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Floer Theory

Problem Set 1

1. Prove the following assertions made in class:

a) The Hessian of a smooth function $f : M \to \mathbb{R}$ at a critical point $p \in M$ of f is a symmetric bilinear form.

b) In local coordinates near p, the Hessian is given by the matrix of second derivatives of f.

c) A critical point p is nondegenerate if and only if it represents a transverse intersection point of the zero section and df in T^*M.

2. Any Morse function on $T^2 = S^1 \times S^1$ must have at least 4 critical points. In contrast, there are smooth functions on T^2 with exactly 3 critical points (the same is true for any surface of positive genus). Give a qualitative description of what such a function could look like. Can you find an explicit formula?
Remark: More generally, on T^n there are functions with exactly $n+1$ critical points. No explicit formulae are known for $n > 3$ (I'm not sure for $n = 3$).

3. Let $f : M \to \mathbb{R}$ be a Morse function on a smooth closed manifold, and let φ_t denote the negative gradient flow of f with respect to some Riemannian metric g.

a) Prove that for every $x \in M$ the limits

$$p_\pm := \lim_{t \to \pm\infty} \varphi_t(x)$$

exist and are critial points of f.

b) Let $x \in M$ be given and consider p_+ as in part (a). Prove that there are constants $a, b > 0$ such that for all $t > 0$ one has

$$d(\varphi_t(x), p_+) \leq a e^{-bt}.$$

Here the constant $b > 0$ can be chosen independently of x (just depending on the linearization of the flow at the point p_+).
Of course, the analogous result also holds for the behaviour as $t \to -\infty$.

Please turn!

4. In exercise 10 on page 51 of the book by Audin and Damian it is asserted that the vector fields with the following two phase portraits cannot be gradient(-like) fields. How does the right picture have to be corrected to make this a true statement?

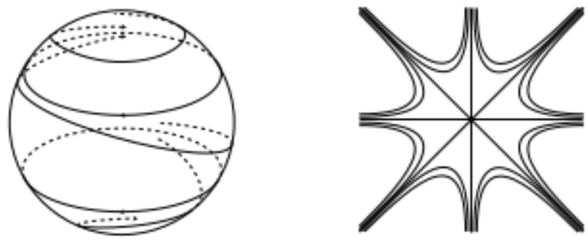

Fig. 2.30