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Winter 2024/25

# Symplectic Geometry

## Problem Set 10

**1.** We consider the action of $U(k)$ on $(\mathbb{C}^k)^n = \mathbb{C}^{kn}$, where we think of an element $Z \in (\mathbb{C}^k)^n$ as a matrix with $n$ columns and $k$ rows and the action is by left multiplication.

**a)** Verify that this action is Hamiltonian with respect to the standard symplectic form on $\mathbb{C}^{kn}$ with moment map $\widetilde{\mu} : (C^k)^n \to \mathfrak{u}(k)^*$ given by

$$\widetilde{\mu}(Z)(\alpha) = -\frac{i}{2} \operatorname{Tr} \overline{Z}^T \alpha Z + \frac{i}{2} \operatorname{Tr}(\alpha).$$

**b)** Prove that $0 \in \mathfrak{u}(k)^*$ is a regular value of $\widetilde{\mu}$, and that

$$\widetilde{\mu}^{-1}(0) = \{Z \in (\mathbb{C}^k)^n \, : \, Z\overline{Z}^T = \mathrm{id}_k\}.$$

In other words, $Z \in \widetilde{\mu}^{-1}(0)$ if and only if the rows of $Z$ form a unitary basis of the $k$-dimensional subspace of $\mathbb{C}^n$ which they span. In particular, $U(k)$ acts freely on $\widetilde{\mu}^{-1}(0)$.
*Hint: It might be helpful to recall that the identity* $\operatorname{Tr} AB = \operatorname{Tr} BA$ *holds for arbitrary (composable) matrices, not just square ones.*

**c)** Conclude that the Marsden-Weinstein quotient exists and is diffeomorphic to $G^{\mathbb{C}}(k, n)$, the Grassmannian of complex $k$-dimensional linear subspaces of $\mathbb{C}^n$.

**2.** We fix $n \geq 2$ and consider the subgroup $L \subseteq SL(n, \mathbb{R})$ of lower triangular matrices with determinant 1.

**a)** Prove that the Lie algebra $\mathfrak{l}$ of $L$ is given by lower triangular matrices of trace 0.

**b)** Let $U \subseteq \mathrm{Mat}(n, \mathbb{R})$ be the subset of *upper* triangular matrix $f$. Prove that the map

$$\begin{aligned} U &\to \mathfrak{l}^* \\ u &\mapsto f_u \qquad \text{where } f_u(\xi) = \operatorname{Tr}(\xi u) \end{aligned}$$

is surjective and identify its kernel.

**Please turn!**

**c)** Prove that the coadjoint action of $L$ on $\mathfrak{l}^*$ has the form

$$\mathrm{Ad}^*_L(f_u) = f_{\pi(LuL^{-1})},$$

where $\pi : \mathrm{Mat}(n, \mathbb{C}) \to U$ is the projection to the upper triangular part.

**d)** Determine the fundamental vector field $X_\xi$ on $\mathfrak{l}^*$ associated to an element $\xi \in \mathfrak{l}$.

**e)** Prove that the coadjoint orbit of the element $f_{u_0} \in \mathfrak{l}^*$, where

$$u_0 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

is the set of elements $f_v \in \mathfrak{l}^*$ associated with matrices of the form

$$v = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & b_{n-2} & a_{n-2} & 0 \\ 0 & 0 & \cdots & 0 & b_{n-1} & a_{n-1} \\ 0 & 0 & \cdots & 0 & 0 & b_n \end{pmatrix}$$

with $\sum b_k = 0$ and $\prod a_k \neq 0$.

**3.** We consider the adjoint and coadjoint actions of the connected Lie group $G$ on its Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$.

**a)** Prove that the value at $\eta \in \mathfrak{g}$ of the fundamental vector field ${}^{\mathfrak{g}}X_\xi$ associated to the element $\xi \in \mathfrak{g}$ via the adjoint action is ${}^{\mathfrak{g}}X_\xi(\eta) = \mathrm{ad}_\xi(\eta)$, where $\mathrm{ad}_\xi(\eta) = [\xi, \eta]$ is the action of $\mathfrak{g}$ on itself induced by the action of $G$ on $\mathfrak{g}$.

**b)** Prove that the value at $f \in \mathfrak{g}^*$ of the fundamental vector field ${}^{\mathfrak{g}^*}X_\xi$ associated to the element $\xi \in \mathfrak{g}$ via the coadjoint action is ${}^{\mathfrak{g}^*}X_\xi(f) = -\mathrm{ad}^*_\xi f$, where $\mathrm{ad}^*_\xi(f)(\eta) = f(\mathrm{ad}_\xi(\eta)) = f([\xi, \eta])$.

By construction, the values of the vector fields ${}^{\mathfrak{g}^*}X_\xi$ at some $f \in \mathfrak{g}^*$ generate the tangent space of the orbit $\mathcal{O}_f \subseteq \mathfrak{g}^*$ of $f$ under the coadjoint action.

**c)** Prove that

$$\omega_f(\mathrm{ad}^*_\xi(f), \mathrm{ad}^*_{\xi'}(f)) := f([\xi, \xi'])$$

is a well-defined non-degenerate skew-symmetric 2-form on $T_f\mathcal{O}_f$.

**d)** Prove that the resulting 2-form $\omega$ on $\mathcal{O}_f$ is closed, so that $(\mathcal{O}_f, \omega)$ is a symplectic manifold.