Universität Hamburg
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Symplectic Geometry

Problem Set 4

Both problems below concern dynamics of Hamiltonian diffeomorphisms on the standard symplectic plane $(\mathbb{R}^2, \omega_{\rm st} = dx \wedge dy)$.

1. **a)** Find explicit autonomous Hamiltonian functions $H_i : \mathbb{R}^2 \to \mathbb{R}$ such that the time-one-maps of the corresponding Hamiltonian flows φ_t^i are

$$\varphi_1^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix} \qquad \text{and} \qquad \varphi_1^2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

b) Prove that $\psi = \varphi_2^1 \circ \varphi_1^1$ cannot be generated by an autonomous Hamiltonian function (and in fact is not the time-one-map of any flow!).
Hint: Assume the contrary and first argue that 0 must be a fixpoint of the flow, then consider the linearization of the flow at this fixpoint to obtain a contradiction.

This clearly illustrates the need for time-dependent Hamiltonians in the definition of $\mathrm{Ham}(M, \omega)$. In fact, general Hamiltonian diffeomorphisms often behave very differently from those generated by an autonomous function - for example they could have dense orbits, which clearly cannot happen in the autonomous case (why?).

2. The goal of this exercise is to gain a little bit of geometric intuition about compactly supported Hamiltonian diffeomorphisms by looking at a specific case.
Consider a Hamiltonian function $H : B^2(0, 10) \to \mathbb{R}$ of the form $H(x, y) = y \cdot \rho(r^2)$, where $r^2 = x^2 + y^2$ and $\rho : [0, \infty) \to [0, 1]$ is a smooth function with the following properties:

$$\rho(t) \equiv 1 \text{ for } 0 \leq t \leq 5, \quad \rho(t) \equiv 0 \text{ for } 8 \leq t, \quad \text{and} \quad \rho'(t) \leq 0 \text{ for all } t \in [0, \infty).$$

a) Compute the Hamiltonian vector field X_H with respect to the symplectic form $\omega = dx \wedge dy$ in terms of x, y, ρ and ρ', and make a rough sketch of it.

b) Give a qualitative description (e.g. give a rough sketch) of the image of the ball $B^2(0, 1)$ under the time-t-map φ_t of the Hamiltonian flow of H for $t = 1$, $t = 10$ and $t = 10^5$!