Topics in Set Theory
Lent Term 2019
Part III of the Mathematical Tripos
University of Cambridge
Prof. Dr. B. Löwe

Example Sheet #3

Course webpage: `https://www.math.uni-hamburg.de/home/loewe/Lent2019/TST_L19.html`

Example Classes.
#1. Monday 4 February, 3:30–5pm, MR5.
#2. Monday 18 February, 3:30–5pm, MR5.
#3. Monday 4 March, 3:30–5pm, MR5.
#4. Thursday 14 March, 3:30-5pm, MR20.
You hand in your work at the beginning of the Example Class.

(23) Let $(\mathbb{P}, \leq)$ be a partial order and $p \in \mathbb{P}$. Show that

  (a) if D is dense below p and $r \leq p$, then D is dense below r;
  (b) if $\{r\,;\, D \text{ is dense below } r\}$ is dense below p, then D is dense below p.

(24) We say that G is $\mathbb{P}$-*antichain generic over* M if for every maximal $\mathbb{P}$-antichain $A \in M$, we have $A \cap G \neq \varnothing$. We call a set B a $\mathbb{P}$-*bar* if for every $p \in \mathbb{P}$ there is a $b \in B$ such that p and b are compatible. We say that G is $\mathbb{P}$-*bar generic over* M if for every $\mathbb{P}$-bar $B \in M$ we have that $B \cap G \neq \varnothing$.

Let $\mathbb{P} \in M$, and G be a filter over $\mathbb{P}$. Show that the following are equivalent:

  (i) G is $\mathbb{P}$-generic over M,
  (ii) G is $\mathbb{P}$-antichain generic over M, and
  (iii) G is $\mathbb{P}$-bar generic over M.

(25) Let M be a transitive model of set theory. Find a splitting partial order $\mathbb{P}$ and a filter H over $\mathbb{P}$ such that $M[H]$ is not a model of ZFC.

(26) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Suppose $\sigma, \tau \in M^{\mathbb{P}}$ and that D is a filter on $\mathbb{P}$. Show that $\mathrm{val}(\sigma \cup \tau, D) = \mathrm{val}(\sigma, D) \cup \mathrm{val}(\tau, D)$.

(27) Let M be a transitive model of set theory, $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order, and $x \in M$. Define $\mathrm{can}(x) := \{(\mathrm{can}(y), p)\,;\, y \in x \wedge p \in \mathbb{P}\}$. Show that if D is a filter, then $\mathrm{val}(\mathrm{can}(x), D) = x$.

(28) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Assume that $\tau \in M^{\mathbb{P}}$ such that for all $(\sigma, p) \in \tau$, we have $\sigma = \check{n}$ for some $n \in \mathbb{N}$. Assume furthermore that G is a generic filter on $\mathbb{P}$. Define

$$\tau^* := \{(\check{n}, p)\,;\, \forall q \in \mathbb{P}((\check{n}, q) \in \tau \to p \perp q)\}.$$

Use the Forcing Theorem to prove that $\mathrm{val}(\tau^*, G) = \omega \backslash \mathrm{val}(\tau, G)$.

(29) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a splitting partial order. Show that the class of names τ such that $\mathbf{1} \Vdash \tau = \check{1}$ is a proper class in M.

(30) Let M be a transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a partial order. Assume that $\tau \in M^{\mathbb{P}}$ and that D is a filter on $\mathbb{P}$. Define

$$\pi := \{(\varrho, p)\,;\, \exists(\sigma, q) \in \tau \exists r(\varrho, r) \in \sigma \wedge p \leq r \wedge p \leq q\}$$

and show that $\mathrm{val}(\pi, D) = \bigcup \mathrm{val}(\tau, D)$. Conclude that $M[D]$ satisfies the Union axiom.

(31) A family of finite sets $\mathcal{D}$ is called a *Δ-system* if there is a finite set R (called the *root of the Δ-system*) such that for all $D, D' \in \mathcal{D}$, if $D \neq D'$, then $D \cap D' = R$. Show that any uncountable family of finite sets contains an uncountable Δ-system.

(*Hint.* Argue that you can assume w.l.o.g. that all elements of the family have the same size and prove the claim by induction on the size of the elements of the family.)

(32) Use (31) in order to prove that for countable Y, the forcing partial order

$$\mathrm{Fn}(X, Y) := \{p\,;\, p \text{ is a partial function from } X \text{ to } Y \text{ with finite domain}\}$$

has the countable chain condition.

(33) Let M be a countable transitive model of set theory and $(\mathbb{P}, \leq, \mathbf{1}) \in M$ be a splitting partial order. Define recursively $M_0 := M$ and $M_{i+1} := M_i[G_i]$ where G_i is $\mathbb{P}$-generic over M_i. Prove that $\bigcup_{i \in \mathbb{N}} M_i$ is not a model of ZFC.