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Symplectic Geometry

Problem Set 8

1. **a)** We consider the Lagrangian embedding $\varphi_1 : S^{n-1} \times S^1 \to (\mathbb{C}^n, \omega_{\mathrm{st}})$ from the lecture given by $\varphi_1(x, e^{it}) = (1 + \epsilon e^{it})x$. Prove the claim made in class that the Maslov index of the loop $\gamma : [0, 2\pi] \to S^{n-1} \times S^1$ given as $t \mapsto (x_o, e^{it})$ for some any $x_0 \in S^{n-1}$ is equal to 2.

b) We now assume $n = 2k$ is even. Then the circle acts freely on S^{n-1} by the diagonal embedding of $S^1 = SO(2) \subseteq SO(2k)$ which simultaneously rotates all two-dimensional factors of $\mathbb{R}^{2k} = \mathbb{R}^2 \times \cdots \times \mathbb{R}^2$. It follows that, for any fixed $x_0 \in S^{n-1}$, the path $\gamma : [0, 2\pi] \to S^{n-1} \times S^1$ given by $t \mapsto (e^{\frac{it}{2}} x_o, e^{\frac{it}{2}})$ connects $(x_0, 1)$ with $(-x_0, -1)$, so it is mapped to a closed loop $\widetilde{\gamma}$ under the Lagrangian immersion $\varphi_2 : S^{n-1} \times S^1 \to (\mathbb{C}^n, \omega_{\mathrm{st}})$ given by $\varphi_2(x, e^{it}) = e^{it}x$. Prove the assertion made in class that the Maslov index of the loop $\widetilde{\gamma}$ in $\varphi_2(S^{n-1} \times S^1)$ (which generates the fundamental group of the image) is equal to n.

2. The goal of this exercise is to prove the assertion made in class that for the round metric on S^n, the symplectic reduction of $(S^*S^n, \omega_{\mathrm{can}})$ by the circle action coming from the geodesic flow is diffeomorphic to the hypersurface $X \subseteq \mathbb{C}P^n$ given in homogeneous coordinates $[z_0 : \ldots : z_n]$ by the equation

$$z_0^2 + z_1^2 + \cdots + z_n^2 = 0. \tag{1}$$

a) Use the fact that the unit cotangent bundle S^*S^n for the round metric can be identified with the subset

$$\{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \ : \ \|x\| = \|v\| = 1, x \perp v\} \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$$

to construct an embedding of S^*S^n into the subset $Z \subseteq \mathbb{C}^{n+1}$ given by (1). We denote the image of this embedding by Y.

b) Prove that the projection $Z \setminus \{0\} \to X$ induced from the projection $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ maps Y surjectively onto X.

c) Prove that (x, v) and (x', v') are mapped to the same point in X if and only if the geodesic circle through x tangent to v is the same as the geodesic circle through x' tangent to v'.

Please turn!

3. **a)** Suppose (M, ω) is an exact symplectic manifold, meaning that $\omega = d\lambda$ is an exact form. Prove that if S^1 acts on M by symplectomorphisms preserving the primitive λ, then the action is Hamiltonian with Hamiltonian function $H : M \to \mathbb{R}$ given by $H(x) = \lambda(X(x))$, where X is the vector field on M whose flow is the given S^1-action.

b) Deduce that if S^1 acts on a manifold Q, then the induced action on the symplectic manifold $(T^*Q, \omega_{\text{can}})$ is Hamiltonian. In particular, if this induced action is free on some level surface of the Hamiltonian function, the quotient space inherits a symplectic structure from ω_{can}.

Remark: The same results will be true for a Hamiltonian group action of an arbitrary group G..