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# Symplectic Geometry

## Problem Set 2

**1.** **a)** Prove that the linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ associated to the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies $A^2 = -\mathbb{1}$ if and only if $d = -a$ and $ad - bc = 1$.

**b)** Deduce that the subset $\mathcal{J} \subseteq \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Sp}(2,\mathbb{R})$ of all such maps has two connected components, one containing $J_{\mathrm{st}}$ and the other containing $-J_{\mathrm{st}}$.

**c)** What is the condition on a map $A$ as above to be tamed by $\omega_{\mathrm{st}}$? To be compatible with $\omega_{\mathrm{st}}$?

**2.** A diffeomorphism $\varphi : Q \to Q'$ between manifolds lifts to a diffeomorphism $\Phi : T^*Q \to T^*Q'$ given by the formula

$$\Phi(x,\alpha) := \left(\varphi(x), \alpha \circ (\varphi_{*,x})^{-1}\right),$$

where $\varphi_{*,x} : T_xQ \to T_{\varphi(x)}Q'$ is the differential of $\varphi$ at $x \in Q$.

**a)** Prove that $\Phi^*(\lambda'_{\mathrm{can}}) = \lambda_{\mathrm{can}}$, and so $\Phi$ is a symplectomorphism from $T^*Q$ to $T^*Q'$!

**b)** Let $Y : Q \to TQ$ be a complete vector field, and denote by $\psi_t$ its flow. Let $X : T^*Q \to T(T^*Q)$ be the vector field generating the corresponding flow $\Psi_t$ on $T^*Q$. Prove that $X$ is the Hamiltonian vector field associated to the function $H : T^*Q \to \mathbb{R}$ defined as

$$H(x,\alpha) := \alpha(Y(x)).$$

**3.** Show that if $\varphi : M \to M$ is any symplectomorphism of $(M,\omega)$ and $H : M \to \mathbb{R}$ is smooth, then the Hamiltonian vector fields of the functions $H$ and $H \circ \varphi^{-1}$ are related by

$$X_{H\circ\varphi^{-1}}(\varphi(x)) = \varphi_*(X_H(x)),$$

where $\varphi_* : TM \to TM$ is the differential of $\varphi$.

**Please turn!**

**4.** A diffeomorphism of a symplectic manifold $(M, \omega)$ is called an *autonomous Hamiltonian diffeomorphism* if it is the time one map of the flow of a Hamiltonian function $H : M \to \mathbb{R}$. The *support* of a diffeomorphism $\varphi : M \to M$ is the closure of the set of all points with $\varphi(x) \neq x$.
In this exercise, we consider the special case $(M, \omega) = (\mathbb{R}^{2n}, \omega_{\text{st}})$.

**a)** Prove that for every pair of points $x, y \in \mathbb{R}^{2n}$ there exists an autonomous Hamiltonian diffeomorphism $\varphi : (\mathbb{R}^{2n}, \omega_{\text{st}}) \to (\mathbb{R}^{2n}, \omega_{\text{st}})$ such that $\varphi(x) = y$.

**b)** Prove that for every point $x \in B^{2n}(0, 1)$ there exists an autonomous Hamiltonian diffeomorphism $\varphi : (\mathbb{R}^{2n}, \omega_{\text{st}}) \to (\mathbb{R}^{2n}, \omega_{\text{st}})$ with $\operatorname{supp} \varphi \subseteq B^{2n}(0, 1)$ such that $\varphi(0) = x$.