I ended last time by claiming that (for closed, symplectically aspherical symplectic manifolds (M,ω)) one can define a Floer complex

$$(CF_*(H,J), \partial)$$

for a generic pair (H,J). As in our discussion of Morse theory, one can set up continuation maps

$$c : (CF(H^+,J^+), \partial) \longrightarrow (CF(H^-,J^-), \partial)$$

between the Floer complexes associated to two choices of Floer data by choosing s-dependent families of Hamiltonians and almost complex structures and then counting solutions to the corresponding Floer equation with index 0.
Similarly, by choosing interpolating 2-parameter families between different such paths one constructs chain homotopies between any two continuation maps, so that the induced map on homology is always the same. Also, compositions of continuation maps are chain homotopic to continuation maps, and the constant data gives the identity $(CF(H,J), \partial) \circlearrowleft$.

Putting all this together, we conclude that up to natural isomorphism, $HF_*(M,\omega) = HF_*(H,J)$ is well-defined.
To complete the discussion, it remains to compute it in a specific example:

<u>Prop:</u> Suppose (H,J) is generic Floer data, where

$$H: M \to \mathbb{R}$$

is autonomous and C^2-small.
Then
$$HF_*(H,J) \cong HM_*(-H),$$
and so Floer homology is isomorphic to singular homology for closed, symplectically aspherical (M,ω).

Sketch of proof:

We list the main steps and comment on them.
For details, see e.g. chapter 10 in Audin–Damian

Step 1: For an autonomous, nondegenerate Hamiltonian $H: M \to \mathbb{R}$ which is sufficiently C^2-small, the only 1-periodic orbits are the constant orbits at the fixpoints.

Pf:
- It is obvious from the definitions that

$$X_{c\cdot H} = c \cdot X_H ,$$

and so the periods of the Hamiltonian $c\cdot H$ are exactly $\frac{1}{c}$ times the periods of H.

- Let $x \in \mathrm{Crit}(H)$, and let $S: T_xM \to T_xM$ be the linear map s.t.

$$\mathrm{Hess}_x H = g_J(S\,\cdot\,,\,\cdot\,)$$

Then the linearized flow of X_H at x, i.e.

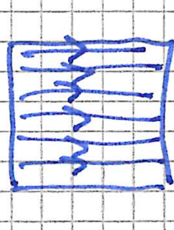
$$\Psi_t = d\varphi_t : T_xM \to T_xM$$

satisfies

$$\dot{\Psi}_t = J S \Psi_t$$

It follows that if $\|S\| < 2\pi$, then the linearized flow does not have 1-periodic orbits. This can be achieved by rescaling H.
But then the fixpoint x is isolated as a 1-periodic orbit.
Conclusion: For each $x \in \mathrm{Crit}(H)$, we find an open nbhd $U_x \subseteq M$ which does not contain 1-periodic orbits other than x.

- On $M \setminus \bigcup U_x$, the flow is nonsingular, and we can cover this compact set by flow charts for X_H (finitely many!).



There is a minimal time T_0 needed

to pass through each one of the flow charts, and so any periodic orbit which is not fixed must have a period of more than T_0.

By another rescaling of H we can arrange $T_0 > 1$.

__Step 2:__ By property (F) of the Conley-Zehnder index we know that for $x \in \mathrm{Crit}(H)$, we have

$$\mu_{CZ}(\Psi_t) = \frac{1}{2}\,\mathrm{sign}(\mathrm{Hess}_x H)$$

$$= \frac{1}{2}\left(\#\,\text{pos. EV} - \#\,\text{neg. EV}\right)$$

$$= \frac{1}{2}\left(2n - 2\,\#\,\text{neg. EV}\right)$$

$$= n - \mathrm{ind}_x H$$

So

$$n + \mu_{CZ}(x) = 2n - \mathrm{ind}_x H = \mathrm{ind}_x -H.$$

__Step 3:__ If $\gamma : \mathbb{R} \to M$ is a gradient flow line for $-H$, then

$$u(s,t) = \gamma(s)$$

satisfies Floer's equation, because $X_H = J \nabla H$ and so

$$\partial_s u + J(u)\left(\partial_t u - X_H(u)\right) = 0$$

$$\iff \quad \partial_s u = J X_H(u)$$

$$= -\nabla H(u) = \nabla(-H)(u).$$

Moreover, one proves that if J is such that $(H, g_J = \omega(\cdot, J\cdot))$ is a Morse-Smale pair, then these solutions are regular as Floer cylinders, i.e. the linearization of Floer's equation at such a solution is surjective.

<u>Step 4:</u> Now consider the sequence of Hamiltonians

$$H_k := \frac{1}{k} H_0, \qquad k \in \mathbb{N}$$

where H_0 is already C^2-small as above. Then if k is sufficiently large, all Floer cylinders for (H_k, J) connecting critical points $x^\pm$ with

$$\mu_{CZ}(x^+) - \mu_{CZ}(x^-) = 1$$

are independent of $t \in S^1$. In particular, this means that for k large enough, the Floer complex for H_k coincides with the Morse complex for $-H_k$.

<u>Sketch of proof:</u> We argue by contradiction, so we assume $x^\pm \in \mathrm{Crit}(H_0)$ satisfy

$$\mu_{CZ}(x^+) - \mu_{CZ}(x^-) = 1$$

and there is a sequence $\{u_{k_n}\}_{n \geq 1}$ of Floer cylinders for H_{k_n} which satisfy $\partial_t u_{k_n} \not\equiv 0$.

Since they connect x^+ and x^-, their energies are all uniformly bounded. They satisfy Floer's equation

$$\partial_s u_{k_n} + J\left(\partial_t u_{k_n} - \frac{1}{k_n} X_{H_0}(u_{k_n})\right) = 0.$$

It follows that

$$v_{k_n}(s,t) = u_{k_n}(k_n \cdot s, k_n \cdot t)$$

satisfy Floer's equation for $H = H_0$.

We know that $\mathcal{M}(x^+, x^-; H, J)$ is compact and discrete, so a subsequence of the v_{k_n} must be constant equal to some $v \in \mathcal{M}(x^+, x^-)$.

We want to argue that v is t-independent.

For that, note that the minimal period of v_{k_n} in the t-coordinate is $\frac{1}{k_n}$ and converges to 0 as $n \to \infty$.

Now given $r \in \mathbb{R}$, we observe

$$\begin{array}{ccc} v_{k_n}(s,t) & = & v_{k_n}\left(s, t + \frac{[r k_n]}{k_n}\right) \leftarrow \text{integer part} \\ \| & & \Big\downarrow n \to \infty \\ v(s,t) & & v(s, t+r) \end{array}$$

Since here $r \in \mathbb{R}$ is arbitrary, the claim that v is independent of $t \in S^1$ follows.
This completes our sketch of the proof of the proposition.

□

Comments:

(1) An alternative argument for relating $HF_*(H,J)$ to Morse homology is to study the so-called PSS-maps (after Piunikhin-Salamon-Schwarz)

$$CM_*(f,g) \longrightarrow CF_*(H,J)$$

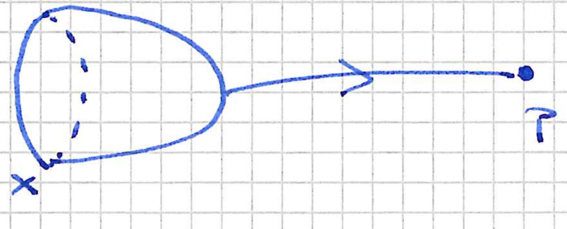
counting

(*)

and

$$CF_*(H,J) \longrightarrow CM_*(f,g)$$

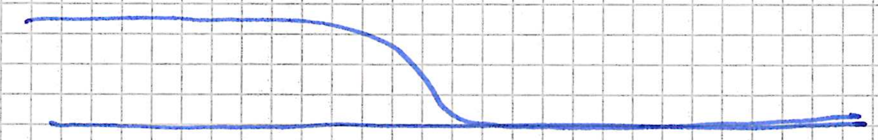
counting



(**)

Here the 1-dimensional pieces are half-infinite gradient flow lines for (f,g), and the surface parts are maps $u : \mathbb{R}^2 \to M$
which in radial coordinates satisfy a Floer-type equation

One way to think of them is as Floer cylinders for an s-dependent family of Hamiltonians $\{H_{s,t}\}$, where for the first map we have



$H_{s,t} = H_t$ for $s \ll 0$ $\qquad$ $H_{s,t} \equiv 0$ for $s \gg 0$

The data for the second map is reversed.
At the end where the Hamiltonian term vanishes we require convergence to a constant, which means that we have a removable singularity there.

Note that for $(f,g) = (H, g_J)$ with H C^2-small and autonomous so that the previous argument works, it turns out that the only "spiked disks" counted by the PSS maps are constant, so in this situation the two arguments coincide.

(2) Note that a posteriori we see that the restriction to contractible periodic orbits was not really a substantial restriction: The invariance proof would also work in other homotopy classes, but since there are no nonconstant periodic orbits for C^2-small Hamiltonians, they would contribute nothing new to the resulting homology.

Next I want to briefly discuss how the basic outline above needs to be adapted when we weaken our assumption of symplectic asphericity.

1. The simplest generalization to deal with is the case
$$\omega|_{\pi_2(M)} = 0 \quad , \text{but} \quad c_1(M)|_{\pi_2(M)} \neq 0 .$$
Here the action functional is well-defined on $\mathcal{L}_0 M$ and bubbles are still excluded, but the grading of periodic orbits generally depends on the choice of capping disk.

<u>Def:</u> The <u>minimal Chern number</u> of (M,ω) is defined as
$$N := \inf \{ k > 0 : \exists u : S^2 \to M \text{ s.t. } \langle c_1(M), [u] \rangle = k \}.$$

From the fact that taking a connected sum of a capping disk with a sphere changes the Conley-Zehnder index by twice the Chern number of the sphere, we see that the index of an orbit without reference to capping disks is only well-defined modulo $2N$.
So Floer homology is now $\mathbb{Z}/_{2N}$-graded instead of $\mathbb{Z}$-graded.
The computation then becomes
$$HF_k(M) \cong \bigoplus_{r \equiv k \bmod 2N} H_r(M) .$$

2. <u>monotone symplectic manifolds</u>

Recall that (M,ω) is called monotone if $\exists \tau > 0$ s.t.
$$\int_{S^2} u^* c_1(M) = \tau \int_{S^2} u^* \omega \quad \text{for all } u : S^2 \to M .$$

Since $c_1(M)$ is integral, we can rescale ω by τ to arrange that the symplectic areas of all spheres are integral as well.

Such a rescaling has no effect on the group

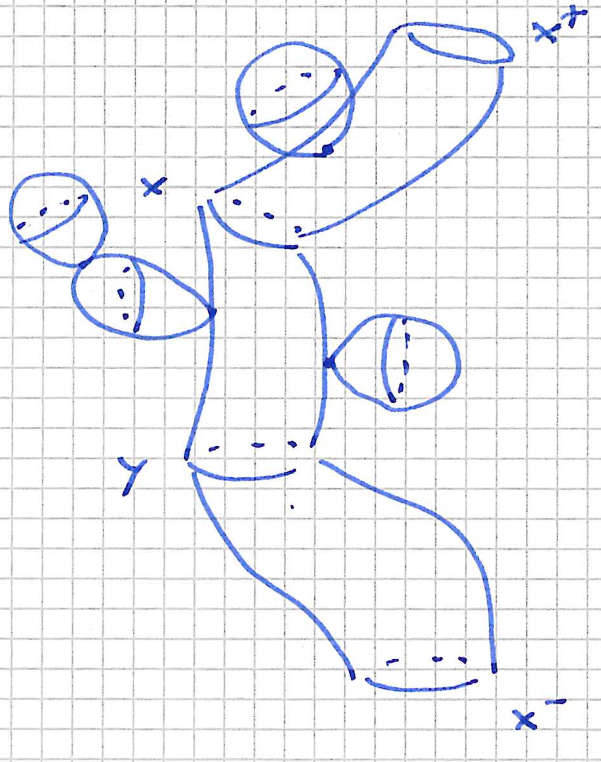
$$\mathrm{Ham}(M,\omega) \subseteq \mathrm{Diff}(M).$$

Now bubbling of J-holomorphic spheres has to be understood, and dealt with.

So suppose (H,J) is regular Floer data, and $x^\pm \in \mathcal{P}(H)$ are contractible 1-periodic orbits.

The moduli space $\widetilde{\mathcal{M}}(x^+, x^-; H, J)$ may now have components of different dimensions, depending on the homotopy class of the cylinders. For the Floer complex, we care in particular about the parts of moduli spaces of dimension 1 (for the definition of ∂) and 2 (for proving $\partial^2 = 0$).

A priori, the limit of a sequence of Floer cylinders of uniformly bounded energy could be a broken Floer cylinder with holomorphic spheres attached:



Floer-Gromov convergence

$$u_k \longrightarrow (u^1, \dots, u^r) \# (v^1, \dots, v^s)$$

$(u^1,\dots,u^r)$: Floer cylinders; $(v^1,\dots,v^s)$: J-hol. spheres

Now since

$$\widetilde{\mathcal{A}}_H(x, v_0 \# v) = \widetilde{\mathcal{A}}_H(x, v_0) + \int_{S^2} v^*\omega$$

and

$$\mu_{CZ}(x, v_0 \# v) = \mu_{CZ}(x, v_0) + 2 \int_{S^2} v^* c_1(M)$$

$$= \mu_{CZ}(x, v_0) + 2\tau \int_{S^2} v^*\omega ,$$

We see that

$$\eta_H(x) := \widetilde{\mathcal{A}}_H(x, v_0) - \frac{1}{2\tau}\mu_{CZ}(x, v_0) \in \mathbb{R}$$

is independent of the capping disk v_0.

But for a Floer cylinder $u: \mathbb{R}\times S^1 \to M$ with $\lim\limits_{s\to\pm\infty} u(s,t) = x^\pm(t)$

we have

$$\begin{aligned} E(u) &= \widetilde{\mathcal{A}}_H(x^+, v^- \# u) - \widetilde{\mathcal{A}}_H(x^-, v^-) \\ &= \eta_H(x^+) - \eta_H(x^-) + \frac{1}{2\tau}\underbrace{\left(\mu_{CZ}(x^+, v^-\# u) - \mu_{CZ}(x^-, v^-)\right)}_{= \dim_u \widetilde{\mathcal{M}}(x^+, x^-)} \end{aligned}$$

So on a component of $\widetilde{\mathcal{M}}(x^+, x^-)$ of fixed dimension the energy is constant (so bounded).

Fix a component of $\widetilde{\mathcal{M}}(x^+, x^-)$ of dimension d, and consider a sequence $\{u_k\}$ of Floer cylinders in this component. By our bubbling analysis, a subsequence will converge to a limit $(u^1, \dots, u^r) \# (v^1, \dots, v^s)$ as above with

$$E(u_k) = \sum_{i=1}^{r} E(u^i) + \sum_{j=1}^{s} E(v^j)$$

$$\eta_H(x^+) - \eta_H(x^-) + \frac{d}{2\tau} = \eta_H(x^+) - \eta_H(x^-) + \frac{1}{2\tau}\sum_i \dim_{u_i}\widetilde{\mathcal{M}}(x_{i-1}, x_i)$$

$$+ \sum_{j=1}^{s} E(v^j).$$

By regularity, we know that $\dim_{u_i}\widetilde{\mathcal{M}}(x_{i-1}, x_i) \geq 1$, and also $E(v^j) > 0$.

$\boxed{d=1}$ The only possible case is ~~..~~ $r=1$ and $s=0$, i.e. no bubbling or breaking can occur and so $\mathcal{M}^1(x^+, x^-) = \widetilde{\mathcal{M}}^1(x^+, x^-)/\mathbb{R}$ is compact for all pairs $x^\pm \in \mathcal{P}(H)$.

$\boxed{d=2}$ Here we consider 2 cases:

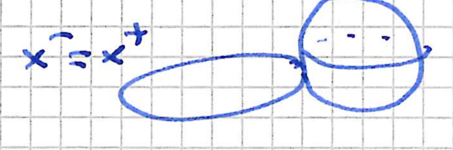
ⓐ If $x^+ \neq x^-$, then in the limit we must have at least one non-trivial Floer cylinder, so

$$\sum_{j=1}^{s} E(w^j) \leq \frac{1}{2\tau}$$

But for each nonconstant map $v: S^2 \to M$ we have
$$E(v) = \frac{1}{\tau} \langle c_1(M), [v] \rangle \geq \frac{N}{\tau} \geq \frac{1}{\tau}.$$

So in this case again bubbling is not possible, and the only options are straight convergence or convergence to a once broken Floer cylinder ($r=2$).

ⓑ The dangerous case is $x^+ = x^-$ and $N=1$. Here we could potentially have a sequence of nontrivial Floer cylinders converging to a union of one constant Floer cylinder
$$u(s,t) = x^+(t)$$
and a holomorphic sphere of Chern number 1.



Their images would have to intersect (as they are a limit of images of Floer cylinders).

But the moduli space of J-holomorphic spheres $v: S^2 \to M$ with $\langle c_1(M), [v] \rangle = c$ has dimension
$$2(n-3) + 2c .$$
So for $c=1$ we get a $2n-4$-dimensional space. The union of images in M is therefore at most $(2n-2)$-dimensional.
So for generic choice of J it will miss the 1-dimensional periodic orbits, again excluding this limiting configuration.
Now the usual argument shows $\partial^2 = 0$.

Again, we conclude

$$HF_*(H,J) \cong \bigoplus_{k \equiv * \bmod 2N} H_k(M;\mathbb{Z}_2).$$