Universität Hamburg
Janko Latschev
Pavel Hájek

Summer 2020

Floer Theory

Problem Set 4

1. Prove that if a 1-periodic orbit of a Hamiltonian system is nondegenerate in the sense defined in the lecture (1 is not an eigenvalue of the linearized return map), then it is a nondegenerate critical point for the action functional.

2. Find holomorphic maps $v : \mathbb{C} \to \mathbb{C}$ for which the inequality

$$(\ell(r))^2 \leq 2\pi r A'(r)$$

we proved in class between the square of the length of $v(\partial B(0,r))$ and the radial derivative of the area of $v(B(0,r))$ is sharp.

3. **a)** Find as many examples as possible of symplectic manifolds that are and that are not symplectically aspherical, beyond those given in the lecture.

b) Try to decide which of your non-aspherical examples are monotone.

4. **a)** Prove that for any smooth closed manifold Q the first Chern class of its cotangent bundle $(T^*Q, \omega_{\text{can}})$ vanishes!
Hint: Write $T^*Q|_Q$ as the complexification of a real vector bundle, and then check what the literature says about Chern classes of such bundles.

b) Prove that for any regular level set $f^{-1}(c) \subset \mathbb{C}^N$ of a holomorphic function $f : \mathbb{C}^N \to \mathbb{C}$ the first Chern class vanishes! How does this generalize to complex submanifolds of higher codimension?

Please turn!

5. Let $S \subset (M, \omega)$ be a hypersurface in a symplectic manifold.

a) Prove that $\ker \omega|_S \subset TS$ is a 1-dimensional subbundle. Any such subbundle is tangent to a foliation of S by 1-dimensional leaves, called characteristics of S.

b) Now suppose that $S = H^{-1}(c)$ is a regular level set of a function $H : M \to \mathbb{R}$. Prove that the Hamiltonian vector field of H gives a trivialization of $\ker \omega|_S$. In particular, simple periodic orbits of X_H (i.e. traversed once, but of any period $T > 0$) are in bijective correspondence to closed characteristics of S.

c) Find the closed characteristics of

(i) a round sphere $S^{2n-1} \subset (\mathbb{R}^{2n}, \omega_{\text{can}})$

(ii) an ellipsoid

$$E(a_1, a_2, \ldots, a_n) = \{z \in \mathbb{C}^n \, : \, \sum_j \frac{\pi |z_j|^2}{a_j} = 1\}$$

with all a_i rationally independent.

6. Express the conditions for a compatible almost complex structure on the symplectization of a contact manifold (V, α) to be convex in the coordinates where the symplectization is given by $(\mathbb{R} \times V, d(e^s \alpha))$!

7. We think of $\mathbb{C}$ as the completion of the ball $\overline{B(0,1)}$, viewed as a Liouville domain with Liouville form $\lambda = \frac{1}{2}(x\,dy - y\,dx)$, and consider the family of Hamiltonians $H_b : \mathbb{C} \to \mathbb{C}$ given as

$$H_b(z) = b|z|^2 - b, \qquad \text{for } b > 0.$$

a) Prove that these Hamiltonians are linear with slope b according to our definitions (this involves writing out the explicit parametrization of the cylindrical end)!

b) What conditions do we have to impose on b in order for H_b to have only nondegenerate 1-periodic orbits? How many 1-periodic orbits are there for such an admissible $b \in (0, \infty)$?

c) Compute $HF_*(H_b, J_0)$ for such Hamiltonians (where J_0 is the standard almost complex structure, i.e. multiplication by i)! Pay special attention to gradings!

d) How much of this discussion generalizes to $\mathbb{C}^n$?