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Winter 2024/25

Symplectic Geometry

Problem Set 9

1. We consider the Lie group $SO(3) = \{A \in GL(3, \mathbb{R}) \,:\, A^T A = \mathrm{id}\}$.

a) Verify that the Lie algebra $\mathfrak{so}(3)$ consists of all matrices $\xi \in \mathrm{Mat}(3, \mathbb{R})$ with the property that $\xi^T + \xi = 0$.

b) Prove that the map $\mathbb{R}^3 \to \mathfrak{so}(3)$, given by

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \mapsto \xi_s := \begin{pmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{pmatrix}$$

is a linear isomorphism with the property that $\xi_s(v) = s \times v$, where $\times$ is the cross product of 3-dimensional vectors.

c) Argue that for $A \in SO(3)$ and $v, w \in \mathbb{R}^3$ we have

$$(Av) \times (Aw) = A(v \times w).$$

d) Deduce that under the above identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the adjoint action of $SO(3)$ on $\mathfrak{so}(3)$ corresponds to the standard action on $\mathbb{R}^3$, i.e.

$$A\xi_s A^{-1} = \xi_{As} \qquad \text{for all } A \in SO(3), s \in \mathbb{R}^3.$$

Hint: Try to avoid actual computations.

It follows that the orbits of the adjoint action of $SO(3)$ on its Lie algebra are the spheres around the origin (and the origin itself).

2. Consider the standard action of the group $SO(3)$ on $\mathbb{R}^3$.

a) Convince yourself that the induced action of $SO(3)$ on $T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$ with coordinates (q, p) is given by

$$A \cdot (q, p) = (Aq, Ap).$$

According to (the generalization of) Problem 3 of Sheet 8, this action is Hamiltonian.

Please turn!

b) Use the results of the previous exercise to see that for $s \in \mathbb{R}^3$ value of the fundamental vector field $X_{\xi_s} \in \mathrm{Vect}(T^*\mathbb{R}^3)$ associated to $\xi_s \in \mathfrak{so}(3)$ at a point $(q,p) \in T^*\mathbb{R}^3$ is given by

$$X_{\xi_s}(q,p) = (s \times q, s \times p).$$

c) Deduce that the map $\mu : T^*\mathbb{R}^3 \to \mathfrak{so}(3)^*$ given by

$$\mu(q,p)(\xi_s) = \langle q \times p, s \rangle$$

is a moment map for the induced action.

3. We now consider the standard action of $SO(3)$ on $S^2 \subseteq \mathbb{R}^3$. The standard area form on S^2 is preserved under this action.

a) Prove that the standard area form on S^2 can be written as

$$\omega_x(u,v) = \langle x \times u, v \rangle \qquad \text{for } v, w \in T_x S^2,$$

where $\langle\, .\, ,\, .\, \rangle$ denotes the standard scalar product on $\mathbb{R}^3$.

b) Prove that with the identification from Problem 1, the infinitesimal action $\mathfrak{so}(3) \to \mathrm{Vect}(S^2)$, $\xi_s \mapsto X_{\xi_s}$ is given by

$$X_{\xi_s}(x) := s \times x.$$

c) Deduce that

$$\omega_x(X_{\xi_s}, v) = \langle s, v \rangle,$$

and conclude that the vector field X_{ξ_s} is Hamiltonian with Hamiltonian function

$$H_{\xi_s}(x) = -\langle s, x \rangle.$$

d) Prove that $H_{\mathrm{Ad}(A^{-1})\xi} = H_{\xi_s} \circ A$ and conclude that the action is Hamiltonian.

e) Describe the moment map $\mu : S^2 \to \mathfrak{so}(3)^*$.

More generally, consider the diagonal action of $SO(3)$ on $(S^2)^n$ by simultaneous rotation of all components, i.e.

$$A \cdot (x_1, \ldots, x_n) = (Ax_1, \ldots, Ax_n).$$

f) Prove that the moment map $\mu : (S^2)^n \to \mathfrak{so}(3)^*$ is simply the sum of the moment maps of the factors.

g) Prove that a point $(x_1, \ldots, x_n) \in (S^2)^n$ is a critical point of the moment map μ if and only if all the x_k are colinear.

h) Deduce that $0 \in \mathfrak{so}(3)^*$ is a regular value of the moment map if and only if n is odd.