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Floer Theory

Problem Set 3

1. In the last lecture on Morse theory I mentioned a construction of operations, which is based on metric trees, i.e. trees with *external vertices* of valency 1 (numbered 1 through k), all other vertices of valency at least 3, and a length $\ell \in (0, \infty)$ for each edge connecting two internal vertices. For a given number k of external vertices, these form a topological space $\mathcal{T}_k$ with trees whose internal vertices are all trivalent forming an open dense subset, and the other trees appearing as limits where the length(s) of some internal edge(s) go to 0.

a) Find all combinatorial types of trees with low numbers ($k \leq 5$, or 6 if you feel adventurous) of external vertices.

b) The stratification by combinatorial type gives $\mathcal{T}_k$ a decomposition into disjoint open cells. Sketch the resulting "cell complex" (there will still be missing boundary) for $k \leq 5$.

c) One can compactify $\mathcal{T}_k$ by adding *broken trees*, where the length(s) of some internal edge(s) become ∞, and this is interpreted as the tree splitting into subtrees. Complete your sketch from (b) by adding the combinatorial types of the trees in the resulting boundary cells.

2. **a)** Let $\Psi : [0,1] \to Sp(2n)$ be a path of symplectic matrices. Prove that

$$\begin{aligned} S : [0,1] &\to \mathrm{Mat}(2n, \mathbb{R}) \\ t &\mapsto -J_0 \dot{\Psi}(t) \Psi(t)^{-1} \end{aligned}$$

is a path of symmetric matrices.

b) Conversely, show that given a path $S : [0,1] \to \mathrm{Sym}(2n)$ of symmetric matrices, the path $\Psi_S : [0,1] \to \mathrm{Mat}(2n, \mathbb{R})$ solving

$$\dot{\Psi}_S(t) = J_0 S(t) \Psi_S(t), \qquad \Psi_S(0) = \mathbb{1}$$

is in $Sp(2n)$.

Please turn!

3. Prove as many of the properties (A)-(H) of the Conley-Zehnder index formulated in the lecture as you can.

4. Suppose $H : M \to \mathbb{R}$ is an autonomous (i.e. time-independent) Hamiltonian.

a) Prove that if a fixpoint of the Hamiltonian flow which corresponds to a critical point of H is nondegenerate in the sense defined in the lecture then it is nondegenerate as a critical point of H. Under what additional condition is the converse true?

b) We now consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_{\text{can}})$. Suppose that $0 \in \mathbb{R}^{2n}$ is a nondegenerate critical point of $H : \mathbb{R}^{2n} \to \mathbb{R}$ and show that if all eigenvalues of $\text{Hess}_0(H)$ have absolute value $< 2\pi$, then there is a neighborhood of 0 in which the only 1-periodic orbit of the Hamiltonian flow is the constant orbit at 0.

c) Use this to show that on a closed symplectic manifold (M, ω), the fixpoints of a Hamiltonian diffeomorphism generated by a sufficiently C^2-small autonomous Hamiltonian $H : M \to \mathbb{R}$ are precisely the critical points of H.

d) Consider the functions $H_c : \mathbb{R}^2 \to \mathbb{R}$, given by $H_c(x, y) = c(x^2 + y^2)$. For which values of $c \in \mathbb{R}$ is $0 \in \mathbb{R}^2$ a nondegenerate orbit of the Hamiltonian flow φ_t^c of H_c? Compute the Conley-Zehnder index of the path $t \mapsto (d\varphi_t^c)_0$ for these values of c (the answer will depend on c).