Analytic setup and outline of proofs for Prop. 1 - 3

We now look at the ingredients and basic steps in the proofs of Propositions 1 - 5, which were the basic inputs in our construction of Morse homology.

To start with, we want to represent the spaces $\mathcal{F}(p,q)$, $\mathcal{K}(p^-,p^+)$ and $\mathcal{H}(p^-,p^+)$ as zero sets of a smooth Fredholm section

$$s: \mathcal{B} \to \mathcal{E}$$

of a Banach bundle over a Banach manifold of maps.

Because the domain $\mathbb{R}$ of all our maps is not compact, some care has to be given to the choice of $\mathcal{B}$ and $\mathcal{E}$ so that we indeed get the desired Fredholm property later on.

<u>Fact 1:</u> Let $f: M \to \mathbb{R}$ be Morse and g <u>any</u> metric. (M closed).
Given $p, q \in \mathrm{Crit}(f)$ there are constants $C_\pm > 0$ and $\delta_\pm > 0$ s.t. for every flow line $\gamma: \mathbb{R} \to M$ of the negative gradient flow of f with

$$\lim_{t \to -\infty} \gamma(t) = p, \quad \lim_{t \to +\infty} \gamma(t) = q$$

there is some $t_0 \in \mathbb{R}$ s.t.

$$d(p, \gamma(t)) \leq C_- e^{\delta_-(t+t_0)}$$

$$d(q, \gamma(t)) \leq C_+ e^{-\delta_+(t-t_0)}$$

Since the differential equations defining our various spaces $\mathcal{F}(p,q)$, $\mathcal{K}(p^-,p^+)$ and $\mathcal{H}(p^-,p^+)$ all look like negative gradient flow outside of a compact subset of $\mathbb{R}$, all their elements have this exponential convergence near $\pm\infty$.
It turns out to be useful to use Banach manifolds of paths which have this kind of convergence already "built in" from the start.

In his book, M. Schwarz uses the following setup:

Compactify $\mathbb{R}$ to $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ with the differentiable structure which makes

$$h : \overline{\mathbb{R}} \longrightarrow [-1,1]$$
$$t \longmapsto \frac{t}{\sqrt{1+t^2}}$$

a diffeomorphism.

A simple computation (see Lemma 2.2. and Cor. 2.4. in Schwarz) shows that every function $f \in C^1(\overline{\mathbb{R}}, \mathbb{R})$ with $f(\pm\infty) = 0$ has the property that $f \in W^{1,2}(\mathbb{R},\mathbb{R})$.

~~We define~~

Let (f,g) be a Morse-Smale pair and let $p,q \in \mathrm{Crit}(f)$. We define

$$C^\infty_{p,q} := \{ u \in C^\infty(\overline{\mathbb{R}}, M) : u(-\infty) = p,\ u(\infty) = q \}$$

Given $u \in C^\infty_{p,q}$, we choose a trivialization ϕ of u^*TM over $\overline{\mathbb{R}}$. With respect to this trivialization, we define

$$W^{1,2}(u^*TM) := (\phi^{-1})_* W^{1,2}(\mathbb{R}, \mathbb{R}^n).$$

Observe that by the Sobolev embedding theorem we have

$$W^{1,2}(\mathbb{R},\mathbb{R}^n) \subset C^0(\mathbb{R},\mathbb{R}^n)$$

$(k=1,\ p=2,\ m=1, \text{ so } p > \frac{m}{k})$

So using the exponential map of the metric g on M, we get

$$(*) \quad \exp_u : W^{1,2}(u^*TM) \longrightarrow C^0(\overline{\mathbb{R}}, M)$$
$$s \longmapsto (t \longmapsto \exp_{u(t)} s(t))$$

We define

$$\mathcal{P}^{1,2}_{p,q} := \{ \exp_u s \in C^0(\overline{\mathbb{R}}, M) : u \in C^\infty_{p,q},\ s \in H^{1,2}(u^*TM) \}$$

Fact 1 (see Schwarz, Sec. 2.1 and Appendix A)

The spaces $\mathcal{P}^{1,2}_{p,q}$ have Banach manifold structures with local charts (*) such that

$$C^\infty_{p,q} \underset{\text{dense}}{\subseteq} \mathcal{P}^{1,2}_{p,q} \subseteq C^0_{p,q}(\overline{\mathbb{R}}, M)$$

Next, one can make sense of the tangent bundle $T\mathcal{P}^{1,2}_{p,q}$ whose characteristic fiber is $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$. We will write simply $T\mathcal{P}^{1,2}_{p,q} = W^{1,2}(\mathcal{P}^{1,2\,*}_{p,q} TM)$.

Similarly, one can define a Banach bundle

$$L^2(\mathcal{P}^{1,2\,*}_{p,q} TM) \longrightarrow \mathcal{P}^{1,2}_{p,q}$$

with characteristic fiber $L^2(\mathbb{R}, \mathbb{R}^n)$.

Fact 2: (prop. 2.8. in Schwarz)

M closed, $f: M \to \mathbb{R}$ Morse, g any metric, $p, q \in \mathrm{Crit}(f)$.
Then the map

$$\mathcal{F}: \mathcal{P}^{1,2}_{p,q} \longrightarrow L^2(\mathcal{P}^{1,2\,*}_{p,q} TM)$$
$$\gamma \longmapsto \dot{\gamma} + (\nabla f) \circ \gamma$$

is a smooth section.
Moreover, the zeroes of $\mathcal{F}$ are exactly the smooth curves $\gamma: \mathbb{R} \to M$ solving the negative gradient flow equation

$$\dot{\gamma} = -(\nabla f) \circ \gamma$$

with $\lim_{t \to -\infty} \gamma(t) = p$, $\lim_{t \to \infty} \gamma(t) = q$.

So we have found

$$\widetilde{\mathcal{F}}(p,q) = \mathcal{F}^{-1}(0).$$

Remark: Similar results hold in the time-dependent case and in the time-dependent parametric case, i.e. for the spaces $\mathcal{K}(p^-, p^+)$ and $\mathcal{H}(p^-, p^+)$.

Now if $F(\gamma)=0$, the linearization of F has the form

$$\begin{aligned} DF_\gamma(\xi) &= \nabla_\xi(\dot\gamma + \nabla f\circ\gamma) \\ &= \nabla_{\dot\gamma}\xi + \nabla_\xi(\nabla f\circ\gamma) \\ &= \dot\xi + A\xi \end{aligned}$$

in some trivialization of $\gamma^* TM$

where $A:\mathbb{R}\to GL(n,\mathbb{R})$ satisfies $A(-\infty)=\mathrm{Hess}_p f$
$A(+\infty)=\mathrm{Hess}_q f$.

Fact 3: (cf. sec. 2.2. of Schwarz)

If f is Morse and $\gamma\in F^{-1}(0)\subseteq \mathcal{P}^{1,2}_{p,q}$, then DF_γ is Fredholm of index

$$\mathrm{ind}\, DF_\gamma = \mathrm{ind}(p) - \mathrm{ind}(q).$$

Morse indices

So far, we have discussed the situation for a fixed metric g. To achieve that our solution spaces are smooth manifolds, we want to make the section F above transverse to the zero section by varying the metric.

To set this up, we need a space of metrics modelled on another Banach space. Here Schwarz chooses a variation on Floer's C^ε-spaces, which are subspaces of C^∞ defined in terms of a sequence

$$\underline{\varepsilon} = (\varepsilon_n)_{n\in\mathbb{N}} \quad \text{of positive real numbers}$$

So suppose $E\to M$ is a Riemannian vector bundle. For a smooth section $s:M\to E$ we define

$$\|s\|_{\underline{\varepsilon}} := \sum_{k\ge 0} \varepsilon_k \max_M \|\nabla^k s\|, \quad \text{where}$$

$$\|\nabla^k s\|(x) = \max\left\{ |\nabla_{v_1}\cdots\nabla_{v_k} s(x)| \;\middle|\; v_1,\dots,v_k\in T_xM \text{ with } |v_i|=1\right\}$$

Now define

$$C^{\varepsilon}(E) := \{ s \in \Gamma(E) = C^{\infty}(M,E) : \|s\|_{\underline{\varepsilon}} < \infty \}.$$

Floer proved that there is a choice of the sequence $\underline{\varepsilon} = (\varepsilon_n)_{n\in\mathbb{N}}$ so that the space $C^{\varepsilon}(E)$ is dense in $L^2(E)$.

Let g_0 be any metric on M. Any other metric can be written as

$$g_A(v,w) = g_0(Av, w),$$

where $A: TM \to TM$ is symmetric w.r.t. g_0, i.e.

$$g_0(Av,w) = g_0(v, Aw).$$

$A = \mathrm{Id}$ corresponds to g_0, so we write $A = \mathrm{Id} + a$, where $a \in \mathrm{Sym}_{g_0}(TM)$. Now we set

$$\mathcal{G} := \{ g_{\mathrm{Id}+a} : \quad a \in C^{\varepsilon}(\mathrm{Sym}_{g_0}(TM)) \}$$

If $\mathcal{U} \subseteq C^{\varepsilon}(\mathrm{Sym}_{g_0}(TM))$ is a sufficiently small nbhd of 0, then

$\mathrm{Id} + \mathcal{U}$ parametrizes a space of metrics $\mathcal{G}'$ near g_0 modelled on the Banach space $C^{\varepsilon}(\mathrm{Sym}_{g_0}(TM))$.

<u>Fact 4:</u> $f: M \to \mathbb{R}$ Morse, $p, q \in \mathrm{Crit}(f)$

(cf. Sec. 2.5. in Schwarz)

Then the linearization of

$$\tilde{\mathcal{F}} : \mathcal{G}' \times \mathcal{P}^{1,2}_{p,q} \longrightarrow L^2(\mathcal{P}^{1,2}_{p,q}{}^{*}TM)$$

at any $(\tilde{g}, \gamma) \in \tilde{\mathcal{F}}(0)$ is surjective.

In particular, there is a dense set of metrics $\tilde{g}$ with respect to which the section

$$\mathcal{F} = \mathcal{F}_{\tilde{g}} : \mathcal{P}^{1,2}_{p,q} \longrightarrow L^2(\mathcal{P}^{1,2}_{p,q}{}^{*}TM)$$

is transverse to the zero section.

<u>Corollary</u>: For any metric $\tilde{g}$ as in Fact 4, the spaces
$$\widetilde{\mathcal{F}}(p,q)$$
are smooth manifolds of dimension
$$\operatorname{ind}(p) - \operatorname{ind}(q)$$
for all $p, q \in \operatorname{Crit}(f)$.

<u>Remark</u>: Again, analogous results hold for the spaces $\mathcal{K}(p^-, p^+)$ and $\mathcal{H}(p^-, p^+)$