Q: Can we extend the result to the zero section in T^*Q?

Note that $Q \subset T^*Q$ does satisfy the assumption
$$\omega|_{\pi_2(T^*Q, Q)} = 0$$
that we used last time. So the only issue that we need to deal with is the noncompactness of T^*Q.

We used the compactness of the ambient symplectic manifold (M, ω) mainly ~~in the~~ (and implicitly) in the discussion of the compactness properties of the moduli spaces of holomorphic strips. Having all relevant holomorphic curves a priori confined to a compact subset of the target symplectic manifold is needed there to convert the gradient bounds coming from the bubbling analysis into C^∞_{loc} convergence, which then gives compactness up to breaking.

Since T^*Q is a Liouville manifold, this confinement can be achieved by using almost complex structures which are convex on the cylindrical end. Then the confinement lemma we discussed in the context of symplectic homology will guarantee that all holomorphic strips will stay near the two Lagrangian submanifolds Q and $\varphi(Q)$, where $\varphi \in Ham_0(T^*Q, \omega_{can})$.

Thm: Let $L_0 = Q \subseteq T^*Q$ be the zero section and $L_1 = \varphi(L_0)$ be some image of Q under a Hamiltonian diffeomorphism such that $L_0 \pitchfork L_1$.
Then for generic $\{J_t\}_{t\in[0,1]}$ convex at infinity the Floer homology $HF(L_0, L_1)$ is well-defined and isomorphic to $H_*(Q; \mathbb{Z}_2)$.

Cor: If $L_1 = \varphi(Q)$ and $L_1 \pitchfork Q$, then the number of intersections of L_1 with $L_0 = Q$ is bounded below by $\sum_{k=0}^{n} b_k(Q; \mathbb{Z}_2)$.

Remarks on gradings in Lagrangian Floer theory:

Last time we associated an integer $\mu(v) \in \mathbb{Z}$ to any map

$$v : [0,1] \times [0,1] \to (M, \omega)$$

with

$$v(0,t) = p \in L_0 \pitchfork L_1 \ , \ v(1,t) = q \in L_0 \pitchfork L_1 \quad \text{for all } t \in [0,1]$$

and

$$v(s,0) \in L_0 \ , \quad v(s,1) \in L_1 \quad \text{for all } s \in [0,1],$$

where L_0 and L_1 are two Lagrangian submanifolds in M.

This was done as follows:

We fixed a trivialization of $(v^* TM, \omega)$ and obtained two paths of Lagrangian subspaces in $\mathbb{R}^{2n}$ starting at $T_p L_0$ and ending at $T_q L_0$:

$c_0 : [0,1] \to \mathcal{L}ag(n)$ simply records $c_0(s) = T_{v(s,0)} L_0$ in our chosen trivialization

$c_1 : [0,1] \to \mathcal{L}ag(n)$ is specified uniquely up to homotopy by the requirement that $c_1(s) \pitchfork T_{v(s,1)} L_1$ for all $s \in [0,1]$ w.r.t. our trivialization.

Then $\mu(v)$ was defined as the Maslov index of the loop $c_1^{-1} * c_0$ in $\mathcal{L}ag(n)$.

Alternatively we can replace c_1 by a path c_1' obtained as follows:

Given two Lagrangian subspaces $W_0, W_1 \leq \mathbb{R}^{2n} \cong \mathbb{C}^n$, one can always find $A \in Sp(2n)$ such that $A(W_0) = \mathbb{R}^n$ and $A(W_1) = i \cdot \mathbb{R}^n$.

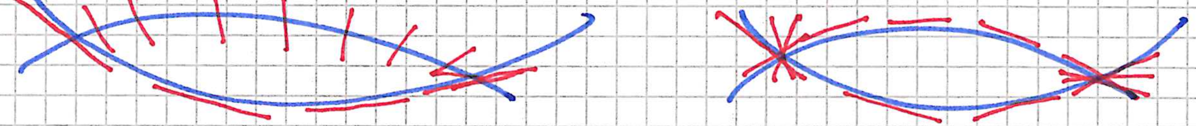
Def: The short homotopy from W_0 to W_1 is the path

$$h : [0,1] \to \mathcal{L}ag(n)$$
$$h(\tau) = A^{-1}\left(\left(e^{-\frac{i\pi\tau}{2}} \cdot \mathbb{R}\right)^n \right)$$

Fact: This path is well-defined up to homotopy

The path c_1' is now given by the concatenation of three paths:
* the short homotopy from $T_p L_0$ to $T_p L_1$
* the path $s \mapsto T_{v(s,1)} L_1$, $s \in [0,1]$
* the inverse of the short homotopy from $T_q L_0$ to $T_q L_1$

<u>Fact:</u> $\mu((c_1')^{-1} * c_0) = \mu(c_1^{-1} * c_0)$.



How could one lift this relative grading to an absolute one?

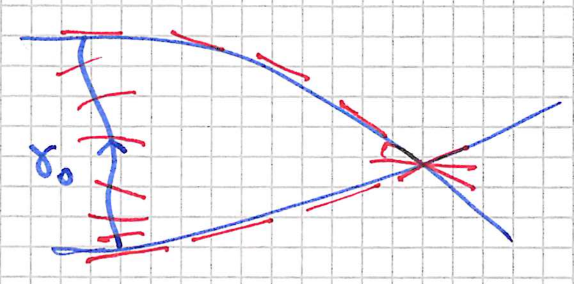
Let (M,ω) and two closed Lagrangian submanifolds $L_0, L_1 \subseteq M$ be given. We assume $L_0 \pitchfork L_1$.

Pick a path $\gamma_0 : [0,1] \to M$ with $\gamma_0(0) \in L_0$ and $\gamma_0(1) \in L_1$ in the component of $\mathcal{P}(L_0,L_1)$ containing ~~the constant~~ a constant path $p \in L_0 \cap L_1$, and fix a ~~path~~ path of Lagrangian subspaces $\tilde{\gamma}_0 : [0,1] \to TM$ with $\tilde{\gamma}_0(t) \in T_{\gamma_0(t)}M$ and $\tilde{\gamma}_0(0) = T_{\gamma_0(0)}L_0$, $\tilde{\gamma}_0(1) = T_{\gamma_0(1)}L_1$.

Now for any homotopy h connecting $\tilde{\gamma}_0$ to the constant path at p, we get a loop of Lagrangian subspaces in $\mathbb{R}^{2n}$ as follows:

Trivialize $(h^* TM, \omega)$, and then concatenate

- the path $s \mapsto T_{h(s,0)}L_0$, $s \in [0,1]$
- the short homotopy from T_pL_0 to T_pL_1
- the path $s \mapsto T_{h(1-s,1)}L_1$, $s \in [0,1]$
- the path $s \mapsto \tilde{\gamma}_0(1-s)$, $s \in [0,1]$



In general, the Maslov index of this loop will depend both on the choice of γ_0 and $\tilde{\gamma}_0$ <u>and</u> the choice of the homotopy h.

However, if $L_1 = \varphi(L_0)$ for a Hamiltonian diffeomorphism $\varphi: M \to M$ and the Maslov index

$$\mu: \pi_2(M, L) \to \mathbb{Z}$$

vanishes, then the above construction associates an integer to each intersection point $p \in L_0 \cap L_1$ which only depends on the choice of γ_0 and $\tilde{\gamma}_0$ but not on the choice of the homotopy.

The proof of this is quite similar to the proof of the well-definedness of the action on $\mathcal{P}_{\gamma_0}(L_0, L_1)$ we gave last time under the assumption that $\omega_{|\pi_2(M,L)} = 0$.

Remark: There are situations when the above assumption

$$\mu_{|\pi_2(M, L_0)} = 0 \quad \text{holds; namely}$$

whenever $\pi_2(M, L_0) = \{0\}$.

For example, this works for the zero section in cotangent bundles of orientable manifolds Q.

In general, the obstruction to integer gradings lies in the values of μ on $\pi_2(M, L_0)$ and $\pi_2(M, L_1)$, as such disks can be used to modify a homotopy h as above. The best one can expect is a grading in $\mathbb{Z}/_N$ where

$$N = \gcd(N_{L_0}, N_{L_1}).$$

Here N_{L_0} and N_{L_1} are the minimal Maslov numbers of the two Lagrangians one wants to consider.

Note that if L_0 and L_1 are both oriented, all Maslov indices of disks will be even, and so all intersection points come with a grading in $\mathbb{Z}/_2$.

A common convention in this case is to call an intersection point even if the short homotopy preserves orientation and odd if it reverses it.

Here are some sources for further reading:

A. Floer A relative Morse index for the symplectic action
CPAM vol. 41, 395-407
1988

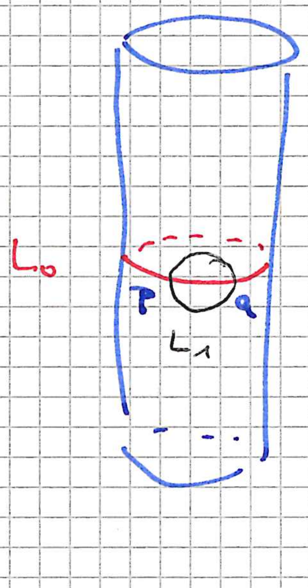
D. Auroux A Beginner's Introduction to Fukaya categories
in: Contact and Symplectic Topology, Springer 2014
also: arxiv/1301.7056

The most systematic treatment of grading issues in Lagrangian Floer theory I am aware of is

P. Seidel Graded Lagrangian submanifolds
Bull. SMF vol. 128, no. 1, 103-149

Even apart from grading issues, disks with boundary on one of the Lagrangian submanifolds are the main obstacle to defining Floer homology. The basic problem is encapsulated in the following example:

Consider $(M,\omega) = (T^*S^1, \omega_{can})$



Let L_0 be the 0-section, and L_1 a contractible circle as drawn.
We have two intersection points p and q, and two half disks which can be parametrized as holomorphic strips

$$u: \mathbb{R}\times[0,1] \to M$$

with $u(\mathbb{R}\times\{i\}) \subseteq L_i$

It turns out that one of them has negative end p and positive end q, and the other one goes from q to p. So we get

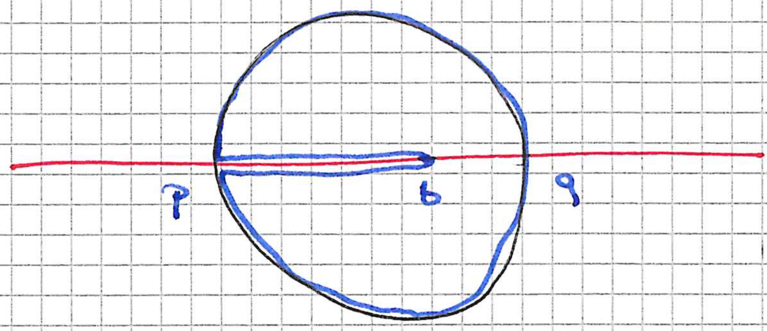
$$\partial p = q \quad \text{and} \quad \partial q = p,$$

which clearly means

$$\partial^2 \neq 0 \,!$$

Where does our proof of $\partial^2 = 0$ go wrong?

Let us try to understand the coefficient of p in $\partial^2 p$.
Our usual strategy would be to consider the moduli space of holomorphic strips from p to p with boundary on L_0 and L_1 and study its compactification.
Geometrically, there is a 1-parameter family of such strips, whose typical element has the following image:



The essential parameter is the position of the branch point b. As b moves toward q, the strips converge to the broken strip consisting of the two half-disks, the upper one going from p to q and the lower one going back.
As b moves to p, the strip degenerates to a disk with boundary on L_1 (it is customary to model this as a configuration consisting of a constant strip at p and a disk bubble).

One way to prevent this problem works in Liouville manifolds. Here a natural assumption which avoids disks like above is to assume that the Lagrangian submanifolds are <u>exact</u>. Recall that this means that the Liouville form λ (which is the primitive of the symplectic form) restricts to each Lagrangian as an exact 1-form. By Stokes' theorem we then have

$$\int_{D^2} u^*\omega = \int_{D^2} u^* d\lambda = \int_{S^1} u^*\lambda = 0$$

for all maps $u : (D^2, S^1) \to (W, L)$.

As nonconstant J-holomorphic disks would have positive symplectic area, they cannot exist in this situation.

So for any two exact closed Lagrangian submanifolds in a Liouville manifold Lagrangian Floer homology can be defined, and shown to be invariant under Hamiltonian isotopy.
This applies in particular to $L_0 = L_1$ (or more precisely, $L_1 = \varphi(L_0)$ for some Hamiltonian diffeomorphism $\varphi \in \mathrm{Ham}(W, d\lambda)$)

Examples:

1. The zero section $Q \subseteq (T^*Q, d\lambda_{can})$ is exact (in fact, the Liouville form λ_{can} vanishes pointwise on it). More generally, the graph of a closed 1-form $\alpha \in \Omega^1(Q)$ is exact if and only if α is exact.

2. In dimension $2n \geq 4$, Lagrangian spheres in Liouville manifolds are always exact because $H^1(S^n) = 0$.

3. An interesting family of Liouville manifolds are obtained as deformations of hypersurfaces with isolated singularities, i.e. manifolds of the form

$$X_{\underline{\alpha},\varepsilon} = \left\{ \sum_{j=0}^{n} z_j^{\alpha_j} = \varepsilon \right\} \subseteq \mathbb{C}^{n+1}$$

for $\varepsilon \neq 0$ and $\underline{\alpha} = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$

These manifolds are (deformation equivalent to) completions of plumbings of cotangent disk bundles of spheres, so they typically contain many Lagrangian spheres.

plumbing: