Universität Hamburg
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Symplectic Geometry

Problem Set 7

1. Let (M, ω) be a symplectic manifold, and suppose the components $H_1, \ldots, H_n$ of a smooth map $H : M \to \mathbb{R}^n$ Poisson commute pairwise. Prove that if $c \in \mathbb{R}^n$ is a regular value of H (so that $H^{-1}(c) \subseteq M$ is a smooth submanifold), then $H^{-1}(c)$ is a *Lagrangian* submanifold.

2. Suppose $S \subseteq M$ is a closed oriented hypersurface and ω_0 and ω_1 are symplectic forms on M whose *restrictions to* S agree, meaning that for the inclusion map $i : S \to M$ we have $i^*\omega_0 = i^*\omega_1$ The goal of this exercise is to prove that suitable neighborhoods of S in (M, ω_0) and in (M, ω_1) are symplectomorphic.

a) Argue that the normal bundle $NS \subseteq TM|_S$ is trivial, i.e. there is a nowhere vanishing section $\sigma : S \to NS$.

b) Suppose $\omega \in \Omega^2(M)$ is a closed 2-form. Prove that ω is symplectic near S if and only if (i) $i^*\omega$ has a 1-dimensional kernel $TS^\perp \subseteq TS$, and (ii) for any nonvanishing (local) section $\xi : S \to TS^\perp$ one has $\omega(\xi(x), \sigma(x)) \neq 0$ for all points $x \in S$ where ξ is defined.

c) Deduce that in our situation the forms $\omega_t = (1-t)\omega_0 + t\omega_1 = \omega_0 + t(\omega_1 - \omega_0)$ are all symplectic in some neighborhood of S.

d) Argue that $\omega_1 - \omega_0$ is exact in a neighborhood of S.

e) Now use a Moser-type argument to construct the desired symplectomorphism between suitable neighborhoods of S in (M, ω_0) and (M, ω_1).

3. Consider $S^n \subseteq \mathbb{R}^{n+1} \cong \mathbb{R} \times \mathbb{R}^n$ with coordinates $(x, y_1 \ldots, y_n)$ on $\mathbb{R}^{n+1}$, so that $S^n = \{(x, y) \in \mathbb{R} \times \mathbb{R}^n \,:\, x^2 + \|y\|^2 = 1\}$.

a) Prove that the map $f : S^n \to \mathbb{C}^n$ given by

$$f(x, y) = (1 + ix) \cdot y$$

is an immersion (meaning that its differential is everywhere injective).[1]

[1] This particular immersion was first considered by Whitney, and so it is usually called the *Whitney immersion*. It turns out to be an important example in differential topology

Please turn!

b) Prove that the Whitney immersion satisfies $f^*\omega_{\mathrm{st}} = 0$. Such immersions are called *Lagrangian immersions.*

c) Draw a picture of the image of f for $n = 1$.

d) Prove that in all dimension $n \geq 1$ the map f is injective except for a single (transverse) double point.

e) Prove that for any $m > 0$, the composition of f with the inclusion $\mathbb{C}^n = \mathbb{C}^n \times \{0\} \hookrightarrow \mathbb{C}^{n+m}$ can be perturbed to an isotropic embedding $S^n \to \mathbb{C}^{n+m}$.

Hint: It's probably easiest to give an explicit construction of the perturbation.

Remark: It is a general fact that any Lagrangian immersion $L \looparrowright (M, \omega)$ can be perturbed to an isotropic embedding $L \hookrightarrow (M, \omega) \times (M', \omega')$ for any positive dimensional symplectic manifold (M', ω').