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Symplectic Geometry

Problem Set 13

1. **a)** Find a sequence $\{u_k\}_{k\geq 1}$ of holomorphic maps

$$u_k : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$$

with the following properties:

- The composition of each u_k with each of the two projections to the factors $\mathbb{C}P^1$ is a biholomorphic map.
- The point $([0:1],[0:1]) \in \mathbb{C}P^1 \times \mathbb{C}P^1$ is contained in $u_k(\mathbb{C}P^1)$ for each $k \geq 1$.
- The images $u_k(\mathbb{C}P^1)$ converge to a subset of $\mathbb{C}P^1 \times CP^1$ of the form $\{p\} \times \mathbb{C}P^1 \cup \mathbb{C}P^1 \times \{q\}$, for suitable points p and q in $\mathbb{C}P^1$.

b) Now find sequences of Möbius tranformations $\varphi_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ and $\psi_k : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that the maps $v_k := u_k \circ \varphi_k$ converge to a holomorphic parametrization of $(\mathbb{C}P^1 \setminus \{p\}) \times \{q\}$ and the maps $w_k := u_k \circ \psi_k$ converge to a holomorphic parametrization of $\{p\} \times (\mathbb{C}P^1 \setminus \{q\})$, both in C^∞_{loc} in the complement of suitable points in the domain $\mathbb{C}P^1$.

2. **a)** Give details of the argument for the statement made in class that the conclusion of the nonsqueezing theorem is false if one replaces symplectic by volume-preserving maps.

b) In the nonsqueezing theorem, it is also important to define the cylinder as a *symplectic* product

$$Z^{2n}(a) = B^2(a) \times (\mathbb{R}^{2n-2}, \omega_{\mathrm{st}}).$$

Indeed, prove that for any $\epsilon > 0$ there is a symplectic embedding of the ball $B^4(1)$ into the *Lagrangian* product

$$Z' := B^2_{x_1,x_2}(0,\epsilon) \times \mathbb{R}^2_{y_1,y_2} \subseteq (\mathbb{R}^4, \omega_{\mathrm{st}})$$

of a ball of radius ϵ in the (x_1, x_2)-plane with the (y_1, y_2)-plane.

Please turn!

3. In problem **2.e)** of problem set 10 you computed the coadjoint orbit of a particular element in the dual of the Lie algebra of the group $\mathcal{L} \subseteq SL(n, \mathbb{R})$ of lower triangular matrices. Problem **3)** of the same problem set discussed the canonical symplectic form on coadjoint orbits. The goal of this exercise is to make that explicit in the example for the simplest case $n = 2$.

a) Compute an explicit expression for the tangent vector $-\operatorname{ad}^*_\xi(f_u) \in T_{f_u}\mathcal{O}_{f_{u_0}}$ associated to the element $\xi \in \mathfrak{sl}(2, \mathbb{R})$.
Hint: As $\mathcal{O}_{f_{u_0}}$ lies in a linear subspace of $\mathfrak{l}^*$, the result should also be an element of that linear subspace.

b) Use the expression in problem **3.c)** to compute the value of the symplectic form at the point $f_u \in \mathcal{O}_{f_{u_0}}$ on the tangent vectors associated to two elements $\xi, \xi' \in \mathfrak{sl}(2, \mathbb{R})$.

c) Use the computations from **a)** and **b)** to conclude that the symplectic form in this example is given in the coordinates of problem **2.e)** by

$$\omega = \frac{1}{a_1} db_1 \wedge da_1.$$

d) If you feel ambitious, try to verify that for general $n \geq 2$ the symplectic form is given by

$$\omega = \sum_{k=1}^{n-1} \frac{1}{a_k} db_k \wedge da_k.$$