Introduction to Floer homology

This course follows up on some topics we already discussed during last term's lectures on symplectic geometry. The motivating questions are related to dynamics of Hamiltonian diffeomorphisms of symplectic manifolds.

Recall: (M, ω) symplectic

$H : \mathbb{R} \times M \to \mathbb{R}$ 1-periodic function

$\rightsquigarrow$ get a vector field X_{H_t} from

$$\omega(X_{H_t}, \cdot) = -dH_t$$

Assume X_H can be integrated to a 1-parameter family of diffeomorphisms φ_t^H of M.
Any diffeo occurring as time-1-map of such a family is called a Hamiltonian diffeomorphism. They form the group $\mathrm{Ham}(M, \omega)$

Conjecture (Arnold) (M, ω) closed
Assume all fixed pts of $\varphi \in \mathrm{Ham}(M, \omega)$ are nondegenerate.
Then
$$\# \mathrm{Fix}(\varphi) \geq \sum_{k=0}^{\dim M} b_k(M)$$

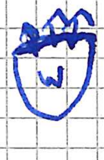
Hamiltonian Floer homology gives an approach to proving this conjecture in many (all?) cases.

Symplectic homology is a variant of this for Liouville domains, and has many interesting algebraic structures. It allows to prove statements on embedability ~~and~~ of Liouville domains (or Lagrangian submanifolds) into other Liouville domains, and has relations to many other topics

$\omega = d\lambda$

$\omega(Y, \cdot) = \lambda$

$L_Y \omega = \omega$



Lagrangian Floer homology grew out of the attempt to prove another conjecture by Arnold:

Conjecture (Arnold) Q closed smooth manifold

$L \subseteq T^*Q$ image of Q under Hamiltonian diffeo of (T^*Q, ω_{can})

Then
$$\# L \cap Q \geq \sum_{k=0}^{\dim Q} b_k(Q)$$
(assuming all intersections are transverse)

Remark: This is the local version of a more general question: (M,ω) symplectic, $Q \subset M$ Lagrangian. Is it true that the above statement holds for L a Hamiltonian ~~displacement~~ image of Q?

Warning: Clearly, this cannot always be true, as any Lagrangian in $(\mathbb{R}^{2n}, \omega_{can})$ can be moved away from itself.

But: This question for the diagonal $M \subseteq M \times M$ is equivalent to the Hamiltonian Arnold conjecture.

Lagrangian Floer homology is the basis for the construction of Fukaya categories

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Both versions are replacements for Morse theory for a certain functional on a suitable path space, and a lot of the ingredients can be understood more readily in the easier context of Morse theory. So regardless which version we concentrate on, we will start with a review of Morse theory, ~~pointing out which~~ using a blueprint that will generalize to the Floer setting.

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Another related theory (not due to Floer) is quantum (co)homology. Roughly, one uses holomorphic spheres to deform the cup product of a symplectic manifold (M,ω) (one also needs to change the coefficient ring in the process).
(In nice cases,) Hamiltonian Floer homology of a closed symplectic manifold and quantum cohomology of the same manifold are isomorphic as rings.

On the other hand, if $L \subseteq (M,\omega)$ is a Lagrangian submanifold s.t. $HF(L,L)$ is defined, then $HF(L,L)$ is a module over $QH(M)$.