ, we can now find a set-theoretic formula $\psi(v_0, v_1)$ and a subset z of κ with the property that $\psi(z, \kappa)$ holds in $\mathrm{H}(\theta)$ and there is no cardinal $\bar{\theta} \in C^{(n)} \cap \kappa$ such that $\psi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$ for some cardinal $\bar{\kappa} < \bar{\theta}$. Since κ is $C^{(n)}$-weakly shrewd, we can find a cardinal $\bar{\theta} \in C^{(n)}$ and a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$ such that $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\bar{\theta})$. Now, define ε to be the least element of $C^{(n)}$ above $\bar{\kappa}$ with the property that $\varphi(z \cap \bar{\kappa}, \bar{\kappa})$ holds in $\mathrm{H}(\varepsilon)$. We then know that $\varepsilon \geq \kappa$ and it is easy to see that the set $\{\varepsilon\}$ is definable by a Σ_{n+1}-formula with parameters $\bar{\kappa}, z \cap \bar{\kappa} \in \mathrm{H}(\kappa)$. This contradicts (3) and we can conclude that (3) implies (1).

These computations complete the proof of the lemma, because (2) obviously implies (3) and Proposition 2.2.(3) shows that (1) implies (2). □

Corollary 4.2. *Given a natural number $n > 0$, every subtle cardinal that is not a limit of subtle cardinals is a stationary limit of strongly inaccessible $C^{(n)}$-weakly shrewd cardinals that are not $C^{(n)}$-strongly unfoldable.*

Proof. Let δ be a subtle cardinal that is not a limit of subtle cardinals and let $\gamma < \delta$ be an ordinal with the property that the interval (γ, δ) does not contain subtle cardinals. Then the set $\{\delta\}$ can be defined by a Σ_2-formula with parameter γ. In this situation, Lemma 4.1 shows that no $C^{(n)}$-weakly shrewd cardinal in the interval (γ, δ) is $C^{(n)}$-strongly unfoldable. An application of Lemma 3.4 now yields the statement of the corollary. □

The next lemma provides us with a tool to derive non-trivial consistency strength from the existence of $C^{(n)}$-weakly shrewd cardinals that are not $C^{(n)}$-strongly unfoldable. An interesting aspect of this result is the fact that, in contrast to most results that provide lower bounds for the consistency strength of certain assumptions, it establishes the existence of large cardinals not only in certain fine-structural inner models (e.g., in the constructible universe L) but also in the inner model HOD. Since the results of this paper show that the given assumption is compatible with the non-existence of strongly inaccessible cardinals, the following lemma reveals that the existence of weakly shrewd cardinals that are not strongly unfoldable has interesting and non-trivial consequences for the behavior of ordinal definable sets.

Lemma 4.3. *Let $n > 0$ be a natural number and let κ be a $C^{(n)}$-weakly shrewd cardinal that is not $C^{(n)}$-strongly unfoldable. If δ is the least ordinal $\varepsilon \geq \kappa$ with the property that the set $\{\varepsilon\}$ is definable by a Σ_{n+1}-formula with parameters in $\mathrm{H}(\kappa)$, then the following statements hold:*

(1) *δ is a cardinal greater than κ with $\mathrm{cof}(\delta) \in \kappa \cup \{\delta\}$.*
(2) *δ is a subtle cardinal in HOD.*

Proof. Fix a Σ_{n+1}-formula $\varphi(v_0, v_1)$ and an element y of $\mathrm{H}(\kappa)$ with the property that δ is the unique set x such that $\varphi(x, y)$ holds. Pick a cardinal $\kappa < \theta \in C^{(n+1)}$ and use Lemma 3.3 to find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $y, \delta \in \mathrm{ran}(j)$. Then $y \in \mathrm{H}(\bar{\kappa})$ with $j(y) = y$. Pick $\varepsilon \in X$ with $j(\varepsilon) = \delta$. Since $\varphi(\delta, y)$ holds in $\mathrm{H}(\theta)$, we then know that $\varphi(\varepsilon, y)$ holds in $\mathrm{H}(\bar{\theta})$ and the fact that $\bar{\theta}$ is an element of $C^{(n)}$ implies that $\varphi(\varepsilon, y)$ also holds in V. This shows that $\delta = \varepsilon$ and hence $\delta > \kappa$. Next, note that the set $\{|\delta|\}$ is Σ_2-definable from the parameter δ and, since $|\delta| \geq \kappa$, the minimality of δ implies that δ is a cardinal. Finally, since the set $\{\mathrm{cof}(\delta)\}$ is Σ_2-definable from the parameter δ, the minimality of δ also ensures that either δ is regular or $\mathrm{cof}(\delta) < \kappa$ holds.

Now, assume that δ is not a subtle cardinal in HOD. Let $\langle \vec{A}, C \rangle$ denote the least pair in the canonical well-ordering of HOD with the property that $\vec{A} = \langle A_\gamma \mid \gamma < \delta \rangle$ is a sequence of length δ with $A_\gamma \subseteq \gamma$ for all $\gamma < \delta$ and C is a closed unbounded subset of δ with the property that $A_\gamma \cap \beta \neq A_\beta$ holds for all $\beta < \gamma$ in C. Since the class of all proper initial segments of the canonical well-ordering of HOD is definable in V by a Σ_2-formula without parameters (see [10, Lemma 13.25]), we then know that, in V, the sets $\{\vec{A}\}$ and $\{C\}$ are both definable by Σ_2-formulas with parameter δ, and hence these sets are also definable by Σ_{n+1}-formulas with parameter y. We then know that $C \cap \kappa \neq \emptyset$, because otherwise $\kappa \leq \min(C) < \delta$ is an ordinal with the property that the set $\{\min(C)\}$ is definable by a Σ_{n+1}-formula with parameters in $\mathrm{H}(\kappa)$.

Claim. *$\kappa \in C$.*

Proof of the Claim. Assume, towards a contradiction, that κ is not an element of C and set $\alpha = \max(C \cap \kappa) < \kappa$ and $\beta = \min(C \setminus \kappa) = \min(C \setminus (\alpha + 1)) < \delta$. Then β is an ordinal greater than κ with the property that the set $\{\beta\}$ is definable by a Σ_{n+1}-formula with parameters $\alpha, y \in \mathrm{H}(\kappa)$ and hence it follows that $\beta \geq \delta$, a contradiction. □

Pick a cardinal $\delta < \theta \in C^{(n+1)}$ and use Lemma 3.3 to find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $y \in \mathrm{ran}(j)$. We then again know that $y \in \mathrm{H}(\bar{\kappa})$ with $j(y) = y$ and, by the correctness properties of $\mathrm{H}(\theta)$ and $\mathrm{H}(\bar{\theta})$, the fact that the sets $\{\vec{A}\}$, $\{C\}$ and $\{\delta\}$ are all definable by Σ_{n+1}-formulas with parameter y implies that $\vec{A}, C, \delta \in X$ with $j(\vec{A}) = \vec{A}$,

$j(C) = C$ and $j(\delta) = \delta$. In this situation, elementarity allows us to conclude that $\bar{\kappa} \in C$ and $A_{\bar{\kappa}} \in X$ with $j(A_{\bar{\kappa}}) = A_\kappa$ and $A_\kappa \cap \bar{\kappa} = j(A_{\bar{\kappa}}) \cap \bar{\kappa} = A_{\bar{\kappa}}$, contradicting the choice of $\vec{A}$ and C. □

As a first application of the above lemma, we observe that, by combining it with Corollary 4.2, we can determine the exact consistency strength of the existence of a $C^{(n)}$-weakly shrewd cardinal that is not $C^{(n)}$-strongly unfoldable. The following corollary directly strengthens [13, Theorem 1.9.(i)]. Note that, in combination with Theorem 2.4, this result shows that, over ZFC, the existence of a $C^{(n)}$-weakly shrewd cardinal that is not $C^{(n)}$-strongly unfoldable has strictly greater consistency strength than the existence of a $C^{(n)}$-weakly shrewd cardinal.

Corollary 4.4. *Given a natural number $n > 0$, the following statements are equiconsistent over* ZFC*:*

(1) *There exists a $C^{(n)}$-weakly shrewd cardinal that is not $C^{(n)}$-strongly unfoldable.*
(2) *There exists a subtle cardinal.* □

Next, we use Lemma 4.3 to show that the principles "Ord *is essentially subtle*" and "Ord *is essentially faint*" are equivalent if only boundedly many cardinals are subtle in HOD.

Proof of Theorem 1.7. Assume, towards a contradiction that Ord is essentially faint and not essentially subtle. By Theorem 2.4, the fact that Ord is not essentially subtle implies that for some natural number $n > 0$, there are no $C^{(n)}$-strongly unfoldable cardinals. Since Ord is essentially faint, Theorem 3.5 now shows that there is a proper class of $C^{(n)}$-weakly shrewd cardinals that are not $C^{(n)}$-strongly unfoldable. In this situation, we can apply Lemma 4.3 to conclude that a proper class of cardinals is subtle in HOD. □

We now continue by using Theorem 1.7 to show that the principle "Ord *is essentially faint*" is not finitely axiomatizable over ZFC.

Proof of Corollary 1.11. Assume, towards a contradiction, that the theory

$$\text{ZFC} + \text{“Ord } \textit{is essentially faint}\text{”}$$

is consistent and there is a sentence ϕ in the language of set theory with the property that the statements (1) and (2) listed in the corollary hold. Define ψ to be the conjunction of ϕ with the statement that no ordinal is a subtle cardinal in HOD.

Claim. *The theory* $\text{ZFC} + \psi$ *is consistent.*

Proof of the Claim. By our assumption and Corollary 1.8, we know that the theory

$$\text{ZFC} + \text{“Ord } \textit{is essentially subtle}\text{”}$$

is consistent and, since subtle cardinals yield set-sized models of this theory, it follows that the theory

$$\text{ZFC} + \text{“Ord } \textit{is essentially subtle}\text{”} + \text{“}\textit{No ordinal is a subtle cardinal in}\text{ HOD”}$$

is also consistent. By (1), this theory proves ψ. □

By (2), we can now apply Theorem 1.7 to conclude that

$$\text{ZFC} + \psi \vdash \text{“Ord } \textit{is essentially subtle}\text{”}$$

holds. In this situation, Proposition 1.9 shows that $\text{ZFC} + \psi$ is inconsistent, contradicting the above claim. □

Another application of Theorem 1.7 yields the following strengthening of Theorem 1.15:

Corollary 4.5. *Given a natural number n, if* Ord *is essentially faint, then unboundedly many ordinals are strongly inaccessible $C^{(n)}$-weakly shrewd cardinals in* HOD.

Proof. If Ord is essentially subtle, then Ord is essentially subtle in HOD and an application of Theorem 2.4 in HOD yields the desired conclusion. Hence, we may assume that Ord is not essentially subtle. Then Theorem 1.7 shows that, in HOD, there is a proper class of subtle cardinals. An application of Lemma 3.4 now allows us to conclude that unboundedly many ordinals are inaccessible $C^{(n)}$-weakly shrewd cardinals in HOD. □

We now continue by using the developed techniques to show that the consistency strength of the inequality of the principles "Ord *is essentially subtle*" and "Ord *is essentially faint*" is equal to the existence of a proper class of subtle cardinals.

Proof of Theorem 1.10. First, assume that

$$\mathrm{ZFC} + \text{“Ord } \textit{is essentially faint}\text{”} \nvdash \text{“Ord } \textit{is essentially subtle}\text{”}$$

holds. Then Theorem 2.4 yields a natural number $n > 0$ with the property that the theory

$$\mathrm{ZFC} + \text{“Ord } \textit{is essentially faint}\text{”} + \text{“}\textit{There are no } C^{(n)}\textit{-strongly unfoldable cardinals}\text{”}$$

is consistent and we can work in a model of this theory. An application of Theorem 1.7 now directly shows that, in HOD, there is a proper class of subtle cardinals. These computations show that (1) implies (3) in the statement of the theorem.

Next, let ϕ denote the set-theoretic sentence stating that there is a proper class of subtle cardinals. A combination of Lemma 3.4 with Theorem 3.5 then shows that the theory $\mathrm{ZFC} + \phi$ proves that Ord is essentially faint. This shows that (3) implies (2) in the statement of the theorem.

Finally, assume that there is a sentence ϕ such that the theory $\mathrm{ZFC} + \phi$ is consistent and

$$\mathrm{ZFC} + \phi \vdash \text{“Ord } \textit{is essentially faint}\text{”}.$$

Then Proposition 1.9 ensures that

$$\mathrm{ZFC} + \phi \nvdash \text{“Ord } \textit{is essentially subtle}\text{”}$$

and hence we can conclude that

$$\mathrm{ZFC} + \text{“Ord } \textit{is essentially faint}\text{”} \nvdash \text{“Ord } \textit{is essentially subtle}\text{”}.$$

This shows that (2) implies (1) in the statement of the theorem. □

In order to expand our analysis of the given class principles, we now explore a phenomenon already unveiled by [13, Theorem 1.11] showing that the existence of weakly shrewd cardinals that are not strongly unfoldable yields reflection properties for statements of arbitrary high complexities. The following lemma strengthens the result from [13] by both weakening the assumption used and establishing the existence of large cardinals that induce the given reflection properties:

Lemma 4.6. *Given natural numbers $n > m > 0$, if κ is a $C^{(m)}$-weakly shrewd cardinal that is not $C^{(m)}$-strongly unfoldable, $\delta > \kappa$ is a cardinal with the property that the set $\{\delta\}$ is definable by a Σ_{m+1}-formula with parameters in $\mathrm{H}(\kappa)$ and $\alpha < \kappa$ is an ordinal, then the interval (α, δ) contains a $C^{(n)}$-weakly shrewd cardinal that is not $C^{(n)}$-strongly unfoldable.*

Proof. Pick $y \in \mathrm{H}(\kappa)$ with the property that the set $\{\delta\}$ is definable by a Σ_{m+1}-formula with parameter y. Since κ is a limit cardinal, we can find a cardinal $\alpha < \rho < \kappa$ with $y \in \mathrm{H}(\rho)$. Moreover, since $n > m$, it follows that the interval (ρ, δ) does not contain an element of $C^{(n+1)}$. Now, assume, towards a contradiction, the interval (ρ, δ) does not contain a $C^{(n)}$-weakly shrewd cardinal that is not $C^{(n)}$-strongly unfoldable. The above observation then allow us to use Lemma 4.1 to conclude that the interval (ρ, δ) does not contain $C^{(n)}$-weakly shrewd cardinals. This shows that for every cardinal μ in the interval (ρ, δ), there exists a set-theoretic formula $\varphi(v_0, v_1)$, a cardinal $\mu < \theta \in C^{(n)}$ and a subset z of μ such that $\varphi(z, \mu)$ holds in $\mathrm{H}(\theta)$ and $\varphi(z \cap \bar{\mu}, \bar{\mu})$ does not hold in $\mathrm{H}(\bar{\theta})$ for all $\bar{\theta} \in C^{(n)}$ and all cardinals $\bar{\mu} < \min(\mu, \bar{\theta})$. We can then find a cardinal $\delta < \zeta \in C^{(1)}$ with the property that, in $\mathrm{H}(\zeta)$, for every cardinal μ in the interval (ρ, δ), there exists a set-theoretic formula $\varphi(v_0, v_1)$, a cardinal $\mu < \theta \in C^{(n)}$ and a subset z of μ such that $\varphi(z, \mu)$ holds in $\mathrm{H}(\theta)$ and $\varphi(z \cap \bar{\mu}, \bar{\mu})$ does not hold in $\mathrm{H}(\bar{\theta})$ for all $\bar{\theta} \in C^{(n)}$ and all cardinals $\bar{\mu} < \min(\mu, \bar{\theta})$. Let η be the least cardinal with this property. Then the set $\{\eta\}$ is definable by a Σ_2-formula with parameters δ, ρ and y. We then directly know that the set $\{\eta\}$ is also definable by a Σ_{m+1}-formula with parameters ρ and y.

Now, pick $\kappa < \vartheta \in C^{(m+1)}$ and use Lemma 3.3 to find $\bar{\vartheta} \in C^{(m)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\vartheta})$, an elementary submodel X of $\mathrm{H}(\bar{\vartheta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\vartheta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $\rho, y \in \mathrm{ran}(j)$. Then $y \in X$ with $j(y) = y$. Moreover, we know that $\rho < \bar{\kappa}$ and therefore $j(\rho) = \rho$. In combination with the fact that all Σ_{m+1}-statements are upwards absolute from X to V, this shows that $\delta, \eta \in X$ with $j(\delta) = \delta$ and $j(\eta) = \eta$. Since $\bar{\kappa}$ is a cardinal in the interval (ρ, δ) in $\mathrm{H}(\eta)$, we can now find a set-theoretic formula $\varphi(v_0, v_1)$, an ordinal $\bar{\kappa} < \theta \in X \cap \eta$ and $z \in \mathcal{P}(\bar{\kappa}) \cap X$ with the property that, in $\mathrm{H}(\eta)$, the ordinal θ is a cardinal in $C^{(n)}$ such that $\varphi(z, \bar{\kappa})$ holds in $\mathrm{H}(\theta)$ and $\varphi(z \cap \mu, \mu)$ does not hold in $\mathrm{H}(\bar{\theta})$ for all $\bar{\theta} \in C^{(n)}$ and all cardinals $\mu < \min(\bar{\kappa}, \bar{\theta})$. The fact that $\mathrm{H}(\eta) \in X$ with $j(\mathrm{H}(\eta)) = \mathrm{H}(\eta)$ then allows us to use the elementarity of j to conclude that, in $\mathrm{H}(\eta)$, the ordinal $j(\theta)$ is a cardinal in $C^{(n)}$ with the property that $\varphi(j(z), \kappa)$ holds in $\mathrm{H}(j(\theta))$ and $\varphi(j(z) \cap \mu, \mu)$ does not hold in $\mathrm{H}(\bar{\theta})$ for all $\bar{\theta} \in C^{(n)}$ and all cardinals $\mu < \min(\kappa, \bar{\theta})$. But, this yields a contradiction, because θ is an element of $C^{(n)}$ in $\mathrm{H}(\eta)$, $\bar{\kappa} < \min(\kappa, \theta)$ and $j(z) \cap \bar{\kappa} = z$. □

As a first application of Lemma 4.6, we show that the principle "Ord *is essentially subtle*" can be axiomatized over the theory ZFC + "Ord *is essentially faint*" in a strong way.

Proof of Proposition 1.12. For each natural number $n > 0$, let ϕ_n denote the sentence in the language of set theory that states that the existence of a proper class of $C^{(n)}$-weakly shrewd cardinals implies the existence of a proper class of $C^{(n)}$-strongly unfoldable cardinals. Define T to be the theory consisting of these sentences. Then Theorem 2.4 shows that T holds in every model of ZFC in which Ord is essentially subtle. Moreover, a combination of Theorem 2.4 and Theorem 3.5 shows that Ord is essentially subtle in every model of ZFC in which Ord is essentially faint and T holds. Finally, assume that we work in a model of ZFC in which the sentence ϕ_m fails for some natural number $m > 0$. Then there exists a proper class of $C^{(m)}$-weakly shrewd cardinals and there are only boundedly many $C^{(m)}$-strongly unfoldable cardinals. In particular, there exists a proper class of $C^{(m)}$-weakly shrewd cardinals that are not $C^{(m)}$-strongly unfoldable. In this situation, a combination of Lemmas 4.1 and 4.6 shows that for every natural number $n > 0$, there exists a proper class of $C^{(n)}$-weakly shrewd cardinals. Hence, we can apply Theorem 3.5 to conclude that Ord is essentially faint in this model. □

By combing Theorem 1.7 with Proposition 1.12, we can now examine the conditions that allow the proposition's statement to be strengthened to result in finite axiomatizations.

Proof of Theorem 1.13. First, assume that the theory

$$\mathrm{ZFC} + \text{“}\textit{There is a proper class of subtle cardinals}\text{”} + \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$$

is inconsistent and let ϕ denote the set-theoretic sentence stating that only boundedly many cardinals are subtle in HOD. Now, if we work in a model V of ZFC in which Ord is essentially subtle, then Ord is essentially subtle in HOD and, since our assumption ensures that HOD has only boundedly many subtle cardinals, we can conclude that ϕ holds in V. Next, if we work in a model of ZFC in which Ord is essentially faint and ϕ holds, then Theorem 1.7 directly shows that Ord is essentially subtle. In combination, this shows that (2) implies (3) in the statement of the theorem.

Next, assume that there is a set-theoretic sentence ϕ satisfying the statements (1a) and (1b) listed in the theorem. Let ψ be the set-theoretic sentence that is given by the conjunction of ϕ and the sentence stating that there is a proper class of subtle cardinals. Then

$$\mathrm{ZFC}\ +\ \psi\ \vdash\ \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}$$

and therefore Proposition 1.9 shows that $\mathrm{ZFC} \vdash \neg\psi$. In this situation, our assumptions imply that

$$\mathrm{ZFC}\ +\ \text{“}\mathrm{Ord}\ \textit{is essentially subtle}\text{”}\ \vdash\ \text{“}\textit{There are only boundedly many subtle cardinals}\text{”}$$

and we can conclude that (3) implies (2) in the statement of the theorem.

Finally, assume that there is a set-theoretic sentence ϕ satisfying the statements (1a) and (1b) listed in the theorem. Let T be the theory given by Proposition 1.12. We then know that

$$\mathrm{ZFC} + \text{“}\mathrm{Ord}\ \textit{is essentially faint}\text{”} + \mathrm{T}\ \vdash\ \phi$$

and hence there is a finite subtheory F of T with

$$\mathrm{ZFC} + \text{“}\mathrm{Ord}\ \textit{is essentially faint}\text{”} + \mathrm{F}\ \vdash\ \phi.$$

Let ψ be the set-theoretic sentence that is given by the disjunction of ϕ with the conjunction of all sentences in F. Our assumptions then ensure that ψ satisfies the statement (1a) of the theorem. Moreover, the definition of ψ implies that

$$\mathrm{ZFC} + \text{“}\mathrm{Ord}\ \textit{is essentially faint}\text{”} + \psi\ \vdash\ \phi$$

and hence our assumption ensure that ψ satisfies statement (1b) of the theorem. Finally, assume that we work in a model of ZFC in which ψ does not hold. Then there exists a sentence in F that does not hold. In this situation, the properties of the theory T allow us to conclude that Ord is essentially faint in this model. This shows that ψ also satisfies statement (1c) of the theorem. In combination, these arguments show that (1) implies (2) in the statement of the theorem and therefore these assumptions are equivalent. □

We end this section by using Lemma 4.6 to show that the principle “Ord *is essentially faint*” does not imply the existence of strongly inaccessible cardinals.

Proof of Theorem 1.14. Work in a model of $\mathrm{ZFC} + \mathrm{V} = \mathrm{L}$ in which a proper class of subtle cardinals exists and no inaccessible cardinal is a limit of subtle cardinals. Let S denote the class of all subtle cardinals. Define R to be the unique class function with domain S that satisfies $R(\min(S)) = \omega$ and $R(\delta) = \sup(S \cap \delta)^+$ for all $\min(S) < \delta \in S$. Our assumptions then ensure that R is a well-defined regressive function. Define $\mathbb{P}$ to be the class partial order that is given by the Easton support product of all partial orders of the form $\mathrm{Add}(R(\delta), \delta)$ with $\delta \in S$ and let G be $\mathbb{P}$-generic over V. Then standard arguments (as in [10, pp. 235–236]) show that $\mathrm{V}[G]$ is a model of ZFC with the same regular cardinals as V.

Claim. *There are no inaccessible cardinals in* $\mathrm{V}[G]$.

Proof of the Claim. Assume, towards a contradiction, that there is an inaccessible κ in $\mathrm{V}[G]$. Set $\delta = \min(S \setminus \kappa) \geq \kappa$. Since κ is not a limit of subtle cardinals in V, we know that $R(\delta) < \kappa$. Moreover, our setup ensures that

$$\left(2^{R(\delta)}\right)^{\mathrm{V}[G]} \;\geq\; \delta \;\geq\; \kappa$$

holds, contradicting the inaccessibility of κ in $\mathrm{V}[G]$. □

Claim. *In* $\mathrm{V}[G]$*, there is a proper class of weakly shrewd cardinals.*

Proof of the Claim. Fix an ordinal $\alpha > \omega$ and pick $\delta \in S$ with $R(\delta) > \alpha$. Set $S_0 = S \cap \delta$ and let $\mathbb{Q}$ denote the Easton support product of all partial orders of the form $\mathrm{Add}(R(\gamma), \gamma)$ with $\gamma \in S_0$, as constructed in V. In addition, let $\mathbb{R}$ denote the partial order $\mathrm{Add}(R(\delta), \delta)$ as constructed in V. A standard factor analysis of the class partial order $\mathbb{P}$ (again, as in [10, pp. 235–236]) now allows us to find an inner model M of $\mathrm{V}[G]$ such that $\mathrm{V} \subseteq M$, $({}^{\delta}\mathrm{V})^M \subseteq \mathrm{V}$ and there are $G_0, G_1 \in \mathrm{V}[G]$ such that $G_0 \times G_1$ is $(\mathbb{Q} \times \mathbb{R})$-generic over M with $\mathrm{V}[G] = M[G_0, G_1]$. Then δ is a subtle cardinal in M and the definition of R in V ensures that δ is the minimal subtle cardinal above $R(\delta)$ in M. In particular, it follows that the set $\{\delta\}$ is definable by a Σ_2-formula with parameter $R(\delta)$ in M.

An application of Lemma 3.4 now shows that, in M, the interval $(R(\delta), \delta)$ contains an inaccessible weakly shrewd cardinal κ. In the following, fix a cardinal $\theta > \delta$ that is an element of $C^{(1)}$ in $\mathrm{V}[G]$ and an element z of $\mathrm{H}(\theta)^{\mathrm{V}[G]}$. Pick a $(\mathbb{Q} \times \mathbb{R})$-name $\dot{z}$ in M with $z = \dot{z}^{G_0 \times G_1}$ and an ordinal $\vartheta > \theta$ that is an element of $C^{(2)}$ in M and satisfies $\dot{z} \in \mathrm{H}(\vartheta)^M$. Proposition 3.2 and Lemma 3.3 now show that, in M, we can find a cardinal $\bar{\vartheta} \in C^{(1)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\vartheta})$, an elementary submodel X of $\mathrm{H}(\bar{\vartheta})$ of cardinality $\bar{\kappa}$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\vartheta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $R(\delta), \theta, \dot{z} \in \mathrm{ran}(j)$. We then know that $R(\delta) < \bar{\kappa}$ and this directly implies that $\delta \in X$ with $j(\delta) = \delta$. In particular, it follows that $\bar{\kappa} < \kappa < \delta < \bar{\vartheta}$. Since the partial order $\mathbb{Q} \times \mathbb{R}$ has cardinality δ in M, we know that the cardinals ϑ and $\bar{\vartheta}$ are elements of $C^{(1)}$ in both $M[G_0]$ and $\mathrm{V}[G]$. Moreover, since X has cardinality $\bar{\kappa} < \delta$ in M, there is a permutation σ of δ in M that extends the injection $j \restriction (X \cap \delta) : X \cap \delta \longrightarrow \delta$. Since $\mathbb{R} = \mathrm{Add}(R(\delta), \delta)^M$, this shows that M also contains an automorphism τ of $\mathbb{R}$ that is induced by the action of σ on the supports of the conditions in $\mathrm{Add}(R(\delta), \delta)$,[6] in the sense that $\mathrm{dom}(\tau(p)) = \sigma[\mathrm{dom}(p)]$ and $\tau(p)(\sigma(\gamma)) = p(\gamma)$ holds for every condition p in $\mathrm{Add}(R(\delta), \delta)$ and all $\gamma \in \mathrm{dom}(p)$. Since our setup ensures that $\bar{\kappa}$ is inaccessible in M and $\mathrm{H}(\bar{\kappa})^M \subseteq X$ with $j \restriction \mathrm{H}(\bar{\kappa})^M = \mathrm{id}_{\mathrm{H}(\bar{\kappa})^M}$, it follows that $\mathrm{Add}(R(\delta), 1)^M \cup \{\mathrm{Add}(R(\delta), 1)^M\} \subseteq X$ and $j \restriction \mathrm{Add}(R(\delta), 1)^M = \mathrm{id}_{\mathrm{Add}(R(\delta),1)^M}$. Moreover, we also know that $\mathbb{R} \in X$ with $j(\mathbb{R}) = \mathbb{R}$. Finally, our definitions ensure that $\tau(p) = j(p)$ holds for every condition p in $\mathbb{R}$ that is an element of X.

Next, since $\mathcal{P}(\mathbb{Q}) \in \mathrm{H}(\bar{\kappa})^M \subseteq X$, we know that G_0 is $\mathbb{Q}$-generic over X, $\mathrm{H}(\vartheta)^M[G_0] = \mathrm{H}(\vartheta)^{M[G_0]}$ and the model $X[G_0]$ is an elementary submodel of $\mathrm{H}(\bar{\vartheta})^{M[G_0]}$. Moreover, the fact that $j \restriction \mathbb{Q} = \mathrm{id}_{\mathbb{Q}}$ implies that there is a canonical lifting

$$j_0 : X[G_0] \longrightarrow \mathrm{H}(\vartheta)^{M[G_0]}$$

of j to $X[G_0]$. Now, set $G_1' = \tau^{-1}[G_1] \subseteq \mathbb{R}$. Since τ is an element of $\mathrm{H}(\vartheta)^M$, it follows that G_1' is $\mathbb{R}$-generic over $M[G_0]$ with

$$\mathrm{H}(\vartheta)^{M[G_0]}[G_1'] \;=\; \mathrm{H}(\vartheta)^{M[G_0]}[G_1] \;=\; \mathrm{H}(\vartheta)^{\mathrm{V}[G]}.$$

[6]Here, we view conditions in partial orders of the form $\mathrm{Add}(\mu, \nu)$ as functions p of cardinality less than μ with $\mathrm{dom}(p) \subseteq \nu$ and $p(\gamma) \in \mathrm{Add}(\mu, 1)$ for every $\gamma \in \mathrm{dom}(p)$.

Set $G_1'' = G_1' \cap X$. Since the partial order $\mathbb{Q}$ has cardinality less than $\bar{\kappa}$ in M and the partial order $\mathbb{R} = \mathrm{Add}(R(\delta), \delta)$ satisfies the $\bar{\kappa}$-chain condition in M, if follows that $\mathbb{R}$ satisfies the $\bar{\kappa}$-chain condition in $M[G_0]$. Together with the fact that $\mathbb{R} \cap X = \mathbb{R} \cap X[G_0]$, this ensures that G_1'' is $\mathbb{R}$-generic over $X[G_0]$ and $X[G_0, G_1'']$ is an elementary submodel of $\mathrm{H}(\bar{\vartheta})^{\mathrm{V}[G]}$. Moreover, since we have ensured that $j_0[G_1''] = j[(\tau^{-1}[G_1]) \cap X] \subseteq G_1$, there is a canonical lifting

$$j_1 : X[G_0, G_1''] \longrightarrow \mathrm{H}(\vartheta)^{\mathrm{V}[G]}$$

of j_0 to $X[G_0, G_1'']$. Pick $\bar{\theta} \in X$ with $j(\bar{\theta}) = \theta$. Since $\bar{\vartheta}$ is an element of $C^{(1)}$ in $\mathrm{V}[G]$, elementarity implies that $\bar{\theta}$ is also an element of this class. Set $Y = \mathrm{H}(\bar{\theta})^{\mathrm{V}[G]} \cap X[G_0, G_1'']$ and define $i = j_1 \restriction Y$. Since $\mathrm{H}(\bar{\theta})^{\mathrm{V}[G]} \in X[G_0, G_1'']$, elementarity ensures that the set Y is an elementary submodel of $\mathrm{H}(\bar{\theta})^{\mathrm{V}[G]}$. Moreover, our setup ensures that $i : Y \longrightarrow \mathrm{H}(\theta)^{\mathrm{V}[G]}$ is an elementary embedding with $i \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $i(\bar{\kappa}) = \kappa$, $i(\delta) = \delta$ and $z \in \mathrm{ran}(i)$. These computations show that, in $\mathrm{V}[G]$, the cardinal $\kappa > \alpha$ is weakly shrewd. □

In combination, the above claims show that $\mathrm{V}[G]$ is a model of ZFC that contains a proper class of weakly shrewd cardinals and in which no strongly inaccessible cardinals exist.

Now, let ϕ denote the sentence in the language of set-theory stating that there are no strongly inaccessible cardinals and there is a proper class of weakly shrewd cardinals. Then ϕ implies that there is a proper class of weakly shrewd cardinals that are not strongly unfoldable and, by Lemma 4.6, this conclusion ensures that for every natural number $n > 0$, there is a proper class of $C^{(n)}$-weakly shrewd cardinals. Therefore, Theorem 3.5 shows that

$$\mathrm{ZFC} + \phi \vdash \text{“}\mathrm{Ord} \textit{ is essentially faint”} + \text{“}\textit{There are no strongly inaccessible cardinals”}.$$

In addition, the above computations show that the consistency of the theory

$$\mathrm{ZFC} + \text{“}\textit{There is a proper class of subtle cardinals”}$$

implies the consistency of $\mathrm{ZFC} + \phi$. Hence, we know that (1) implies (3) in the statement of the theorem.

Next, if we work in a model of ZFC in which Ord is essentially faint and there are no strongly inaccessible cardinals, then Theorem 2.4 shows that Ord is not essentially subtle and Theorem 1.7 allows us to conclude that a proper class of cardinals in subtle in HOD. In particular, we know that (2) implies (1) in the statement of the theorem.

This completes the proof of the theorem, because the remaining implication from (3) to (2) in its statement is trivial. □

5. Abstract logics

In the remainder of this paper, we will connect the theory developed above with the study of structural properties of abstract logics. We start by recalling the relevant definitions. Our setup is based on the definitions given in [6, Section 2.5] and our presentation closely follows those provided in [4], [5] and [9].

Definition 5.1. (1) A *language* is a tuple

$$\tau \ = \ \langle \mathfrak{C}_\tau, \mathfrak{F}_\tau, \mathfrak{R}_\tau, \mathfrak{a}_\tau \rangle,$$

where $\mathfrak{C}_\tau$, $\mathfrak{F}_\tau$ and $\mathfrak{R}_\tau$ are pairwise disjoint sets and $\mathfrak{a}_\tau : \mathfrak{F}_\tau \cup \mathfrak{R}_\tau \longrightarrow \omega \setminus \{0\}$ is a function. We then call $\mathfrak{C}_\tau$ the *set of constant symbols* of τ, $\mathfrak{F}_\tau$ the *set of function symbols* of τ, $\mathfrak{R}_\tau$ the *set of relation symbols* of τ, $\mathfrak{a}_\tau$ the *arity function* of τ and $\mathfrak{S}_\tau = \mathfrak{C}_\tau \cup \mathfrak{F}_\tau \cup \mathfrak{R}_\tau$ the set of *logical symbols* of τ.

(2) Given a language τ, a *τ-structure* is a tuple

$$M \ = \ \langle \mathrm{dom}(M), \langle c^M \mid c \in \mathfrak{C}_\tau \rangle, \langle f^M \mid f \in \mathfrak{F}_\tau \rangle, \langle r^M \mid r \in \mathfrak{R}_\tau \rangle \rangle,$$

where $\mathrm{dom}(M)$ is a non-empty set, each c^M is an element of $\mathrm{dom}(M)$, each f^M is an $\mathfrak{a}_\tau(f)$-ary function on $\mathrm{dom}(M)$ and each r^M is an $\mathfrak{a}_\tau(r)$-ary relation on $\mathrm{dom}(M)$.

(3) Given a language τ, we let Str_τ denote the class of all τ-structures.

(4) Given a language τ and $M, N \in Str_\tau$, an *isomorphism* between M and N is a bijection $\pi : \mathrm{dom}(M) \longrightarrow \mathrm{dom}(N)$ satisfying:
   (a) If $c \in \mathfrak{C}_\tau$, then $\pi(c^M) = c^N$.
   (b) If $f \in \mathfrak{F}_\tau$ and $x_0, \ldots, x_{\mathfrak{a}_\tau(f)-1} \in \mathrm{dom}(M)$, then

$$\pi(f^M(x_0, \ldots, x_{\mathfrak{a}_\tau(f)-1})) \ = \ f^N(\pi(x_0), \ldots, \pi(x_{\mathfrak{a}_\tau(f)-1})).$$

   (c) If $r \in \mathfrak{R}_\tau$ and $x_0, \ldots, x_{\mathfrak{a}_\tau(r)-1} \in \mathrm{dom}(M)$, then

$$r^M(x_0, \ldots, x_{\mathfrak{a}_\tau(r)-1}) \iff r^N(\pi(x_0), \ldots, \pi(x_{\mathfrak{a}_\tau(r)-1})).$$

(5) Given a language τ and $M, N \in Str_\tau$, the structure M is a *substructure* of the structure N if $\mathrm{dom}(M) \subseteq \mathrm{dom}(N)$, $c^M = c^N$ for all $c \in \mathfrak{C}_\tau$, $f^M = f^N \restriction \mathrm{dom}(M)^{\mathfrak{a}_\tau(f)}$ for all $f \in \mathfrak{F}$ and $r^M = r^N \cap \mathrm{dom}(M)^{\mathfrak{a}_\tau(r)}$ for all $r \in \mathfrak{R}$.

(6) A language σ is a *sublanguage* of a language τ if $\mathfrak{C}_\sigma \subseteq \mathfrak{C}_\tau$, $\mathfrak{F}_\sigma \subseteq \mathfrak{F}_\tau$, $\mathfrak{R}_\sigma \subseteq \mathfrak{R}_\tau$ and $\mathfrak{a}_\sigma = \mathfrak{a}_\tau \restriction (\mathfrak{F}_\sigma \cup \mathfrak{R}_\sigma)$.

(7) Given a sublanguage σ of a language τ and $M \in Str_\tau$, the *σ-reduct* $M \restriction \sigma$ of M is the unique element N of Str_σ with $\mathrm{dom}(N) = \mathrm{dom}(M)$, $c^N = c^M$ for all $c \in \mathfrak{C}_\sigma$, $f^N = f^M$ for all $f \in \mathfrak{F}_\sigma$ and $r^N = r^M$ for all $r \in \mathfrak{R}_\sigma$.

(8) A *renaming* of a language σ into a language τ is a bijection

$$u : \mathfrak{S}_\sigma \longrightarrow \mathfrak{S}_\tau$$

satisfying $u[\mathfrak{C}_\sigma] = \mathfrak{C}_\tau$, $u[\mathfrak{F}_\sigma] = \mathfrak{F}_\tau$, $u[\mathfrak{R}_\sigma] = \mathfrak{R}_\tau$ and $\mathfrak{a}_\sigma(s) = \mathfrak{a}_\tau(u(s))$ for all $s \in \mathfrak{F}_\sigma \cup \mathfrak{R}_\sigma$.

(9) Given a renaming u of a language σ into a language τ, we let $u^* : Str_\sigma \longrightarrow Str_\tau$ denote the bijective class function with the property that for all $M \in Str_\sigma$, we have $\mathrm{dom}(u^*(M)) = \mathrm{dom}(M)$ and $u(s)^{u^*(M)} = s^M$ for all $s \in \mathfrak{S}_\sigma$.

Note that, if u is a renaming of a language σ into a language τ, then the inverse u^{-1} of u is a renaming of τ into σ with the property that $(u^{-1})^* = (u^*)^{-1}$ holds.

We are now ready to introduce the notion of an abstract logic:

Definition 5.2. An *abstract logic* consists of a class function $\mathcal{L}$ and a binary class relation $\models_{\mathcal{L}}$ satisfying the following statements:

(1) The domain of $\mathcal{L}$ is the class of all languages.

(2) If $M \models_{\mathcal{L}} \phi$ holds, then there exists a language τ such that $M \in Str_\tau$ and $\phi \in \mathcal{L}(\tau)$.

(3) (Monotonicity) If σ is a sublanguage of τ, then $\mathcal{L}(\sigma) \subseteq \mathcal{L}(\tau)$.

(4) (Expansion) If σ is a sublanguage of τ, $M \in Str_\tau$ and $\phi \in \mathcal{L}(\sigma)$, then

$$M \models_{\mathcal{L}} \phi \iff M \restriction \sigma \models_{\mathcal{L}} \phi.$$

(5) (Isomorphism) Given a language τ and isomorphic τ-structures M and N, we have

$$M \models_{\mathcal{L}} \phi \iff N \models_{\mathcal{L}} \phi$$

for all $\phi \in \mathcal{L}(\tau)$.

(6) (Renaming) Every renaming u of a language σ into a language τ induces a unique bijection $u_* : \mathcal{L}(\sigma) \longrightarrow \mathcal{L}(\tau)$ with the property that
$$M \models_{\mathcal{L}} \phi \iff u^*(M) \models_{\mathcal{L}} u_*(\phi)$$
holds for every $M \in Str_\sigma$ and every $\phi \in \mathcal{L}(\sigma)$.
(7) (Occurrence number) There exists an infinite cardinal o with the property that for every language τ and every $\phi \in \mathcal{L}(\tau)$, there exists a sublanguage σ of τ with $|\mathfrak{S}_\sigma| < o$ and $\phi \in \mathcal{L}(\sigma)$. The least such cardinal is called the *occurrence number* of the logic.

In the following, we just write $\mathcal{L}$ to denote an abstract logic. We then call the corresponding relation $\models_{\mathcal{L}}$ the *satisfaction relation* of L and, given a language τ, we say that $\mathcal{L}(\tau)$ is the set of *τ-sentences*. In addition, we write $o_{\mathcal{L}}$ to denote the occurrence number of an abstract logic $\mathcal{L}$. Note that the uniqueness of u_* in Item (6) of Definition 5.2 implies that $(u^{-1})_* = (u_*)^{-1}$ holds for every renaming u.

The above definitions now allow us to generalize fundamental concepts from first-order logic to all abstract logics.

Definition 5.3. Let $\mathcal{L}$ be an abstract logic.

(1) An *$\mathcal{L}$-theory* is a set T with the property that $T \subseteq \mathcal{L}(\tau)$ for some language τ.
(2) An $\mathcal{L}$-theory T is *satisfiable* if there is a language τ with $T \subseteq \mathcal{L}(\tau)$ and $M \in Str_\tau$ with $M \models_{\mathcal{L}} \phi$ for all $\phi \in T$.
(3) Given an infinite cardinal κ, an $\mathcal{L}$-theory T is *$<\kappa$-satisfiable* if every subset of T of cardinality less than κ is satisfiable.
(4) An infinite cardinal κ is a *strong compactness cardinal for $\mathcal{L}$* if every $<\kappa$-satisfiable $\mathcal{L}$-theory is satisfiable.
(5) An infinite cardinal κ is a *weak compactness cardinal for $\mathcal{L}$* if every $<\kappa$-satisfiable $\mathcal{L}$-theory of cardinality κ is satisfiable.
(6) Given a language τ and $N \in Str_\tau$, an *$\mathcal{L}$-elementary substructure* of N is a substructure M of N with the property that
$$M \models \phi \iff N \models \phi$$
holds for all $\phi \in \mathcal{L}(\tau)$.
(7) An uncountable cardinal κ is a *Löwenheim–Skolem–Tarski number* for $\mathcal{L}$ if for every language τ with $|\mathfrak{S}_\tau| < \kappa$ and every τ-structure N, there exists an $\mathcal{L}$-elementary substructure M of N with $|\mathrm{dom}(M)| < \kappa$.
(8) An uncountable cardinal κ is a *weak Löwenheim–Skolem–Tarski number* for $\mathcal{L}$ if for every language τ with the property that $|\mathfrak{S}_\tau| < \kappa$ and every τ-structure N with $|\mathrm{dom}(N)| = \kappa$, there exists an $\mathcal{L}$-elementary substructure M of N with $|\mathrm{dom}(M)| < \kappa$.
(9) An uncountable cardinal κ is a *strict Löwenheim–Skolem–Tarski number* for $\mathcal{L}$ if for every language τ with the property that $|\mathfrak{S}_\tau| < \kappa$, every τ-structure N with $|\mathrm{dom}(N)| = \kappa$ and every $\phi \in \mathcal{L}(\tau)$ with $N \models_{\mathcal{L}} \phi$, there exists a substructure M of N with $|\mathrm{dom}(M)| < \kappa$ and $M \models_{\mathcal{L}} \phi$.

In [5, Section 4], it is verified that Makowsky's Theorem 1.1 holds true with respect to the above definitions. In another direction, an unpublished result of Stavi shows that the validity of Vopěnka's Principle is also equivalent to the fact that analogs of the Downward Löwenheim–Skolem Theorem hold for all abstract logics (see [15, Theorem 6]). For sake of completeness and to motivate the notions introduced in Items (8) and (9) of Definition 5.3, we present a proof of Stavi's result.

Theorem 5.4 (Stavi)**.** *The following schemes are equivalent over* ZFC*:*

(1) *Vopěnka's Principle.*
(2) *Every abstract logic has a Löwenheim–Skolem–Tarski number.*
(3) *For every abstract logic $\mathcal{L}$, there exists an uncountable cardinal κ with the property that for every language τ with $|\mathfrak{S}_\tau| < \kappa$, every τ-structure N and every $\phi \in \mathcal{L}(\tau)$ with $N \models_{\mathcal{L}} \phi$, there exists a substructure M of N with $|\mathrm{dom}(M)| < \kappa$ and $M \models_{\mathcal{L}} \phi$.*

Proof. First, assume that (1) holds and fix an abstract logic $\mathcal{L}$. Then there is a natural number $n > 0$, a Σ_n-formula $\varphi(v_0, v_1, v_2)$, a Σ_n-formula $\psi(v_0, v_1, v_2)$ and a set z with the property that the function $\mathcal{L}$ is defined by the formula $\varphi(v_0, v_1, v_2)$ and the parameter z and the relation $\models_{\mathcal{L}}$ is defined by the formula $\psi(v_0, v_1, v_2)$ and the parameter z. By [1, Corollary 4.15], we can now find a $C^{(n)}$*-extendible cardinal* κ (i.e., a cardinal κ with the property that for every $\eta > \kappa$, there exists an ordinal ζ and an elementary embedding $j : \mathrm{V}_\eta \longrightarrow \mathrm{V}_\zeta$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \eta$ and $j(\kappa) \in C^{(n)}$) with $z \in \mathrm{H}(\kappa)$. Now, fix a language σ with $|\mathfrak{S}_\sigma| < \kappa$ and a σ-structure K. Then there exists a renaming u of σ into a language $\tau \in \mathrm{V}_\kappa$. Since [1, Proposition 3.4] shows that $\kappa \in C^{(n+2)}$, we know that $\mathcal{L}(\tau)$ is also an element of V_κ. Set $M = u^*(K)$. Pick $\kappa < \eta \in C^{(n+2)}$ with $M \in \mathrm{V}_\eta$. Then, there is an ordinal ζ and an elementary embedding $j : \mathrm{V}_\eta \longrightarrow \mathrm{V}_\zeta$ with $\mathrm{crit}\,(j) = \kappa$, $j(\kappa) > \eta$ and $j(\kappa) \in C^{(n)}$. Since $\tau \in \mathrm{V}_\kappa$ and $j(\tau) = \tau$, we now know that $j(M)$ is a τ-structure with $s^{j(M)} = j(s^M)$ for every $s \in \mathfrak{S}_\tau$. This shows that $j(M)$ has a substructure N with $\mathrm{dom}(N) = j[\mathrm{dom}(M)]$. We then know that the map $j \restriction \mathrm{dom}(M)$ is an isomorphism between M and N and this isomorphism is an element of V_ζ. In this situation, the correctness properties of V_η, the elementarity of j and Item (5) in Definition 5.2 ensure that

$$\mathrm{V}_\zeta \;\models\; \psi(M, \phi, z) \longleftrightarrow \psi(N, \phi, z) \tag{A}$$

holds for all $\phi \in \mathcal{L}(\tau)$. Moreover, since κ and η are both elements of $C^{(n)}$, we know that the formula $\psi(v_0, v_1, v_2)$ is absolute between V_κ and V_η. Elementarity then ensures that this formula is also absolute between $\mathrm{V}_{j(\kappa)}$ and V_ζ. Moreover, since $j(\kappa)$ is also an element of $C^{(n)}$, it follows that $\psi(v_0, v_1, v_2)$ is absolute between V_η and V_ζ. For all $\phi \in \mathcal{L}(\tau)$, we then have

$$\mathrm{V}_\zeta \models \psi(j(M), \phi, z) \iff \mathrm{V}_\eta \models \psi(M, \phi, z) \iff \mathrm{V}_\zeta \models \psi(M, \phi, z) \iff \mathrm{V}_\zeta \models \psi(N, \phi, z),$$

where the first equivalence is given by elementarity of j, the second equivalence follows from the above absoluteness considerations and the third equivalence is given by (A). The elementarity of j now yields a substructure S of M with $|\mathrm{dom}(S)| < \kappa$ and the property that

$$\mathrm{V}_\eta \;\models\; \psi(M, \phi, z) \longleftrightarrow \psi(S, \phi, z)$$

holds for all ϕ in $\mathcal{L}(\tau)$. Since η is an element of $C^{(n)}$, this shows that S is an $\mathcal{L}$-elementary submodel of M. Set $R = (u^*)^{-1}(M) \in Str_\sigma$. Then R is a submodel of K with $|\mathrm{dom}(R)| < \kappa$. Moreover, we can apply Item (6) in Definition 5.2 to conclude that R is an $\mathcal{L}$-elementary submodel of K. We can conclude that κ is a Löwenheim–Skolem–Tarski number for $\mathcal{L}$. These computations show that (1) implies (2).

Now, assume that (3) holds and fix a proper class Γ of graphs. Then there is an abstract logic $\mathcal{L}_\Gamma$ such that the following statements hold:

- For every language τ, the set $\mathcal{L}_\Gamma(\tau)$ consists of all first-order sentences of τ together with a distinguished sentence ϕ_r for every binary relation symbol r in τ.
- Given is a language τ, $M \in Str_\tau$ and a first-order sentence $\phi \in \mathcal{L}_\Gamma(\tau)$, then $M \models_{\mathcal{L}_\Gamma} \phi$ holds if and only if ϕ holds in M (in the sense of first-order structures).

- Given is a language τ, $M \in Str_\tau$ and a binary relation symbol r in τ, then $M \models_{\mathcal{L}_\Gamma} \phi_r$ holds if and only if $\langle \mathrm{dom}(M), r^M \rangle$ is a graph that is isomorphic to a graph in Γ.

Our assumption now yields an uncountable cardinal κ with the property that for every language τ with $|\mathfrak{S}_\tau| < \kappa$, every τ-structure N and every $\phi \in \mathcal{L}_\Gamma(\tau)$ with $N \models_{\mathcal{L}_\Gamma} \phi$, there exists a substructure M of N with $|\mathrm{dom}(M)| < \kappa$ and $M \models_{\mathcal{L}_\Gamma} \phi$. Since Γ is a proper class, we can fix a graph G in Γ that has cardinality greater than κ. Let σ denote a language with a single binary relation symbol e and let τ be a language extending σ that adds functions symbols for first-order Skolem functions to σ. View G as a σ-structure and let G' be a τ-structure whose σ-reduct is G and that interprets the new function symbols as the corresponding first-order Skolem functions. Then Item (4) in Definition 5.2 ensures that $G' \models_{\mathcal{L}_\Gamma} \phi_e$ holds and, since $|\mathfrak{S}_\tau| < \kappa$, we can use our assumption to find a substructure H' of G' with $|H'| < \kappa$ and $H' \models_{\mathcal{L}_\Gamma} \phi_e$. Let H denote the σ-reduct of H'. Our setup now ensures that H is a first-order elementary submodel of G and $H \models_{\mathcal{L}_\Gamma} \phi_e$ holds. This shows that $\langle \mathrm{dom}(H), e^H \rangle$ is a graph that is isomorphic to an element K of Γ. Since $\mathrm{dom}(H)$ has cardinality less than κ, we know that $K \neq G$ and K embeds into G. These computations show that (1) holds.

This completes the proof of the theorem, because (2) obviously implies (3). □

Motivated by Theorem 5.4, we will also use the developed theory to prove the following analogs of Theorems 1.4 and 1.6 for strict Löwenheim–Skolem–Tarski numbers in the next section:

Theorem 5.5. (1) *The following schemes of sentences are equivalent over* ZFC*:*
   (a) Ord *is essentially faint.*
   (b) *Every abstract logic has a strict Löwenheim–Skolem–Tarski number.*

(2) *The following schemes of sentences are equivalent over* ZFC*:*
   (a) Ord *is essentially subtle.*
   (b) *Every abstract logic has a stationary class of strict Löwenheim–Skolem–Tarski numbers.*
   (c) *Every abstract logic has a stationary class of weak Löwenheim–Skolem–Tarski numbers.*

In the last part of the next section, we will consider the question of whether the first equivalence can also be extended to characterize the existence of weak Löwenheim–Skolem–Tarski numbers for all abstract logics.

6. Structural properties of abstract logics

This section contains the proofs of Theorems 1.4, 1.6 and 5.5. The following lemma shows how strong structural properties of abstract logics can be derived from the existence of $C^{(n)}$-weakly shrewd cardinals.

Lemma 6.1. *Let $n > 0$ be a natural number, let $\varphi(v_0, v_1, v_2)$ and $\psi(v_0, v_1, v_2)$ be Σ_n-formulas and let z be a set with the property that there is an abstract logic $\mathcal{L}$ such that the function $\mathcal{L}$ is defined by the formula $\varphi(v_0, v_1, v_2)$ and the parameter z and the relation $\models_{\mathcal{L}}$ is defined by the formula $\psi(v_0, v_1, v_2)$ and the parameter z. Then the following statements hold:*

(1) *There exists a cardinal $\mu \in C^{(1)}$ such that the following statements hold:*
   (a) *$\mu > o_{\mathcal{L}}$ and $z \in \mathrm{H}(\mu)$.*
   (b) *If σ is a language that is an element of $\mathrm{H}(o_{\mathcal{L}})$, then $\mathcal{L}(\sigma) \in \mathrm{H}(\mu)$.*
   (c) *If σ is a language that is an element of $\mathrm{H}(o_{\mathcal{L}})$ and π is a non-trivial permutation of $\mathcal{L}(\sigma)$, then there exists a σ-structure $M_{\sigma,\pi} \in \mathrm{H}(\mu)$ and $\phi_{\sigma,\pi} \in \mathcal{L}(\sigma)$ with*

$$M_{\sigma,\pi} \models_{\mathcal{L}} \phi_{\sigma,\pi} \iff \neg(M_{\sigma,\pi} \models_{\mathcal{L}} \pi(\phi_{\sigma,\pi})). \tag{B}$$

(2) *If μ is a cardinal with the properties listed in* (1)*, σ is a language in $\mathrm{H}(o_{\mathcal{L}})$ and u is a renaming of σ into a language τ, then u_* is the unique bijection ϖ between $\mathcal{L}(\sigma)$ and $\mathcal{L}(\tau)$ with the property that*

$$M \models_{\mathcal{L}} \phi \iff u^*(M) \models_{\mathcal{L}} \varpi(\phi) \tag{C}$$

holds for every σ-structure M in $\mathrm{H}(\mu)$ and every $\phi \in \mathcal{L}(\sigma)$.

(3) *If μ is a cardinal with the properties listed in* (1)*, $\kappa > \mu$ is a $C^{(n)}$-weakly shrewd cardinal and τ is a language with $|\mathfrak{S}_\tau| \leq \kappa$ and $\mathfrak{S}_\tau \subseteq \mathrm{H}(\kappa)$, then for every cardinal $\kappa < \theta \in C^{(n+2)}$ and every $y \in \mathrm{H}(\theta)$, there exists*

- *a cardinal $\bar{\theta} \in C^{(n)}$,*
- *a cardinal $\mu < \bar{\kappa} < \min(\kappa, \bar{\theta})$,*
- *a sublanguage $\bar{\tau}$ of τ with $|\mathfrak{S}_{\bar{\tau}}| \leq \bar{\kappa}$ and $\mathfrak{S}_{\bar{\tau}} = \mathfrak{S}_\tau \cap \mathrm{H}(\bar{\kappa})$,*
- *an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} \cup \{\bar{\kappa}, \bar{\tau}, \mathcal{L}(\bar{\tau}), z\} \subseteq X$ and*
- *an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(z) = z$, $j(\bar{\tau}) = \tau$, $j(\mathcal{L}(\bar{\tau})) = \mathcal{L}(\tau)$, $j \restriction (\mathcal{L}(\bar{\tau}) \cap X) = \mathrm{id}_{\mathcal{L}(\bar{\tau}) \cap X}$ and $y \in \mathrm{ran}(j)$.*

(4) *If μ is a cardinal with the properties listed in* (1)*, then every $C^{(n)}$-weakly shrewd cardinal greater than μ is a weak compactness cardinal for $\mathcal{L}$.*

(5) *If μ is a cardinal with the properties listed in* (1)*, then every $C^{(n)}$-weakly shrewd cardinal greater than μ is a strict Löwenheim–Skolem–Tarski number for $\mathcal{L}$.*

(6) *If μ is a cardinal with the properties listed in* (1)*, then every $C^{(n)}$-weakly shrewd cardinal $\kappa > \mu$ with the property that $\nu^{<o_{\mathcal{L}}} \leq \kappa$ holds for all $\nu < \kappa$ is a weak Löwenheim–Skolem–Tarski number for $\mathcal{L}$.*

Proof. (1) Note that, given a language σ, an application of (6) in Definition 5.2 to the trivial renaming of σ into itself ensures that the identity function on $\mathcal{L}(\sigma)$ is the unique permutation ι of $\mathcal{L}(\sigma)$ with the property that

$$M \models_{\mathcal{L}} \phi \iff M \models_{\mathcal{L}} \iota(\phi)$$

holds for every σ-structure M and every $\phi \in \mathcal{L}(\sigma)$. Hence, for every non-trival permuation π of $\mathcal{L}(\sigma)$, there exists a σ-structure $M_{\sigma,\pi}$ and $\phi_{\sigma,\pi} \in \mathcal{L}(\sigma)$ with (B). Using this observation, it is now easy to find a cardinal μ with the desired properties.

(2) Let μ be a cardinal as in (1), let σ be a language in $\mathrm{H}(o_{\mathcal{L}})$, let u be a renaming of σ into a language τ and let ϖ be a bijection between $\mathcal{L}(\sigma)$ and $\mathcal{L}(\tau)$ such that (C) holds. Assume, towards a contradiction that $\varpi \neq u_*$. Then $\pi = (u_*)^{-1} \circ \varpi$ is a non-trivial permutation of $\mathcal{L}(\sigma)$ and, since $M_{\sigma,\pi}$ is an element of $\mathrm{H}(\mu)$, our assumptions on ϖ ensure that

$$M_{\sigma,\pi} \models_{\mathcal{L}} \phi_{\sigma,\pi} \iff u^*(M_{\sigma,\pi}) \models_{\mathcal{L}} \varpi(\phi_{\sigma,\pi}) \iff M_{\sigma,\pi} \models_{\mathcal{L}} \pi(\phi_{\sigma,\pi}),$$

contradicting the definitions of $M_{\sigma,\pi}$ and $\phi_{\sigma,\pi}$. This shows that ϖ is equal to u_*.

(3) Fix a cardinal μ as in (1), a $C^{(n)}$-weakly shrewd cardinal $\kappa > \mu$, a cardinal $\kappa < \theta \in C^{(n+2)}$, an element y of $\mathrm{H}(\theta)$ and a language τ with $|\mathfrak{S}_\tau| \leq \kappa$ and $\mathfrak{S}_\tau \subseteq \mathrm{H}(\kappa)$. Using Lemma 3.3, we can find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\bar{\kappa} < \min(\kappa, \bar{\theta})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $\mu, \tau, y \in \mathrm{ran}(j)$. Then $\mu < \bar{\kappa}$ and the fact that $\mu \in C^{(1)}$ implies that $\mathrm{H}(\mu) \cup \{\mathrm{H}(\mu)\} \subseteq X$ with $j \restriction \mathrm{H}(\mu) = \mathrm{id}_{\mathrm{H}(\mu)}$. In particular, we know that $z \in X$ with $j(z) = z$. Moreover, our setup ensures that there is a language $\bar{\tau}$ in X with $j(\bar{\tau}) = \tau$, $\mathfrak{S}_{\bar{\tau}} \subseteq \mathrm{H}(\bar{\kappa})$ and $|\mathfrak{S}_{\bar{\tau}}| \leq \bar{\kappa}$. It then follows that $j \restriction \mathfrak{S}_{\bar{\tau}} = \mathrm{id}_{\mathfrak{S}_{\bar{\tau}}}$ and $\bar{\tau}$ is a sublanguage of τ with $\mathfrak{S}_{\bar{\tau}} = \mathfrak{S}_\tau \cap \mathrm{H}(\bar{\kappa})$. Moreover, since $\mathcal{L}(\tau)$ is the unique set ℓ such that $\varphi(\tau, \ell, z)$ holds and $\mathcal{L}(\bar{\tau})$ is the unique set $\bar{\ell}$ such that $\varphi(\bar{\tau}, \bar{\ell}, z)$ holds, the correctness properties of $\mathrm{H}(\theta)$ and $\mathrm{H}(\bar{\theta})$ imply that $\mathcal{L}(\bar{\tau}) \in X$ with $j(\mathcal{L}(\bar{\tau})) = \mathcal{L}(\tau)$.

Now, assume, towards a contradiction, that there is $\phi_* \in \mathcal{L}(\bar{\tau}) \cap X$ with $j(\phi_*) \neq \phi_*$. The correctness properties of $\mathrm{H}(\bar{\theta})$ and (7) in Definition 5.2 then allow us to find a sublanguage τ_0 of $\bar{\tau}$ in X with $\phi_* \in \mathcal{L}(\tau_0)$ and $|\mathfrak{S}_{\tau_0}| < o_{\mathcal{L}}$. We then know that τ_0 is an element of $\mathrm{H}(\bar{\kappa}) \cap X$ with $j(\tau_0) = \tau_0$ and $\mathcal{L}(\tau_0)$ is an element of X with $j(\mathcal{L}(\tau_0)) = \mathcal{L}(\tau_0)$. This implies that $j(\phi_*)$ is an element of $\mathcal{L}(\tau_0)$. In this situation, we can now pick a language τ_1 that is an element of $\mathrm{H}(o_{\mathcal{L}})$ and a renaming u of τ_1 into τ_0 that is an element of $\mathrm{H}(\bar{\kappa}) \cap X$. It follows that $j(u) = u$. Let u^* denote the canonical bijection between Str_{τ_1} and Str_{τ_0} induced by u. Since τ_0 and u are both elements of $\mathrm{H}(\bar{\kappa}) \cap X$, it follows that $u^*(M) \in \mathrm{H}(\bar{\kappa}) \cap X$ holds for every τ_1-structure M in $\mathrm{H}(\mu)$. In particular, we know that $j(u^*(M)) = u^*(M)$ holds for every τ_1-structure M in $\mathrm{H}(\mu)$. Let $u_* : \mathcal{L}(\tau_1) \longrightarrow \mathcal{L}(\tau_0)$ denote the unique bijection given by (6) in Definition 5.2.

Claim. *The map u_* is an element of X.*

Proof of the Claim. First, since θ is an element of $C^{(n+2)}$, if follows that u_* is an element of $\mathrm{H}(\theta)$. The elementarity of j and the correctness properties of $\mathrm{H}(\bar{\theta})$ then imply that there is a bijection $\varpi : \mathcal{L}(\tau_1) \longrightarrow \mathcal{L}(\tau_0)$ in X with the property that

$$M \models_{\mathcal{L}} \phi \iff u^*(M) \models_{\mathcal{L}} \varpi(\phi)$$

holds for all $M \in Str_{\tau_1} \cap X$ and all $\phi \in \mathcal{L}(\tau_1)$. Since $\mathrm{H}(\mu) \subseteq X$, we can now apply (2) to conclude that $\varpi = u_*$. □

Since our setup ensures that $\mathcal{L}(\tau_1) \subseteq X$, we now know that $\mathcal{L}(\tau_0) = u_*[\mathcal{L}(\tau_1)] \subseteq X$ and the fact that $j(\mathcal{L}(\tau_0)) = \mathcal{L}(\tau_0)$ implies that

$$j \restriction \mathcal{L}(\tau_0) : \mathcal{L}(\tau_0) \longrightarrow \mathcal{L}(\tau_0)$$

is a bijection. If we now define

$$\pi \;=\; (u_*)^{-1} \circ (j \restriction \mathcal{L}(\tau_0)) \circ u_* : \mathcal{L}(\tau_1) \longrightarrow \mathcal{L}(\tau_1),$$

then we have

$$\pi((u_*)^{-1}(\phi_*)) \;=\; (u_*)^{-1}(j(\phi_*)) \;\neq\; (u_*)^{-1}(\phi_*)$$

and this shows that π is a non-trivial permutation of $\mathcal{L}(\tau_1)$. Since $M_{\tau_1,\pi} \in \mathrm{H}(\mu) \subseteq \mathrm{H}(\bar{\kappa}) \cap X$, we know that $j(M_{\tau_1,\pi}) = M_{\tau_1,\pi}$. Moreover, earlier computations show that $u^*(M_{\tau_1,\pi})$ is an element of X with $j(u^*(M_{\tau_1,\pi})) = u^*(M_{\tau_1,\pi})$. Since $u_*(\phi_{\tau_1,\pi}) \in \mathcal{L}(\tau_0) \subseteq X$, we can now conclude that

$$\begin{aligned}
M_{\tau_1,\pi} \models_{\mathcal{L}} \phi_{\tau_1,\pi} &\iff u^*(M_{\tau_1,\pi}) \models_{\mathcal{L}} u_*(\phi_{\tau_1,\pi}) \\
&\iff X \models \psi(u^*(M_{\tau_1,\pi}), u_*(\phi_{\tau_1,\pi}), z) \\
&\iff \mathrm{H}(\theta) \models \psi(u^*(M_{\tau_1,\pi}), (j \circ u_*)(\phi_{\tau_1,\pi}), z) \\
&\iff u^*(M_{\tau_1,\pi}) \models_{\mathcal{L}} (u_* \circ \pi)(\phi_{\tau_1,\pi}) \\
&\iff M_{\tau_1,\pi} \models_{\mathcal{L}} \pi(\phi_{\tau_1,\pi}),
\end{aligned}$$

where the first and fifth equivalence is given by (6) in Definition 5.2, the second equivalence is given by the correctness properties of $\mathrm{H}(\bar{\theta})$, the third equivalence is given by the elementarity of j and the fourth equivalence is given by the correctness properties of $\mathrm{H}(\theta)$. The above equivalences now contradict the defining properties of $M_{\tau_1,\pi}$ and $\phi_{\tau_1,\pi}$. This shows that $j \restriction (\mathcal{L}(\bar{\tau}) \cap X) = \mathrm{id}_{\mathcal{L}(\bar{\tau}) \cap X}$.

(4) Let μ be a cardinal as in (1), let $\kappa > \mu$ be a $C^{(n)}$-weakly shrewd cardinal and let U be a $<\kappa$-satisfiable $\mathcal{L}$-theory of cardinality κ. Since $o_{\mathcal{L}} < \kappa$, we can find a language σ with $|\mathfrak{S}_\sigma| \leq \kappa$ and $U \subseteq \mathcal{L}(\sigma)$. Then there is a renaming u of σ into a language τ with $\mathfrak{S}_\tau \subseteq \mathrm{H}(\kappa)$. Set $T = u_*[U]$. Then (6) in Definition 5.2 ensures that T is also $<\kappa$-satisfiable. Now, pick a cardinal $\kappa < \theta \in C^{(n+2)}$ with

$T \in \mathrm{H}(\theta)$ and use (3) to find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\mu < \bar{\kappa} < \min(\kappa, \bar{\theta})$, a sublanguage $\bar{\tau}$ of τ with $|\mathfrak{S}_{\bar{\tau}}| \leq \bar{\kappa}$ and $\mathfrak{S}_{\bar{\tau}} = \mathfrak{S}_\tau \cap \mathrm{H}(\bar{\kappa})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} \cup \{\bar{\kappa}, \bar{\tau}, \mathcal{L}(\bar{\tau}), z\} \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(z) = z$, $j(\bar{\tau}) = \tau$, $j(\mathcal{L}(\bar{\tau})) = \mathcal{L}(\tau)$, $j \restriction (\mathcal{L}(\bar{\tau}) \cap X) = \mathrm{id}_{\mathcal{L}(\bar{\tau}) \cap X}$ and $T \in \mathrm{ran}(j)$. Pick $\bar{T} \in X$ with $j(\bar{T}) = T$. Then $\bar{T}$ is a subset of $\mathcal{L}(\bar{\tau})$ of cardinality $\bar{\kappa}$ and the fact that $\bar{\kappa} \subseteq X$ ensures that $\bar{T} \subseteq X$. Moreover, since $j \restriction (\mathcal{L}(\bar{\tau}) \cap X) = \mathrm{id}_{\mathcal{L}(\bar{\tau}) \cap X}$, it follows that $\bar{T} = j[\bar{T}] \subseteq T$ and hence $\bar{T}$ is satisfiable. This shows that there is a $\bar{\tau}$-structure M with the property that $\psi(M, \phi, z)$ holds for all ϕ in $\bar{T}$. The correctness properties of $\mathrm{H}(\bar{\theta})$ now ensure that this statement holds in X and hence we know that, in $\mathrm{H}(\theta)$, there is a τ-structure N with the property that $\psi(N, \phi, z)$ holds for all ϕ in T. Using the correctness properties of $\mathrm{H}(\theta)$, we can now conclude that T is satisfiable and, by (6) in Definition 5.2, we also know that U is satisfiable.

(5) Let μ be a cardinal as in (1), let $\kappa > \mu$ be a $C^{(n)}$-weakly shrewd cardinal, let σ be a language with the property that $|\mathfrak{S}_\sigma| < \kappa$, let I be a σ-structure with $|\mathrm{dom}(I)| = \kappa$ and let ϕ be an element of $\mathcal{L}(\sigma)$ with $I \models_{\mathcal{L}} \phi$. Let u be a renaming of σ into a language $\tau \in \mathrm{H}(\kappa)$. Set $K = u^*(I)$ and $\phi_1 = u_*(\phi)$. Then $K \models_{\mathcal{L}} \phi_1$ and there is an isomorphism π between K and a τ-structure N with $\mathrm{dom}(N) = \kappa$ and $N \models_{\mathcal{L}} \phi_1$. Fix a cardinal $\kappa < \theta \in C^{(n+2)}$. Then K is an element of $\mathrm{H}(\theta)$ and we can use (3) to find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\mu < \bar{\kappa} < \min(\kappa, \bar{\theta})$, a sublanguage $\bar{\tau}$ of τ with $|\mathfrak{S}_{\bar{\tau}}| \leq \bar{\kappa}$ and $\mathfrak{S}_{\bar{\tau}} = \mathfrak{S}_\tau \cap \mathrm{H}(\bar{\kappa})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} \cup \{\bar{\kappa}, \bar{\tau}, \mathcal{L}(\bar{\tau}), z\} \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(z) = z$, $j(\bar{\tau}) = \tau$, $j(\mathcal{L}(\bar{\tau})) = \mathcal{L}(\tau)$, $j \restriction (\mathcal{L}(\bar{\tau}) \cap X) = \mathrm{id}_{\mathcal{L}(\bar{\tau}) \cap X}$ and $N, \phi_1 \in \mathrm{ran}(j)$. The fact that τ is an element of $\mathrm{H}(\kappa)$ then implies that $\tau = \bar{\tau} \in \mathrm{H}(\bar{\kappa})$, $\mathfrak{S}_\tau \subseteq X$ and $j \restriction \mathfrak{S}_\tau = \mathrm{id}_{\mathfrak{S}_\tau}$. Pick $M \in X$ with $j(M) = N$. Then M is a τ-structure with $\mathrm{dom}(M) = \bar{\kappa}$ and it follows that M is a substructure of N. Next, pick ϕ_0 in X with $j(\phi_0) = \phi_1$. Then $\phi_0 \in \mathcal{L}(\tau)$ and the fact that $j \restriction (\mathcal{L}(\tau) \cap X) = \mathrm{id}_{\mathcal{L}(\tau) \cap X}$ implies that $\phi_0 = \phi_1 \in X$. Moreover, using the elementarity of j and the correctness properties of $\mathrm{H}(\theta)$ and $\mathrm{H}(\bar{\theta})$, we know that $M \models_{\mathcal{L}} \phi_1$ holds. Then $\pi^{-1} \restriction \bar{\kappa}$ is an isomorphism between M and a substructure J of K, and we can apply (5) in Definition 5.2 to show that $J \models_{\mathcal{L}} \phi_1$. Finally, we can use (5) in Definition 5.2 to conclude that $H = (u^*)^{-1}(J)$ is a substructure of I with $|\mathrm{dom}(H)| < \kappa$ and $H \models_{\mathcal{L}} \phi$.

(6) Given a cardinal μ as in (1), a $C^{(n)}$-weakly shrewd cardinal $\kappa > \mu$ satisfying $\nu^{<o_{\mathcal{L}}} \leq \kappa$ for all $\nu < \kappa$, a language σ with $|\mathfrak{S}_\sigma| < \kappa$ and a σ-structure I with $|\mathrm{dom}(I)| = \kappa$, we proceed as in (5) to first find a renaming u of σ into a language $\tau \in \mathrm{H}(\kappa)$ and an isomorphism π between $u^*(I)$ and a τ-structure N with $\mathrm{dom}(N) = \kappa$. As above, fix a cardinal $\kappa < \theta \in C^{(n+2)}$ and use (3) to find a cardinal $\bar{\theta} \in C^{(n)}$, a cardinal $\mu < \bar{\kappa} < \min(\kappa, \bar{\theta})$ with $\tau \in \mathrm{H}(\bar{\kappa})$, an elementary submodel X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} \cup \{\bar{\kappa}, \tau, \mathcal{L}(\tau), z\} \subseteq X$ and an elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$, $j(z) = z$, $j(\tau) = \tau$, $j(\mathcal{L}(\tau)) = \mathcal{L}(\tau)$, $j \restriction (\mathcal{L}(\tau) \cap X) = \mathrm{id}_{\mathcal{L}(\tau) \cap X}$ and $N \in \mathrm{ran}(j)$. This setup then ensures that $\nu^{<o_{\mathcal{L}}} \leq \bar{\kappa}$ holds for all $\nu < \bar{\kappa}$.

Claim. $\mathcal{L}(\tau) \subseteq X$.

Proof of the Claim. Fix $\phi \in \mathcal{L}(\tau)$. By (7) in Definition 5.2, there is a sublanguage τ_0 of τ with $|\mathfrak{S}_{\tau_0}| < o_{\mathcal{L}}$ and $\phi \in \mathcal{L}(\tau_0)$. Since $|\mathfrak{S}_\tau|^{<o_{\mathcal{L}}} \leq \bar{\kappa} \subseteq X$, it follows that τ_0 is an element of $\mathrm{H}(\bar{\kappa}) \cap X$ and this implies that $j(\tau_0) = \tau_0$ and $j(\mathcal{L}(\tau_0)) = \mathcal{L}(\tau_0)$. Now, we can find a language $\tau_1 \in \mathrm{H}(o_{\mathcal{L}})$ and a renaming v of τ_1 into τ_0 that is an element of X. Our setup then ensures that $\mathcal{L}(\tau_1) \in \mathrm{H}(\mu) \subseteq X$ and $j(\mathcal{L}(\tau_1)) = \mathcal{L}(\tau_1)$. Since (2) shows that v_* is the unique bijection between $\mathcal{L}(\tau_1)$ and $\mathcal{L}(\tau_0)$ with the property that (C) holds for all τ_1-structures M in $\mathrm{H}(\mu)$ and every $\phi \in \mathcal{L}(\tau_1)$, the fact that all parameters of this existence statement are elements of X that are fixed by j ensures that v_* is an

element of X. In particular, since $\mathcal{L}(\tau_1) \subseteq \mathrm{H}(\mu) \subseteq X$, we know that $\mathcal{L}(\tau_0) \subseteq X$ and hence ϕ is an element of X. □

Now, pick $M \in X$ with $j(M) = N$. Then M is a τ-structure with $\mathrm{dom}(M) = \bar{\kappa}$. Moreover, since $j \restriction \mathfrak{S}_\tau = \mathrm{id}_{\mathfrak{S}_\tau}$, we know that M is a substructure of N. Fix $\phi \in \mathcal{L}(\tau)$. Then $\phi \in X$ with $j(\phi) = \phi$ and we can conclude that

$$M \models_{\mathcal{L}} \phi \iff X \models \psi(M, \phi, z) \iff \mathrm{H}(\theta) \models \psi(N, \phi, z) \iff N \models_{\mathcal{L}} \phi.$$

This shows that M is an $\mathcal{L}$-elementary submodel of N and, by arguing as in the proof of (5), this shows that there is an $\mathcal{L}$-elementary submodel H of I with $|\mathrm{dom}(H)| < \kappa$. □

By combining the above lemma with Theorems 2.4 and 3.5, we obtain the forward implications in Theorems 1.4, 1.6 and 5.5:

Corollary 6.2. (1) *If* Ord *is essentially faint and $\mathcal{L}$ is an abstract logic, then there is a proper class of cardinals that are both weak compactness cardinals for $\mathcal{L}$ and strict Löwenheim–Skolem–Tarski numbers for $\mathcal{L}$.*

(2) *If* Ord *is essentially subtle and $\mathcal{L}$ is an abstract logic, then the class of cardinals that are both weak compactness cardinals for $\mathcal{L}$ and weak Löwenheim–Skolem–Tarski numbers for $\mathcal{L}$ is stationary.*

Proof. Given a natural number $n > 0$, assume that $\mathcal{L}$ is an abstract logic with the property that the function $\mathcal{L}$ and the relation $\models_{\mathcal{L}}$ are definable by Σ_n-formulas with parameters. Then (4) and (5) of Lemma 6.1 show that all sufficiently large $C^{(n)}$-weakly shrewd cardinals are both weak compactness cardinals for $\mathcal{L}$ and strict Löwenheim–Skolem–Tarski numbers for $\mathcal{L}$. Theorem 3.5 then shows that the assumption that Ord is essentially faint implies that there is a proper class of cardinals that are both weak compactness cardinals for $\mathcal{L}$ and strict Löwenheim–Skolem–Tarski numbers for $\mathcal{L}$.

Now, let $\mathcal{L}$ be an abstract logic and let C be a closed unbounded class of ordinals. Then there is a natural number $n > 0$ such that both $\mathcal{L}$ and C are definable by Σ_n-formulas with parameters. Since every sufficiently large cardinal in $C^{(n)}$ is an element of C, we can combine the above computations with Lemma 4.1 to conclude that every sufficiently large $C^{(n)}$-strongly unfoldable cardinal is an element of C that is both a weak compactness cardinal for $\mathcal{L}$ and a weak Löwenheim–Skolem–Tarski number for $\mathcal{L}$. In particular, Theorem 2.4 shows that the assumption that Ord is essentially subtle implies that every closed unbounded class of ordinals contains a cardinal that is both a weak compactness cardinal for $\mathcal{L}$ and a weak Löwenheim–Skolem–Tarski number for $\mathcal{L}$. □

By combining the above corollary with arguments contained in [5], we can also use the obtained results to separate the principle "Ord *is subtle*" from the principle "Ord *is essentially subtle*":

Corollary 6.3. *If the theory*

$$\mathrm{ZFC} \ + \ \text{“Ord } \textit{is subtle}\text{”}$$

is consistent, then this theory does not prove that Ord *is essentially subtle.*

Proof. The proof of [5, Theorem 5.6] shows that, if the above theory is consistent, then it is consistent with the assumption that there is no weak compactness cardinal for second-order logic. An application of Corollary 6.2 then shows that this theory does not prove that Ord is essentially subtle. □

We now work towards proving the backward implications in Theorems 1.4, 1.6 and 5.5. Given a natural number $n > 0$ and a cardinal κ that is not $C^{(n)}$-weakly shrewd, we let θ^n_κ denote the

minimal element θ of $C^{(n)}$ greater than κ with the property that there exists $z \in \mathrm{H}(\theta)$ such that for all $\bar{\theta} \in C^{(n)}$, all cardinals $\bar{\kappa} < \min(\kappa, \bar{\theta})$ and all elementary submodels X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$, there is no elementary embedding $j : X \longrightarrow \mathrm{H}(\theta)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$. In addition, given a natural number $n > 0$ and a cardinal κ that is not $C^{(n)}$-weakly shrewd, we let Z^n_κ denote the non-empty set of all elements z of $\mathrm{H}(\theta^n_\kappa)$ with the property that for all $\bar{\theta} \in C^{(n)}$, all cardinals $\bar{\kappa} < \min(\kappa, \bar{\theta})$ and all elementary submodels X of $\mathrm{H}(\bar{\theta})$ with $\bar{\kappa} + 1 \subseteq X$, there is no elementary embedding $j : X \longrightarrow \mathrm{H}(\theta^n_\kappa)$ with $j \restriction \bar{\kappa} = \mathrm{id}_{\bar{\kappa}}$, $j(\bar{\kappa}) = \kappa$ and $z \in \mathrm{ran}(j)$. In the following, let σ denote the language consisting of a binary relation symbol E and four constant symbols d, e, f and g. Given a natural number m, a natural number $n > 0$ and a cardinal $\rho \in C^{(1)}$ of uncountable cofinality, we define $\mathcal{C}^{m,n}_\rho$ to be the class of all σ-structures M with the property that there exists a (necessarily unique) cardinal $\rho < \kappa_M \in \mathrm{Lim}(C^{(m)})$ such that the following statements hold:

- There exists a cardinal $\vartheta > \kappa_M$ with the property that $\vartheta > \theta^n_\mu$ holds for every cardinal μ in $C^{(m)} \cap (\rho, \kappa_M]$ that is not $C^{(n)}$-weakly shrewd and $\mathrm{dom}(M)$ is an elementary submodel of $\mathrm{H}(\vartheta)$ of cardinality κ_M with $\kappa_M + 1 \subseteq \mathrm{dom}(M)$.
- $E^M = \in \restriction (\mathrm{dom}(M) \times \mathrm{dom}(M))$, $d^M = \kappa_M$ and $e^M = \rho$.
- f^M and g^M are functions whose domains are the set of all cardinals in $C^{(m)} \cap (\rho, \kappa_M]$ that are not $C^{(n)}$-weakly shrewd.
- If $\mu \in \mathrm{dom}(f^M)$, then $f^M(\mu)$ is an elementary submodel of $\mathrm{H}(\theta^n_\mu)$ of cardinality μ with $\mu + 1 \subseteq f^M(\mu)$ and $g^M(\mu) \in Z^n_\mu \cap f^M(\mu)$.

The following lemma shows how the above class of structures can be used to derive the existence of $C^{(n)}$-weakly shrewd cardinals from strong structural properties of abstract logics:

Lemma 6.4. *Given a natural number m, a natural number $n > 0$ and a cardinal $\rho \in C^{(1)}$ of uncountable cofinality, assume that $\mathcal{L}$ is an abstract logic with the property that the following statements hold whenever τ is a language extending σ:*

1. *The set $\mathcal{L}(\tau)$ contains all first-order sentences of τ, a distinguished sentence ϕ_* and distinguished sentences $\phi_{<\xi}$ and $\phi_{\geq\xi}$ for every ordinal $\xi \leq \rho$.*
2. *Given $M \in \mathit{Str}_\tau$ and a first-order sentence $\phi \in \mathcal{L}(\tau)$, $M \models_{\mathcal{L}} \phi$ holds if and only if ϕ holds in M (in the sense of first-order structures).*
3. *Given $M \in \mathit{Str}_\tau$, $M \models_{\mathcal{L}} \phi_*$ holds if and only if $M \restriction \sigma$ is isomorphic to a structure in $\mathcal{C}^{m,n}_\rho$.*
4. *Given $\xi \leq \rho$ and $M \in \mathit{Str}_\tau$, $M \models_{\mathcal{L}} \phi_{<\xi}$ holds if and only if $\langle \mathrm{dom}(M), E^M \rangle$ is a well-order of order-type less than ξ.*
5. *Given $\xi \leq \rho$ and $M \in \mathit{Str}_\tau$, $M \models_{\mathcal{L}} \phi_{\geq\xi}$ holds if and only if $\langle \mathrm{dom}(M), E^M \rangle$ is a well-order of order-type at least ξ.*

If $\kappa \in \mathrm{Lim}(C^{(m)})$ is either a weak compactness cardinal for $\mathcal{L}$ or a strict Löwenheim–Skolem–Tarski number for $\mathcal{L}$, then $\kappa > \rho$ and the set $C^{(m)} \cap (\rho, \kappa]$ contains a $C^{(n)}$-weakly shrewd cardinal μ with the property that $\delta^\eta < \mu$ holds for all $\delta < \mu$ and $\eta \leq \rho$ with $\delta^\eta < \kappa$.

Proof. First, assume towards a contradiction, that $\kappa \leq \rho$. Define

$$T \;=\; \{\phi_{<\kappa}\} \;\cup\; \{\phi_{\geq\xi} \mid \xi < \kappa\} \;\subseteq\; \mathcal{L}(\sigma).$$

Then T is an unsatisfiable $\mathcal{L}$-theory that is $<\kappa$-satisfiable. This shows that κ is not a weak compactness cardinal for $\mathcal{L}$. If we now pick a σ-structure N with $\mathrm{dom}(N) = \kappa$ and $E^N = \in \restriction (\kappa \times \kappa)$, then $N \models_{\mathcal{L}} \phi_{\geq\kappa}$ holds and $M \models_{\mathcal{L}} \phi_{\geq\kappa}$ fails for every substructure M of N with $|\mathrm{dom}(M)| < \kappa$. This shows that κ is also not a strict Löwenheim–Skolem–Tarski number for $\mathcal{L}$, contradicting our initial assumptions. Therefore, we know that $\kappa > \rho$ and we can pick $M \in \mathcal{C}^{m,n}_\rho$ with $\kappa_M = \kappa$.

Claim. *There exist σ-structures H and K in $\mathcal{C}^{m,n}_\rho$ with $\kappa \in \{\kappa_H, \kappa_K\}$ and the property that there is an elementary embedding $j : \mathrm{dom}(H) \longrightarrow \mathrm{dom}(K)$ satisfying $j(\kappa_H) = \kappa_K$, $j(\rho) = \rho$, $j \restriction (\kappa_H + 1) \neq \mathrm{id}_{\kappa_H+1}$, $j(f^H) = f^K$ and $j(g^H) = g^K$.*

Proof of the Claim. First, assume that κ is a weak compactness cardinal for $\mathcal{L}$. Let τ denote the language that extends σ by a constant symbol c and a constant symbol c_x for every $x \in \mathrm{dom}(M)$. Next, let T denote the $\mathcal{L}(\tau)$-theory consisting of the union of the first-order elementary diagram of M using the constant symbols c_x and the set

$$\{\phi_*\} \ \cup \ \{c \, E \, d\} \ \cup \ \{c \neq c_\alpha \mid \alpha < \kappa\}.$$

This theory has cardinality κ and τ-expansions of the structure M witness that it is $<\kappa$-satisfiable. Hence, our assumption implies that T is satisfiable. Our setup and (5) in Definition 5.2 now ensure that T has a model whose σ-reduct N is an element of $\mathcal{C}^{m,n}_\rho$. We can now find an elementary embedding $j : \mathrm{dom}(M) \longrightarrow \mathrm{dom}(N)$ with $j(\kappa) = \kappa_N \geq \kappa$, $j(\rho) = \rho$, $c^N \in \kappa_N \setminus j[\kappa]$, $j(f^M) = f^N$ and $j(g^M) = g^N$. In particular, we know that $j \restriction (\kappa + 1) \neq \mathrm{id}_{\kappa+1}$.

Now, assume that κ is a strict Löwenheim–Skolem–Tarski number for $\mathcal{L}$. Let τ be a language extending σ that adds functions symbols for first-order Skolem functions to σ and let N be a τ-structure such that $N \restriction \sigma = M$ and N interprets all new function symbols as Skolem functions. Then $N \models_{\mathcal{L}} \phi_*$ and $|\mathrm{dom}(N)| = \kappa$. Therefore, our assumptions yield a substructure K of N with $K \models_{\mathcal{L}} \phi_*$ and $|\mathrm{dom}(K)| < \kappa$. We can now find a σ-structure H in $\mathcal{C}^{m,n}_\rho$ that is isomorphic to $K \restriction \sigma$. Our setup now ensures that there is an elementary embedding $j : \mathrm{dom}(H) \longrightarrow \mathrm{dom}(M)$ with $j(\kappa_H) = \kappa$, $j(\rho) = \rho$, $j(f^H) = f^M$ and $j(g^H) = g^M$. Moreover, since $\kappa_H = |\mathrm{dom}(H)| < \kappa$, we know that $j \restriction (\kappa_H + 1) \neq \mathrm{id}_{\kappa_H+1}$. □

Fix σ-structures H and K in $\mathcal{C}^{m,n}_\rho$ and an elementary embedding $j : \mathrm{dom}(H) \longrightarrow \mathrm{dom}(K)$ as in the above claim. Since $j \restriction (\kappa_H + 1) \neq \mathrm{id}_{\kappa_H+1}$, we can find an ordinal $\mu \leq \kappa_H$ with $j \restriction \mu = \mathrm{id}_\mu$ and $j(\mu) > \mu$. The fact that $\mathrm{dom}(H)$ is an elementary submodel of $\mathrm{H}(\vartheta)$ for some cardinal $\vartheta > \kappa$ then implies that μ is a regular cardinal.

Claim. $\mu > \rho$.

Proof of the Claim. Since ρ is an element of $C^{(1)} \cap \kappa_H$, we know that $\mathrm{V}_\rho \cup \{\mathrm{V}_\rho\} \subseteq \mathrm{dom}(H)$. Moreover, since $j(\rho) = \rho$, it follows that $j(\mathrm{V}_\rho) = \mathrm{V}_\rho$ and $j \restriction \mathrm{V}_\rho : \mathrm{V}_\rho \longrightarrow \mathrm{V}_\rho$. In this situation, the fact that ρ has uncountable cofinality now allows us to use the *Kunen Inconsistency* to conclude that $j \restriction \mathrm{V}_\rho = \mathrm{id}_{\mathrm{V}_\rho}$. □

Claim. $\mu \in C^{(m)}$.

Proof of the Claim. Assume, towards a contradiction, that $\mu \notin C^{(m)}$. Set $\xi = \min(C^{(m)} \setminus \mu) < \kappa_H$. Then there is an ordinal $\eta < \mu$ with the property that the ordinal ξ is definable in $\mathrm{dom}(H)$ by a formula using the parameters κ_H and η and the same formula with the parameters κ_K and η defines ξ in $\mathrm{dom}(K)$. It follows that $j(\xi) = \xi > \mu$. Since $C^{(0)}$ is the class of all ordinals, we now know $m > 0$ and this implies that $\kappa_H \in C^{(1)}$ and $\mathrm{V}_{\kappa_H} \subseteq \mathrm{dom}(H)$. The fact that $j(\mathrm{V}_{\xi+2}) = \mathrm{V}_{\xi+2}$ then implies that $j \restriction \mathrm{V}_{\xi+2} : \mathrm{V}_{\xi+2} \longrightarrow \mathrm{V}_{\xi+2}$ is a non-trivial elementary embedding, contradicting the *Kunen Inconsistency*. □

Claim. *If $\delta < \mu$ and $\eta \leq \rho$ with $\delta^\eta < \kappa$, then $\delta^\eta < \mu$.*

Proof of the Claim. Assume, towards a contradiction, that there are cardinals $\delta < \mu$ and $\eta \leq \rho$ with $\mu \leq \delta^\eta < \kappa$. Then $\mathrm{dom}(H)$ contains an injection $\iota : \mu \longrightarrow {}^\eta\delta$. Moreover, since $\delta, \eta < \mu$,

$j(\kappa_H) = \kappa_K$ and $\kappa \in \{\kappa_H, \kappa_K\}$, the elementarity of j ensures that $\delta^\eta < \kappa_H$ and hence we know that ${}^\eta\delta \subseteq \mathrm{dom}(H)$. Set $s = j(\iota)(\mu) \in {}^\eta\delta$. Since $j(s) = s$, we can now find an $\alpha < \mu$ with $\iota(\alpha) = s$. But then $j(\iota)(\alpha) = j(\iota(\alpha)) = j(s) = j(\iota)(\mu)$, contradicting the injectivity of ι. □

Since $\kappa_H \leq j(\kappa_H) = \kappa_K$ and $\kappa \in \{\kappa_H, \kappa_K\}$, we know that $\mu \leq \kappa_H \leq \kappa$ and the above computations show that the cardinal μ is an element of the set $C^{(m)} \cap (\rho, \kappa]$. Assume, towards a contradiction, that μ is not a $C^{(n)}$-weakly shrewd cardinal. Then $\mu \in \mathrm{dom}(f^H)$. Set $\nu = j(\mu) > \mu$. Then ν is a regular cardinal in the interval $(\rho, \kappa_K]$ and elementarity implies that $\nu \in \mathrm{dom}(f^K)$. Set $X = f^H(\mu) \subseteq \mathrm{dom}(H)$ and $z = g^N(\nu) = j(g^M(\mu)) \in Z^n_\nu$. Then θ^n_μ is an element of $C^{(n)}$, $\mu < \min(\nu, \theta^n_\mu)$, X is an elementary submodel of $\mathrm{H}(\theta^n_\mu)$ with $\mu + 1 \subseteq X$ and $i = j \restriction X : X \longrightarrow \mathrm{H}(\theta^n_\nu)$ is an elementary embedding with $i \restriction \mu = \mathrm{id}_\mu$, $i(\mu) = \nu$ and $z \in \mathrm{ran}(i)$. But, this contradicts the fact that z is an element of Z^n_ν. This yields the conclusion of the lemma. □

We can now also combine the above lemma with Theorems 2.4 and 3.5 to derive the backward implications of Theorems 1.4, 1.6 and 5.5. This completes the proofs of these results.

Corollary 6.5. (1) *If every abstract logic has either a weak compactness cardinal or a strict Löwenheim–Skolem–Tarski number, then* Ord *is essentially faint.*

(2) *If every abstract logic has, for every natural number n, either a weak compactness cardinal that is an element of $C^{(n)}$ or a strict Löwenheim–Skolem–Tarski number that is an element of $C^{(n)}$, then* Ord *is essentially subtle.*

Proof. (1) Fix a natural number $n > 0$ and a cardinal ρ in $C^{(1)}$ of uncountable cofinality. Set $m = 0$. Then there exists an abstract logic $\mathcal{L}$ that satisfies the statements (1)–(5) listed in Lemma 6.4 for m, n and ρ. The fact that $C^{(m)}$ is the class of all ordinals now allows us to apply the lemma to find a $C^{(n)}$-weakly shrewd cardinal greater than ρ.

These computations show that for every natural number $n > 0$, there is a proper class of $C^{(n)}$-weakly shrewd cardinals. In this situation, Theorem 3.5 allows us to conclude that Ord is essentially faint.

(2) Given a natural number $n > 0$ and a cardinal $\rho \in C^{(1)}$ of uncountable cofinality, there exists an abstract logic $\mathcal{L}$ that satisfies the statements (1)–(5) listed in Lemma 6.4 for $m = n + 1$, n and ρ. Since $C^{(n+2)} \subseteq \mathrm{Lim}(C^{(n+1)})$, our assumptions allows us to use the lemma to find a $C^{(n)}$-weakly shrewd cardinal greater than ρ that is an element of $C^{(n+1)}$. We can then combine Theorem 2.4 with Lemma 4.1 to conclude that Ord is essentially subtle. □

Remark 6.6. In [8], Fuchino and Sakai consider the possibility of characterizing the existence of weak compactness cardinals for second-order logic $\mathcal{L}^2$ through the existence of certain large cardinals, analogous to Magidor's characterization of the existence of strong compactness cardinals for $\mathcal{L}^2$ through the existence of extendible cardinals in [14]. For this purpose, they introduce the concept of *weak extendibility* as a natural candidate for a large cardinal property providing such a characterization. Here, a cardinal κ is defined to be *weakly extendible* if $2^{<\kappa} = \kappa$ holds and for every ordinal $\theta > \kappa$ and every elementary submodel X of V_θ of cardinality κ with $\kappa + 1 \subseteq X$, there exists an ordinal $\vartheta > \kappa$ and an elementary embedding $j : X \longrightarrow \mathrm{V}_\vartheta$ with $j \restriction \kappa = \mathrm{id}_\kappa$ and $j(\kappa) > \kappa$. The results of [8] show that all weakly extendible cardinals are weakly compact and therefore strongly inaccessible.

Fuchino and Sakai prove that a cardinal κ is a weak compactness cardinal for $\mathcal{L}^2$ if and only if for every ordinal $\theta > \kappa$ and every elementary submodel X of V_θ of cardinality κ with $\kappa + 1 \subseteq X$, there exists an ordinal $\vartheta > \kappa$ and an elementary embedding $j : X \longrightarrow \mathrm{V}_\vartheta$ with $j(\kappa) > \sup(j[\kappa])$. This

equivalence directly implies that every weakly extendible cardinal is a weak compactness cardinal for $\mathcal{L}^2$. Moreover, a standard application of Jensen's *Covering Lemma* (as in [10, Theorem 18.27]) allows us to conclude that the assumption $\mathrm{V} = \mathrm{L}$ causes all weak compactness cardinals for $\mathcal{L}^2$ to be weakly extendible. In addition, a standard argument shows that, if κ is the least weak compactness cardinal for $\mathcal{L}^2$ and $2^{<\kappa} = \kappa$ holds, then κ is weakly extendible. These implications directly raise the question of whether the existence of a weak compactness cardinal for $\mathcal{L}^2$ provably implies the existence of a weakly extendible cardinal. A combination of Theorems 1.6 and 1.14 now provides a negative answer to this question by showing that weak compactness cardinals for $\mathcal{L}^2$ can exist in models of ZFC without strongly inaccessible cardinals.

In the remainder of this section, we consider the question whether it is possible to extend the conclusion of Theorem 5.5.(1) to obtain a characterization of the existence of weak Löwenheim–Skolem–Tarski numbers for all abstract logics through the assumption that Ord is essentially faint.

Theorem 6.7. *The following schemes are equivalent over* ZFC*:*

(1) *Every abstract logic has a weak Löwenheim–Skolem–Tarski number.*
(2) *For every natural number n and every cardinal η, there is a $C^{(n)}$-weakly shrewd cardinal κ with $\delta^\eta < \kappa$ for all $\delta < \kappa$.*

Proof. First, assume that (1) holds, $n > 0$ is a natural number and μ is a cardinal. Pick a cardinal ρ in $C^{(1)}$ above μ. Then there is an abstract logic $\mathcal{L}$ that satisfies statements (1)–(5) listed in Lemma 6.4 for $m = 0$, n and ρ together with the following statements:

(6) If τ is a language, then $\mathcal{L}(\tau)$ contains the set $\mathcal{L}_{\rho,\omega}(\tau)$ of all sentences of the infinitary logic $\mathcal{L}_{\rho,\omega}$ that extends first-order logic by allowing conjunctions and disjunctions of less than ρ-many formulas.
(7) Given a language τ, $M \in Str_\tau$ and $\phi \in \mathcal{L}_{\rho,\omega}(\tau)$, then $M \models_{\mathcal{L}} \phi$ if and only if $M \models_{\mathcal{L}_{\rho,\omega}} \phi$.

Our assumption now ensures that there is a weak Löwenheim–Skolem–Tarski number κ for $\mathcal{L}$.

Claim. *If $\delta < \kappa$, then $\delta^\eta < \kappa$.*

Proof of the Claim. Assume, towards a contradiction, that there exists a cardinal $\eta \leq \delta < \kappa$ with $\delta^\eta \geq \kappa$. Let τ denote a language that is obtained from σ by first adding a constant symbol c_γ for each $\gamma \leq \delta$ and then closing the language under Skolem functions for first order formulas. Pick a sufficiently large cardinal $\theta > \kappa$ and a τ-structure N such that $\mathrm{dom}(N)$ is an elementary submodel of $\mathrm{H}(\theta)$ of cardinality κ with $\kappa + 1 \subseteq \mathrm{dom}(N)$, $E^N = \in \restriction (\mathrm{dom}(N) \times \mathrm{dom}(N))$, $c^N_\gamma = \gamma$ for all $\gamma \leq \delta$ and all new function symbols are interpreted as the corresponding Skolem functions. By elementarity, we know that the set $\mathrm{dom}(N) \cap {}^\eta\delta$ has cardinality κ. Since κ is a weak Löwenheim–Skolem–Tarski number for $\mathcal{L}$, we can find an $\mathcal{L}$-elementary submodel M of N with $|\mathrm{dom}(M)| < \kappa$. Then $\mathrm{dom}(M)$ is an elementary submodel of $\mathrm{H}(\theta)$ with $\delta + 1 \subseteq \mathrm{dom}(M)$. Moreover, we know that there is a function $s \in N \cap {}^\eta\delta$ that is not an element of M. But, the set $\mathcal{L}_{\rho,\omega}(\tau)$ contains a sentence ϕ_s that states that there is a function from c_η to c_δ that sends c_γ to $c_{s(\gamma)}$ for all $\gamma < \eta$. It then follows that $N \models_{\mathcal{L}} \phi_s$ holds and $M \models_{\mathcal{L}} \phi_s$ fails, contradicting the fact that M is an $\mathcal{L}$-elementary submodel of N. □

An application of Lemma 6.4 now shows that the interval $(\rho, \kappa]$ contains a $C^{(n)}$-weakly shrewd cardinal μ with the property that $\delta^\eta < \mu$ holds for all $\delta < \mu$ with $\delta^\eta < \kappa$. By the above claim, we can conclude that $\delta^\eta < \mu$ holds for all $\delta < \mu$. This shows that (2) holds in this case.

Now, assume that (2) holds. Pick an abstract logic $\mathcal{L}$ and a natural number $n > 0$ such that $\mathcal{L}$ is definable by a Σ_n-formula with parameters. We can now apply (6) of Lemma 6.1 to see that

every sufficiently large $C^{(n)}$-weakly shrewd cardinal κ with the property that $\nu^{<o_{\mathcal{L}}} \leq \kappa$ holds for all $\nu < \kappa$ is a weak Löwenheim–Skolem–Tarski number for $\mathcal{L}$. Since our assumption ensures that there is a proper class of such cardinals, we can conclude that (1) holds in this case. □

The above result leaves open the question of whether the existence of weak Löwenheim–Skolem–Tarski numbers for all abstract logics is provably equivalent to the assumption that Ord is essentially faint (see Question 7.1 in the next section). In the following, we will use Theorem 6.7 to severely restrict the class of models of set theory in which the given principles are not equivalent. For this purpose, recall that the *Singular Cardinal Hypothesis* SCH states that $\kappa^{\mathrm{cof}(\kappa)} = \kappa^+$ holds for every singular cardinal κ with $2^{\mathrm{cof}(\kappa)} < \kappa$. In the following, we say that the SCH *eventually holds* if this implication holds for eventually all singular cardinals.

Corollary 6.8. *The following schemes are equivalent over* ZFC + "SCH *eventually holds*"*:*

(1) *Every abstract logic has a weak Löwenheim–Skolem–Tarski number.*
(2) Ord *is essentially faint.*

Proof. In one direction, a combination of Theorem 3.5 with Theorem 6.7 directly shows that (1) implies (2). In the other direction, note that, if the SCH eventually holds, then the proof of [10, Theorem 5.22] shows that for every cardinal μ, all sufficiently large weakly inaccessible cardinals κ have the property that $\nu^{<\mu} < \kappa$ holds for all $\nu < \kappa$. In particular, it follows that all sufficiently large $C^{(n)}$-weakly shrewd cardinals satisfy the assumptions of (6) in Lemma 6.1 and therefore these cardinals are weak Löwenheim–Skolem–Tarski numbers for abstract logics definable by Σ_n-formulas. This shows that (2) implies (1). □

Remember that a classical result of Solovay in [20] shows that the existence of a strongly compact cardinal implies that the SCH eventually holds. In particular, the existence of such a cardinal entails that Ord is essentially faint if and only if every abstract logic has a weak Löwenheim–Skolem–Tarski number. In another direction, Corollary 6.8 also shows that the inequivalence of the two principles implies the consistency of large cardinal assumptions in the region of measurable cardinals of high Mitchell order. This follows from the fact that this inequivalence requires a failure of the SCH at a proper class of singular cardinals and such a failure is known to imply the existence of such large cardinals in canonical inner models (see, for example, [18]).

7. Open questions

We end this paper by discussing two questions raised by the above results.

First, while the existence of strict Löwenheim–Skolem–Tarski numbers for all abstract logics was shown to be equivalent to the assumption that Ord is essentially faint in Theorem 5.5, it is unclear whether these principles are also equivalent to the existence of weak Löwenheim–Skolem–Tarski numbers for all abstract logics. More specifically, Theorem 6.7 leaves open the possibility that the equivalence stated in Theorem 3.5 can be strengthened so that the existence of $C^{(n)}$-weakly shrewd cardinals satisfying the cardinal arithmetic assumptions listed in Theorem 6.7 can be derived from the principle "Ord *is essentially faint*".

Question 7.1. *Assume that* Ord *is essentially faint. Given a natural number $n > 0$ and a cardinal η, is there is a $C^{(n)}$-weakly shrewd cardinal κ with $\delta^\eta < \kappa$ for all $\delta < \kappa$?*

As discussed at the end of Section 6, a negative answer to Question 7.1 would require substantial large cardinal assumptions. However, it is not clear if the given fragment of the SCH outright follows from the assumption that Ord is essentially faint.

Second, we ask whether the analysis carried out to separate the principle "Ord *is essentially subtle*" from the principle "Ord *is essentially faint*" can be extended to obtain characterizations of the consistency of stronger subtlety properties of the class of ordinals through the existence and non-existence of set-theoretic sentences with certain provable consequences. More specifically, Theorem 1.10 shows that the consistency of the class-version of the large cardinal property of being a regular limit of subtle cardinals is equivalent to the existence of a consistent sentence implying that Ord is essentially faint. In addition, Theorem 1.13 provides a characterization of the consistency of the class-version of subtle limits of subtle cardinals through the non-existence of a sentence axiomatizing the principle "Ord *is essentially subtle*" over the principle "Ord *is essentially faint*". These results raise the question whether the consistency of the class-version of the next stronger large cardinal property, i.e., the property of being a stationary limit of subtle cardinals, can be characterized in a similar manner.

Question 7.2. *Are there analogs of Theorems 1.10 and 1.13 that characterize the consistency of the theory*

$$\text{ZFC} + \text{"}\textit{The class of subtle cardinals is stationary in}\ \text{Ord}\text{"}?$$

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