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Winter 2024

Symplectic Geometry

Problem Set 5

1. Let $\varphi_t : (M, \omega) \to (M, \omega)$ be the family of diffeomorphisms determined by the time-dependent Hamiltonian function $H : [0, 1] \times M \to \mathbb{R}$ via

$$\dot{\varphi}_t = X_{H_t} \circ \varphi_t.$$

a) For each $c \in (0, 1)$, write φ_c as time one map of a family of diffeomorphisms determined by a new Hamiltonian function built from H.

b) Find a time-dependent Hamiltonian function G whose time one map is $(\varphi_1)^{-1}$.

c) Now suppose ψ is the time one map of a second family ψ_t determined by $F : [0, 1] \times M \to \mathbb{R}$. Find a time-dependent Hamiltonian function K (suitably built out of H and F) with time one map $\psi \circ \varphi$.

In summary, you have shown that the subset of Hamiltonian diffeomorphisms inside $\mathrm{Symp}_0(M, \omega)$ is connected and closed under taking inverses and under multiplication, so it forms a connected subgroup $\mathrm{Ham}(M, \omega) \subseteq \mathrm{Symp}_0(M, \omega)$ of the identity component of the group of symplectomorphisms.

2. Give examples of closed submanifolds of $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ which are isotropic or coisotropic or Lagrangian or symplectic with respect to the standard symplectic structure $\omega_{\mathrm{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ on T^4! Can you find some that are not tori?

3. Let (M, ω) be a symplectic manifold and $S \subset M$ a closed oriented hypersurface.

a) Prove that

$$L := TS^{\perp_\omega} = \{v \in TS \mid \omega(v, w) = 0 \text{ for all } w \in TS \text{ with } \pi(v) = \pi(w)\}$$

is a 1-dimensional subbundle of $TS \xrightarrow{\pi} S$ which inherits an orientation from S.

b) Prove that if $S = H^{-1}(c)$ for a regular value $c \in \mathbb{R}$ of a function $H : M \to \mathbb{R}$, then the restriction of X_H to S is a section of L.

Please turn!

Any one-dimensional subbundle of the tangent bundle of a manifold S is integrable, i.e. it is tangent to a family of 1-dimensional submanifolds of S. In the situation above, this family consists of the flow lines of X_H as in **b)**. It is called *the characteristic foliation of the hypersurface* $S \subset (M, \omega)$.

c) Describe the subbundle L and the characteristic foliation for

$$S_{a,b} = \{(z_1, z_2) \mid \frac{|z_1|^2}{a^2} + \frac{|z_2|^2}{b^2} = 1\} \subseteq \mathbb{C}^2 \cong (\mathbb{R}^4, \omega_{\mathrm{st}}),$$

where $a, b > 0$ (Consider the three cases: $a = b$, $\frac{a}{b} \in \mathbb{Q} \setminus \{1\}$ and $\frac{a}{b} \notin \mathbb{Q}$).

d) Conclude that there is no symplectomorphism $\varphi : (\mathbb{R}^4, \omega_{\mathrm{st}}) \to (\mathbb{R}^4, \omega_{\mathrm{st}})$ which maps the standard sphere $S^{2n-1} = S_{1,1}$ onto $S_{a,b}$ for $(a, b) \neq (1, 1)$.