Infinite Games
Lent Term 2020
Part III of the Mathematical Tripos
University of Cambridge
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Example Sheet #1

Examples Classes.
#1. Monday 3 February 2020, 3:30-5pm, MR14.
#2: Monday 17 February 2020, 3:30-5pm, MR14.
#3: Monday 2 March 2020, 3:30-5pm, MR14.
#4: Monday 16 March 2020, 3:30-5pm, MR14.
You hand in your work at the beginning of the Example Class.

Notation. In the following, M is the set of possible moves, $M^{<\omega}$ the set of positions, and M^{ω} the set of plays. If $s \in M^{<\omega}$ and $x \in M^{\omega}$, we write sx for the concatenation of s and x, i.e., $sx(n) := s(n)$ for $n < |s|$ and $sx(n) := x(n - |s|)$ for $n \geq |s|$. If $m \in M$, we write mx for the concatenation of the length one sequence with element m and the infinite sequence x.

1. If $x \in M^{\omega}$, we defined strategies σ_x and τ_x as the strategies for player I or player II, respectively, that play x independently of what the other player is doing. These strategies were called *blindfolded*. Show that a strategy σ is a winning strategy for player I in the game $\mathrm{G}(A)$ if and only if it wins against all blindfolded strategies for player II (i.e., $\sigma * \tau_x \in A$ for all $x \in M^{\omega}$).

2. If σ is a strategy, we defined the I-strategic tree T^{I}_{σ} and the II-strategic tree T^{II}_{σ}. Show that for all strategies σ and τ, we have that $[T^{\mathrm{I}}_{\sigma}] \cap [T^{\mathrm{II}}_{\tau}] = \{\sigma * \tau\}$.

3. If $A, B \subseteq M^{\omega}$ are disjoint, we define the winning conditions for the game $\mathrm{G}(A, B)$ as follows: if the play of the game is $x \in M^{\omega}$, then player I wins if $x \in A$, player II wins if $x \in B$, and otherwise the game is a *draw*. A strategy σ is *winning or drawing in* $\mathrm{G}(A, B)$ *for player I* if for all strategies τ, we have that $\sigma * \tau \in A$ or $\sigma * \tau \notin B$, respectively. A strategy τ is *winning or drawing in* $\mathrm{G}(A, B)$ *for player II* if for all strategies σ, we have that $\sigma * \tau \in B$ or $\sigma * \tau \notin A$, respectively.

   Show that if A and $M^{\omega}\backslash B$ are determined, then at least one of the two players has a drawing strategy in $\mathrm{G}(A, B)$ and that both players have a drawing strategy if and only if none of them has a winning strategy.

4. If $T \subseteq M^{<\omega}$ is a tree on M and $A \subseteq [T]$ is a payoff set, we define the game $\mathrm{G}(A; T)$ as follows: define $A_T := \{x \in M^{\omega}\,;\, x \in A \text{ or } x \notin [T] \text{ and the least } n \text{ such that } x{\upharpoonright}n \notin T \text{ is odd}\}$ and if the play of the game is $x \in M^{\omega}$, then player I wins if $x \in A_T$. Observe that $\mathrm{G}(A; T)$ is a game with rules described by the tree T: if a rule is broken (i.e., a position outside of the tree is played), the first player to do so loses the game. Clearly, $\mathrm{G}(A; T)$ and $\mathrm{G}(A_T)$ are the same game. Use this idea to describe games in which

   (a) the two players have different move sets (player I can make moves in M_{I} and player II can make moves in M_{II}) or

   (b) the players do not play alternatingly, but according to a move function $\mu : M^{<\omega} \to \{\mathrm{I}, \mathrm{II}\}$ that determines who has to move in position $p \in M^{<\omega}$

   as games of the form $\mathrm{G}(A)$ for some appropriate set A.

5. If G and H are graphs, the *Ramsey game* $\mathrm{Ramsey}(G,H)$ is the positional game in which players pick edges of G and the first one whose collected edge set contains a subgraph isomorphic to H wins. As usual, K_n and K_ω denote the complete graphs with n and countably many vertices, respectively. Fix $s \in \mathbb{N}$ and show that the following are equivalent:

    (a) player I has a winning strategy in $\mathrm{Ramsey}(\mathrm{K}_\omega, \mathrm{K}_s)$ and

    (b) there is a $T \in \mathbb{N}$ such that for all sufficiently large n, player I has a winning strategy in $\mathrm{Ramsey}(\mathrm{K}_n, \mathrm{K}_s)$ in less than T moves.

6. Let X and Y be sets. By $\mathsf{AC}_Y(X)$ we denote the *axiom of choice for Y-parametrised families of subsets of X*, i.e., the statement that for each family $\{A_y\,;y\in Y\}$ of non-empty subsets of X there is a choice function $c : Y \to X$ with $c(y) \in A_y$ for all $y \in Y$.

    Show (in ZF without the axiom of choice) that $\mathsf{AC}_\omega(\mathbb{R})$ implies that $\aleph_1$ is a regular cardinal.

7. Let X be a set and $A \subseteq X \times X$. A partial function $f : X \to X$ is called a *uniformisation of A* if for each $x \in X$, if there is some $y \in X$ such that $(x, y) \in A$, then $x \in \mathrm{dom}(f)$ and for all $x \in \mathrm{dom}(f)$, we have that $(x, f(x)) \in A$. We say that X *satisfies the uniformisation principle* if each subset $A \subseteq X \times X$ has a uniformisation.

    Show (in ZF without the axiom of choice) that $\mathsf{AC}_X(X)$ implies that X has the uniformisation principle.

8. Show (in ZF without the axiom of choice) that the determinacy of all finite games on a set of moves M implies $\mathsf{AC}_M(M)$.

9. Let $\kappa := |M|$ and show that $\kappa^+ = \{\alpha\,;$ there is a wellfounded tree T on M such that $\mathrm{ht}(T) = \alpha\}$.

10. Construct a set $A \subseteq M^\omega$ such that player I has a winning strategy in both $\mathrm{G}(A)$ and $\mathrm{G}(M^\omega\backslash A)$.

11. If $A \subseteq M^\omega$ and $m \in M$, let $A_m := \{x \in M^\omega\,; mx \in A\}$. Assume that for every $m \in M$, the set A_m is determined. Prove that $M^\omega\backslash A$ is determined.

12. Assume AC and show that

    (a) there is a determined set whose complement is not determined,

    (b) there are determined sets whose intersection is not determined, and

    (c) there are determined sets whose union is not determined.

13. In the proof of Zermelo's theorem, the winning strategy was "stay on positions with your label"; in the proof of the Gale-Stewart theorem, the winning strategy for player I was "stay on positions with your label and reduce the age of the position". Show that, in general, "stay on positions with your label" is not enough in the Gale-Stewart proof, but that if the payoff set A is clopen (i.e., both A and its complement are open), then it is enough.

14. Let $A \subseteq \omega^\omega$ be the difference of two open sets (i.e., there are open sets P and Q such that $A = P\backslash Q$). Show that A is determined.