UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

2003/2004; 2nd Trimester
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Homework Set # 8 Deadline: Thursday, March 11th, 2004

Exercise 8.1 (Regular Ordinal Operations).

Let Φ be a function-like formula. We call Φ an **ordinal operation** if

(1) For all $\alpha \in \mathrm{Ord}$ there is a β such that $\Phi(\alpha, \beta)$.
(2) If $\Phi(\alpha, \beta)$, then α and β are ordinals.

If Φ is an ordinal operation, then we write $\Phi(\alpha)$ for the unique β such that $\Phi(\alpha, \beta)$.
An ordinal operation Φ is called **regular** if

(1) If $\alpha < \alpha^*$, then $\Phi(\alpha) < \Phi(\alpha^*)$.
(2) If λ is a limit ordinal, then $\Phi(\lambda) = \bigcup\{\Phi(\alpha)\,;\,\alpha < \lambda\}$.

Show that for every regular ordinal operation Φ and every ordinal α there is an ordinal $\gamma > \alpha$ such that $\Phi(\gamma) = \gamma$.
Show that for each α and β there is some $\gamma > \alpha$ such that $\beta^\gamma = \gamma$ (ordinal exponentiation).
Why is this supposed to remind you of ε_0?

Exercise 8.2 (Singular Cardinals; = ***x12.18** in *Moschovakis*)

Show that for each regular cardinal κ there is a singular cardinal $\lambda > \kappa$ such that $\mathrm{cf}(\lambda) = \kappa$.

N.B. The simple solution $\lambda := \aleph_\kappa$ could run into trouble if $\kappa = \aleph_\kappa$.