Universität Hamburg
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Symplectic Geometry

Problem Set 11

1. The goal of this exercise is to derive formulas for the Fubini-Study form ω_{FS} on $\mathbb{C}P^n$ in both homogeneous and inhomogeneoous coordinates. In an earlier lecture we defined ω_{FS} as the unique 2-form satisfying

$$\pi^*\omega_{\mathrm{FS}} = \iota^*\omega_{\mathrm{st}},$$

where $\pi : S^{2n+1} \to \mathbb{C}P^n$ and $\iota : S^{2n+1} \to \mathbb{C}^{n+1}$ are the standard projection and the standard embedding.

a) Prove that the 2-form

$$\widetilde{\omega} = \frac{i}{2}\left(\sum_j \frac{dz_j \wedge d\bar{z}_j}{\|z\|^2} - \sum_{j,k} \frac{\bar{z}_j z_k dz_j \wedge d\bar{z}_k}{\|z\|^4}\right)$$

on $\mathbb{C}^{n+1} \setminus \{0\}$ is invariant under the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1} \setminus \{0\}$ by rescaling.

b) Prove that at a given point $z \in \mathbb{C}^{n+1} \setminus \{0\}$ the complex line $\mathrm{span}_{\mathbb{C}}(z) \subseteq T_z\mathbb{C}^{n+1}$ is in the kernel of $\widetilde{\omega}$.

Together these two assertions mean that $\widetilde{\omega}$ is a lift from $\mathbb{C}P^n$, i.e. there is a unique 2-form $\tau \in \Omega^2(\mathbb{C}P^n)$ with $\pi^*\tau = \widetilde{\omega}$, where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ is the quotient projection.

c) Prove that the restriction $\iota^*\widetilde{\omega}$ to the unit sphere agrees with the restriction $\iota^*\omega_{\mathrm{st}}$.

In particular, this shows that $\tau = \omega_{\mathrm{FS}}$ is the Fubini-Study form on $\mathbb{C}P^n$, so that $\widetilde{\omega}$ gives an expression for this form in homogeneous coordinates.

d) The open sets $U_k = \{[z_0 : \ldots : z_n] \in \mathbb{C}P^n \mid z_k \neq 0\}$ cover $\mathbb{C}P^n$. Prove that the maps

$$\begin{aligned} \psi_k : U_k &\to \mathbb{C}^n \\ [z_0 : \ldots : z_n] &\mapsto \left(\frac{z_0}{z_k}, \ldots, \widehat{\frac{z_k}{z_k}}, \ldots, \frac{z_n}{z_k}\right) \end{aligned}$$

give complex charts for $\mathbb{C}P^n$, i.e. the transition maps are holomorphic. What is $\mathbb{C}P^n \setminus U_k$ diffeomorphic to? What is $U_k \cap U_j$ for $k \neq j$ diffeomorphic to?

Please turn!

e) Derive an expression for the Fubini-Study form in inhomogeneous coordinates $\zeta_r := \frac{z_r}{z_0}$ on the open set $U_0 \subset \mathbb{C}P^n$.

f) Compute an explicit expression for the 2-form $\eta = \frac{i}{2}\partial\bar{\partial}(\log(1 + \|\zeta\|^2))$ on $\mathbb{C}^n$ and compare with the result for $(\psi_0^{-1})^*\omega_{\mathrm{FS}}$ you obtained in **d)**.

g) Compute the integral

$$\int_{\mathbb{C}P^1} \omega_{\mathrm{FS}}.$$

2. For a function $a : \mathbb{R}^4 \to \mathbb{R}$, we consider the almost complex structure J_a on the manifold $M = \mathbb{R}^4$ which in the standard global coordinates (x_1, y_1, x_2, y_2) has the form

$$J_a(p) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a(p) & 0 & 0 & -1 \\ 0 & -a(p) & 1 & 0 \end{pmatrix}, \text{ i.e. } J_a\left(\frac{\partial}{\partial x_1}\right) = \frac{\partial}{\partial y_1} + a(p)\frac{\partial}{\partial x_2} \quad \text{etc.}$$

In particular, for $a \equiv 0$ we get the standard complex structure on $\mathbb{R}^4$.

a) Prove that if $|a(p)| < 2$ for all $p \in \mathbb{R}^4$, then J_a is tamed by the standard symplectic form $\omega_{\mathrm{st}} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$!
Hint: Recall that the taming condition just means that $\omega(v, Jv) > 0$ for all nonzero v, but ω need not be J-invariant, so that the bilinear form $\omega(\,.\,, J\,.\,)$ need not be symmetric.

b) Prove that for any almost complex structure J (not just J_a) and any two vector fields X and Y one has $N_J(JX, Y) = -J(N_J(X, Y))$.

c) Under which conditions on the function a is the almost complex structure J_a on $\mathbb{R}^4$ integrable?
Hint: It follows from **b)** *that the value of N_{J_a} on any two vectors $v, w \in T_p\mathbb{R}^4$, is determined by $N_{J_a}\left(\frac{\partial}{\partial x_1}(p), \frac{\partial}{\partial x_2}(p)\right)$. Compute this to find the right condition on the function a.*

3. We consider an almost complex structure J on an open neighborhood U of $0 \in \mathbb{R}^{2n}$.

a) Prove that if $f : U \to \mathbb{C}$ is J-holomorphic, .i.e. we have $df \circ J = J_{\mathrm{st}} \circ df$, then for each point $p \in U$ the differential df_p has either rank 0 or rank 2.

b) Prove that the Lie bracket of any two sections of $\ker df \subseteq TU$ is is again in $\ker df$.

See next page!

c) Prove that the image of the Nijenhuis tensor N_J is contained in $\ker df$.

d) Consider the case $n = 2$ (i.e. $U \subseteq \mathbb{R}^4$) and construct an almost complex structure J such that there are *no nonconstant J-holomorphic functions* $f : U \to \mathbb{C}$.

Hint: You can find such a J in the form

$$J_{b,c}(p) = \begin{pmatrix} 0 & -\frac{1}{b} & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{c} \\ 0 & 0 & c & 0 \end{pmatrix}$$

for suitable functions $b : U \to \mathbb{R}$ and $c : U \to \mathbb{R}$. Choose these functions such that (i) N_J has 2-dimensional image and (ii) this image is not closed under taking Lie brackets. This does the trick (why?). Observe that you could even arrange for J to be arbitrarily close to J_{st}.