Introduction to symplectic geometry

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Summary of topics

1. Linear symplectic geometry (treated as background for the main topics)
   (a) the different types of linear subspaces and their normal forms
   (b) linear symplectic maps and their properties
   (c) relations between $\mathrm{Sp}(2n)$, $U(n)$ and $O(2n)$
   (d) compatible complex structures, contractibility of $\mathcal{J}(\omega_{\mathrm{st}})$
   (e) relation to hermitian metrics

2. Basics of symplectic manifolds
   (a) basic definitions and examples
   (b) symplectic and Hamiltonian diffeomorphisms, inlcuding their basic properties and construction of examples with prescribed behavior (when possible)
   (c) integrability, statement of Arnold-Liouville theorem, examples
   (d) Moser's argument with applications, especially Darboux' theorem
   (e) (local) generating functions for symplectomorphisms
   (f) special submanifolds (isotropic, Lagrangian, coisotropic)
   (g) Lagrangian neighborhood theorem, simple consequences
   (h) integrable vs. non-integrable complex structures
   (i) Kähler manifolds: definition, examples, Kähler forms as $(1,1)$-forms, Kähler potentials

3. Symplectic group actions and reduction
   (a) Hamiltonian Lie group actions and their moment maps
   (b) Marsden-Weinstein reduction: basic procedure and examples

4. J-holomorphic curves
   (a) definition, equivalent forms of defining equations, energy, examples
   (b) spaces of J-holomorphic curves, statement of regularity
   (c) compactness of the moduli space: basic phenomena (what happens with uniform gradient bounds? what is bubbling?)
   (d) statement of the non-squeezing theorem, strategy of the proof

Advice for your exam preparation

Of course you will need to know the statements and proofs of the main results. Make sure you also know and understand *many* examples. For instance, in the exam I might ask you to write down a function on $\mathbb{R}^{2n}$ whose Hamiltonian flow will have a prescribed effect, or to give an example of a J-holomorphic curve with certain properties, or ...

Other questions I like to ask include: What happens to a given theorem when you leave out one of the assumptions? Do you know counterexamples? What is a simple situation where a given theorem or construction is useful? How is it proven?

As you prepare for the exam, look back at the exercises, as they often give a valuable second perspective on topics covered in the lecture. Also, you might find it useful to look up the treatment of the topics we covered in the textbooks by A. Cannas da Silva or by D. McDuff and D. Salamon.

You may choose to start the exam with a topic from the following list:

- **Moser's method and its applications**
- **Hamiltonian group actions**
- **J-holomorphic curves**

The duration of the exam is between 25 and 30 minutes.