Universität Hamburg
Janko Latschev Winter 2024

Symplectic Geometry

Problem Set 6

1. Consider the map $\varphi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ given in local coordinates (q, p) on the domain and (x, y) on the target as

$$x = \sqrt{2p}\cos q, \quad y = \sqrt{2p}\sin q.$$

a) Prove that $\varphi^*(dx \wedge dy) = dp \wedge dq$.

b) Find a generating function $F = F(q, x)$ for φ.
Hint: You are looking for a function with $\frac{\partial F}{\partial x} = y$ *and* $\frac{\partial F}{\partial q} = p$. *Use the explicit form of* φ *to rewrite the right hand sides of these equations as functions of* q *and* x, *and then find a function* F *having these partial derivatives.*

c) At which points does your solution to **b)** fail to work? Why is that, and how could you sidestep the issue?

d) Consider the Hamiltonian function $H(x, y) = \frac{x^2+y^2}{a^2}$. What form does it take in the (q, p)-coordinates? What does that mean for the Hamiltonian flow?

2. Let $S \subseteq \mathbb{R}^N$ be a closed submanifold of dimension n. Its normal bundle can be identified with the subset

$$TS^\perp := \{(x, v) \in S \times \mathbb{R}^N \, : \, v \perp T_x S\} \subseteq S \times \mathbb{R}^N.$$

Consider the map $\Phi : TS^\perp \to \mathbb{R}^N$ given by $\Phi(x, v) = x + v$.

a) Give a geometric interpretation of the map Φ and compute its derivative at a point $(x, 0) \in TS^\perp$.

b) Conclude that at all points $p = (x, 0) \in TS^\perp$ the map Φ is a local diffeomorphism, i.e. it maps a neighborhood of $p \in TS^\perp$ diffeomorphically onto a neighborhood of $x \in \mathbb{R}^N$.

c) Argue that there exists $\varepsilon > 0$ such the restriction of Φ to $TS^\perp_\varepsilon := \{(x, v) \in TS^\perp \, : \, \|v\| < \varepsilon\}$ is an embedding.

Please turn!

3. Suppose $S \subseteq \mathbb{R}^n$ is a smooth strictly convex hypersurface (necessarily diffeomorphic to S^{n-1}). Let

$$\Delta_k := \{(x_1, \dots, x_k) \in S \times \dots \times S \mid x_k = x_1 \text{ or } \exists\, 1 \leq i < k \,:\, x_i = x_{i+1}\} \subseteq S \times \dots \times S.$$

a) Prove that a critical point of the function

$$\begin{aligned} f_k : S \times \dots \times S \setminus \Delta_k &\to \mathbb{R} \\ (x_1, \dots, x_k) &\mapsto \|x_2 - x_1\| + \dots + \|x_k - x_{k-1}\| + \|x_1 - x_k\|. \end{aligned}$$

corresponds to a k-periodic orbit of the billard map for the domain bounded by S.

b) Prove that for each $k \geq 2$ there exists a k-periodic billard trajectory.

c) If $(x_1, \dots, x_k) \in S \times \dots \times S$ corresponds to a k-periodic billard trajectory, then so do $(x_2, \dots, x_k, x_1)$ and $(x_k, x_{k-1}, \dots, x_1)$ (the reversed sequence). Can you prove the existence of two geometrically distinct k-periodic trajectories, i.e. periodic trajectories that are not related by such "trivial" modifiations?

Hint: If you have trouble getting started, you should try the case $n = 2$.