Next we want to analyse how the chain map c depends on the chosen family (f_t, g_t).

So we fix two Morse-Smale pairs (f^-, g^-) and (f^+, g^+) and consider two families

$$(f^0_t, g^0_t) \quad \text{and} \quad (f^1_t, g^1_t)$$

connecting them.

We then pick a 2-parameter-family $(f_{t,s}, g_{t,s})$ with $s \in [0,1]$ such that

$$f_{t,s} = \begin{cases} f^- & \text{for } t \ll 0 \text{ and all } s \in \mathbb{R} \\ f^+ & \text{for } t \gg 0 \text{ and all } s \in \mathbb{R} \\ f^0_t & \text{for } s = 0 \\ f^1_t & \text{for } s = 1 \end{cases}$$

and similarly for $g_{t,s}$.

Now for $p^\pm \in \mathrm{Crit}(f^\pm)$ we define

$$\mathcal{H}(p^-, p^+) := \left\{ \gamma : \mathbb{R} \to M \;\middle|\; \exists\, s \in [0,1] \text{ with } \dot\gamma(t) = X_{t,s}(\gamma(t)) \text{ and } \lim_{t \to \pm\infty} \gamma(t) = p^\pm \right\}$$

Proposition 3: For a generic 2-parameter family $(f_{t,s}, g_{t,s})$ the spaces $\mathcal{H}(p^-, p^+)$ are manifolds of dimension

$$\mathrm{ind}(p^-) - \mathrm{ind}(p^+) + 1.$$

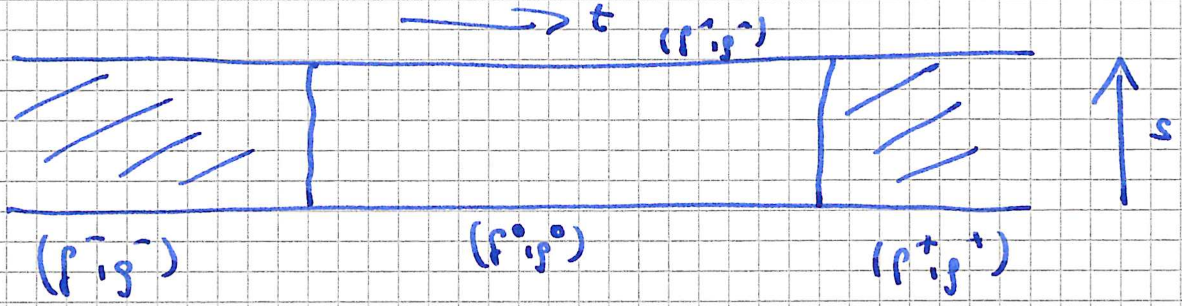
Moreover:

a) If $\mathrm{ind}(p^-) = \mathrm{ind}(p^+) - 1$, then $\mathcal{H}(p^-, p^+)$ is compact.

b) If $\mathrm{ind}(p^-) = \mathrm{ind}(p^+)$, then $\mathcal{H}(p^-, p^+)$ has a natural compactification to a 1-manifold $\overline{\mathcal{H}}(p^-, p^+)$ with boundary s.t.

$$\partial \overline{\mathcal{H}}(p^-, p^+) \underset{\text{bij}}{\cong} \mathcal{K}^0(p^-, p^+) \cup \mathcal{K}^1(p^-, p^+)$$
$$\cup \bigcup_{\mathrm{ind}\, p' = \mathrm{ind}\, p^- - 1} \mathcal{F}(p^-, p') \times \mathcal{H}(p', p^+)$$
$$\cup \bigcup_{\mathrm{ind}\, p'' = \mathrm{ind}\, p^+ + 1} \mathcal{H}(p^-, p'') \times \mathcal{F}(p'', p^+)$$

Schematically, the situation is as follows:



As a consequence of Proposition 3, the map

$$h = h^{(f_{t,s}, g_{t,s})} : CM_*(f^-, g^-) \longrightarrow CM_{*+1}(f^+, g^+)$$

$$h(p^-) = \sum_{\substack{\mathrm{ind}(p^+) \\ = \mathrm{ind}(p^-)+1}} \#_2 \mathcal{H}(p^-, p^+)$$

is a chain homotopy between c^0 and c^1:

$$c^1 - c^0 = \partial^+ h + h \partial^-$$

In particular, all continuation maps from $CM_*(f^-, g^-)$ to $CM_*(f^+, g^+)$ induce the same map on homology.

<u>Important observation:</u>

A composition of continuation maps

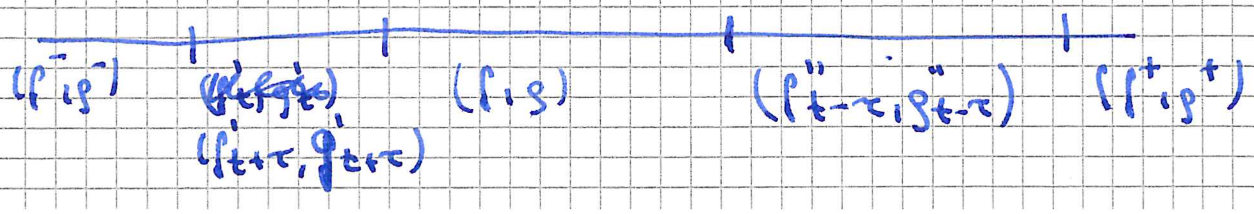
$$c' : CM_*(f^-, g^-) \to CM_*(f, g) \quad \text{and}$$

$$c'' : CM_*(f, g) \to CM_*(f^+, g^+)$$

can be viewed as a limit of continuation maps

$$c : CM_*(f^-, g^-) \longrightarrow CM_*(f^+, g^+)$$

obtained by concatenating the "interesting parts" of the families (f_t', g_t') and (f_t'', g_t'') used to define c' and c'' with a suitable shift:



Considering the special case $(f^-, g^-) = (f^+, g^+)$, with an arbitrary (f, g) inbetween, and recalling that all continuation maps on the same complex are chain homotopic to the identity, we conclude

Cor: The homologies of the various Morse complexes $(CM_*(f,g), \partial^{(f,g)})$ for different Morse-Smale pairs (f,g) are isomorphic (with the isomorphisms induced by chain maps in a unique chain-homotopy class for each pair).

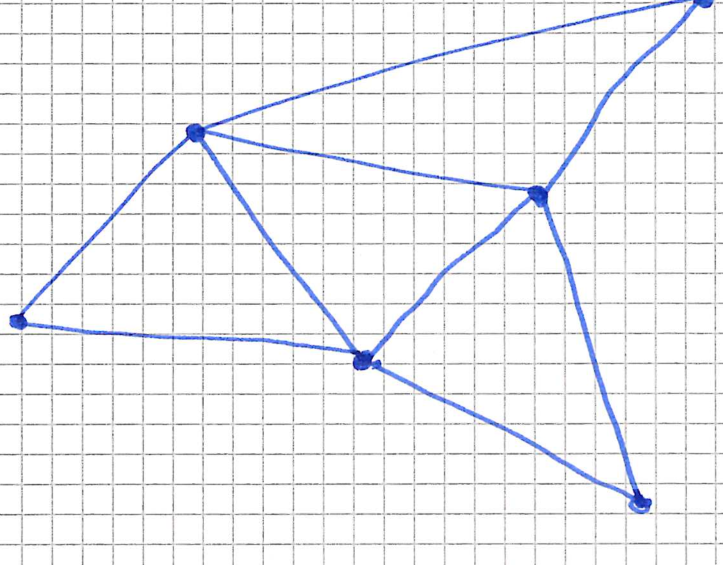
At this point we know that the Morse homology

$$HM_*(M; \mathbb{Z}_2) := H_*(CM_*(f,g))$$

is well-defined up to isomorphism.

Now we can return to the question:
What does $HM_*(M; \mathbb{Z}_2)$ compute?

There are many ways to get the answer.
One simple way is to use the fact that every smooth manifold admits a triangulation. Then one can build a Morse function and a metric s.t. the critical points of index k are exactly the barycenters of the k-simplices, and the negative gradient flow is tangent to all the simplices

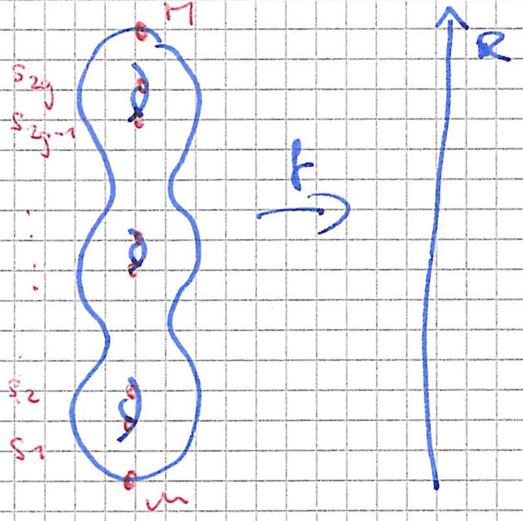


In this case $(CM_*(f,g), \partial^{(f,g)})$ is isomorphic to the simplicial chain complex corresponding to the triangulation, and so

$$HM_*(M;\mathbb{Z}_2) \cong H_*(CM_*(f,g),\partial)$$
$$\cong H_*^{simp}(M;\mathbb{Z}_2) \cong H_*(M;\mathbb{Z}_2).$$

Before we get into the discussion of how one proves Prop 1-3, here are some further comments and examples:

(1) On a surface of genus $g \geq 0$, we can consider a "height function" similar to the one on the torus:



The function will have
1 minimum
1 maximum
$2g$ saddle points

There are two connecting orbits from the maximum to each of the saddle points and also two from each of the saddle points to the minimum.

Choose g so that (f,g) is Morse-Smale

It follows that the mod 2 counts are all 0, and so we get a simple computation of the homology of these surfaces.

(2) On $\mathbb{CP}^n$, we consider homogeneous coordinates $[z_0 : \dots : z_n]$.

<u>Exercise:</u> The function

$$f([z_0:\dots:z_n]) := \sum_{k=1}^{n} k \frac{|z_k|^2}{\|z\|^2}$$

is Morse with one critical point of index $2k$ for each $k \in \{0,1,\dots,n\}$.

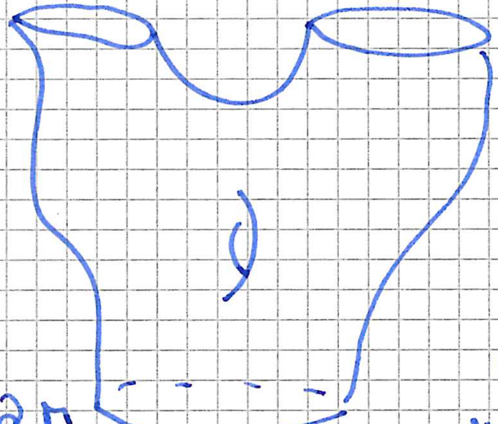
Here all boundary maps are zero for degree reasons (there are no critical points of index difference 1), and so again we get a simple computation of $H_*(\mathbb{CP}^n;\mathbb{Z}_2)$.

③ Exercise: What happens when we restrict the function from the previous example to $\mathbb{RP}^n \subseteq \mathbb{CP}^n$?

④ There are many variations on the construction. For example, suppose M is not closed, but instead has boundary, which we assume is divided into two pieces (either of which could be empty or consist of several pieces):

$$\partial M = \partial_+ M \sqcup \partial_- M$$

$\partial_+ M$



$\partial_- M$

Now we can look at Morse functions $f: M \to \mathbb{R}$ s.t. $\partial_- M$ and $\partial_+ M$ are (components of) regular level sets, with $\partial_- M$ a local minimum and $\partial_+ M$ a local maximum.

If we only allow homotopies $\{f_t\}$ and homotopies of homotopies $\{f_{t,s}\}$ with the same constraint, then we still get a well-defined theory that computes

$$H_*(M, \partial_- M; \mathbb{Z}_2).$$

⑤ Poincaré duality is very easy to see from the point of view of Morse theory:
If (f,g) is Morse-Smale, then so is $(-f,g)$. All the critical points agree, but the indices change from k for f to $\dim M - k$ for $-f$.
Also, all trajectories stay the same except for a change of direction, so

$$\mathcal{F}^f(p,q) \underset{\text{bij}}{\cong} \mathcal{F}^{-f}(q,p).$$

We conclude that
$$(CM_*(f,g), \partial^{(f,g)}) \cong (CM_{\dim M - *}(-f,g), \partial^{(-f,g)})^*$$

But we also have the chain isomorphism
$$(CM_*(f,g), \partial) \cong (CM_*(-f,g), \partial)$$
given by continuation.
So we find that
$$HM_*(M;\mathbb{Z}_2) \cong HM^{\dim M - *}(M;\mathbb{Z}_2)$$

or, in the situation with boundary considered in 4 above,
$$HM_*(M, \partial_- M; \mathbb{Z}_2) \cong HM^{\dim M - *}(M, \partial_+ M; \mathbb{Z}_2)$$

6 One can lift the discussion to $\mathbb{Z}$-coefficients. In order to do this, one needs to orient the spaces $\mathcal{F}(p,q)$, $\mathcal{K}(p^-,p^+)$ and $\mathcal{H}(p^-,p^+)$ and upgrade the statements on the compactifications of 1-dimensional components to include orientations.

Here is the underlying geometric idea to assign signs to isolated flow lines, i.e. points in a 0-dimensional component of $\mathcal{F}(p,q)$:

Given a Morse-Smale pair (f,g), we choose (arbitrarily!) orientations of all the unstable manifolds $W^u(p)$, where $p \in \mathrm{Crit}(f)$. This amounts to choosing an orientation of $T_p W^u(p)$.

This choice also gives a coorientation to $W^s(p)$ for all $p \in \mathrm{Crit}(f)$.

Fact from differential topology:
The transverse intersection of an oriented submanifold with a cooriented submanifold inherits a natural orientation:
A basis $\{b_1, \dots, b_r\}$ of $T_x(W^u(p) \cap W^s(q))$ is positive, if the ordered collection of vectors in

$T_x W^u(p)$ needed to complete it to a positively oriented basis of $T_x W^u(p)$ project to a positively oriented basis of
$$N_x W^s(p) = T_xM / T_x W^s(p) .$$

If $W^u(p) \cap W^s(q)$ has dimension 1, i.e. consists of isolated flow lines, we can compare this orientation with the one given by the flow. This determines a sign for each flow line, and the claim is that counting flow lines with these signs gives a well-defined boundary map over $\mathbb{Z}$.

This ends our outline of Morse theory. For more examples and applications, look at the book by Audin and Damian.

Next time we will talk about the analytic setup needed to prove Propositions 1 – 3, essentially following Schwarz.