UNIVERSITEIT VAN AMSTERDAM
INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Axiomatic Set Theory

(Axiomatische Verzamelingentheorie)

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Homework Set # 11

For these exercises, use the "inductive computation of κ^λ" (Theorem 5.20 in Jech's book): Cardinal exponentiation κ^λ can be computed by induction on κ as follows:

(1) If $\kappa \le \lambda$, then $\kappa^\lambda = 2^\lambda$.
(2) If there is some $\mu < \kappa$ such that $\mu^\lambda \ge \kappa$, then $\kappa^\lambda = \mu^\lambda$.
(3) If $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then
 (a) if $\mathrm{cf}(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$, and
 (b) if $\mathrm{cf}(\kappa) \le \lambda$, then $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$.

Exercise 11.1 (Cardinal Arithmetic with exponent $\aleph_1$)

(1) Show that $(\aleph_n)^{\aleph_1} = \aleph_n \cdot 2^{\aleph_1}$.
(2) Show that $(\aleph_\omega)^{\aleph_1} = (\aleph_\omega)^{\aleph_0} \cdot 2^{\aleph_1}$.
(3) Show that $(\aleph_{\omega+1})^{\aleph_1} = (\aleph_{\omega+1})^{\aleph_0} \cdot 2^{\aleph_1}$.

Exercise 11.2 (Cardinal Arithmetic under the assumption $2^{\aleph_1} = \aleph_2$)
In this exercise, we work in $\mathsf{ZFC} + 2^{\aleph_1} = \aleph_2$. Keep the computations of # 5.1 in mind.

(1) Compute $(\aleph_n)^{\aleph_1}$ for all values of $n \in \omega$. ("Computation" means: determine α such that $(\aleph_n)^{\aleph_1} = \aleph_\alpha$.)
(2) Show that $(\aleph_\omega)^{\aleph_1} = (\aleph_\omega)^{\aleph_0}$.
(3) Show that $(\aleph_\omega)^{\aleph_0} \neq \aleph_{\aleph_1}$. (**Hint.** What is $((\aleph_\omega)^{\aleph_0})^{\aleph_1}$? What is $(\aleph_{\aleph_1})^{\aleph_1}$?)
(4) Assume in addition that $(\aleph_\omega)^{\aleph_0} > \aleph_{\aleph_1}$. Show that under this additional assumption, we get $(\aleph_\omega)^{\aleph_0} = (\aleph_{\aleph_1})^{\aleph_1}$.