Compactness modulo bubbling and breaking

Besides regularity of the moduli spaces, we also need to understand their compactifications.
The starting point here is that for any Floer cylinder we have

$$E(u) = \iint_{\mathbb{R}\times S^1} \|\partial_s u\|^2 \, ds\,dt = \mathcal{A}_H(x^+) - \mathcal{A}_H(x^-)$$

i.e. within a moduli space $\widetilde{\mathcal{M}}(x^+; x^-; H, J)$ the energy is uniformly bounded. (For this statement to be true without our standing assumption of symplectic asphericity, we need to also fix the homotopy class of the cylinder).

There is one essential difference to the Morse case now: $W^{1,2}$-maps from the 1-dimensional domain $\mathbb{R}$ are continuous by the Sobolev embedding theorem. But now our domain $\mathbb{R}\times S^1$ is 2-dimensional, which is the borderline case for the Sobolev inequalities. So interesting new phenomena can (and in general will happen).

The strategy remains the same: we will try to get C^∞_{loc}-convergence of a subsequence of a given sequence $\{u_k\}_{k\geq 1}$ in $\widetilde{\mathcal{M}}(x^+, x^-)$. The main tool for that was the Arzela-Ascoli theorem, which in turn relies on bounds on the derivatives of our map.
These were trivial in the Morse case, but need some work (and generally will fail!) in the Floer setting.

Prop: (M,ω) symplectically aspherical, $H: \mathbb{R}\times M \to \mathbb{R}$ nondegenerate
let $x^\pm \in \mathcal{P}(H)$ be given.
Suppose $\{u_k\}_{k\geq 1}$ is a sequence in $\widetilde{\mathcal{M}}(x^+, x^-)$ s.t. the images are contained in a fixed compact subset of M.
Then

$$\sup_k \|\partial_s u_k\|_\infty < \infty. \qquad (*)$$

<u>Rem:</u> The statement also holds without assuming $x^\pm$ to be nondegenerate, but we will have no use for that, and the proof would get more complicated.

<u>Pf:</u> The proof is a bubbling argument, i.e. we assume that (*) fails and derive a contradiction by constructing a nonconstant holomorphic sphere in that case.

If (*) fails, we can pass to a subsequence and then find points $z_k \in \mathbb{R}\times S^1$ s.t.

$$M_k := |\partial_s u_k(z_k)| \xrightarrow[k\to\infty]{} \infty$$

and

$$M_k = \max_{\mathbb{R}\times S^1} \|\partial_s u_k\|$$

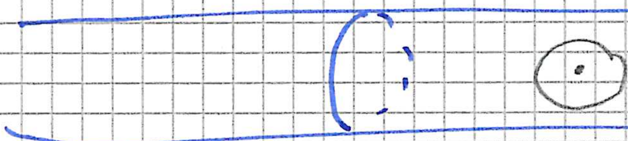
(This maximum exists because $|\partial_s u| \to 0$ as $s \to \pm\infty$ for Floer cylinders)

Now pick $\varepsilon_k \to 0$ such that $\varepsilon_k M_k \to \infty$, for example $\varepsilon_k = (\sqrt{M_k})^{-1}$ would work.

Wlog we may assume $\varepsilon_k \ll 1$, so that

$$B(z_k, \varepsilon_k) \subseteq \mathbb{R}\times S^1$$

is an actual ball for all k.



$B(z_k, \varepsilon_k)$

Now we define

$$v_k : B(0, \varepsilon_k M_k) \longrightarrow M$$

as

$$v_k(z) = u_k\left(z_k + \frac{z}{M_k}\right).$$

Then clearly

$$\partial_s v_k(z) = \frac{1}{M_k}\, \partial_s u_k\left(z_k + \frac{z}{M_k}\right) \quad \text{and}$$

$$\partial_t v_k(z) = \frac{1}{M_k}\, \partial_t u_k\left(z + \frac{z}{M_k}\right).$$

From our choice of M_k, we conclude that

$$|\partial_s v_k| \leq 1 \quad \text{and} \quad |\partial_s v_k(0)| = 1.$$

So the v_k have uniformly bounded derivatives. They satisfy the equation

$$\partial_s v_k + J(v_k)\left(\partial_t v_k - \frac{1}{M_k} X_H(v_k)\right) = 0$$

By our assumption on the images of the u_k staying in a compact subset of M, we can pass to a further subsequence to arrange

$$v_k(0) \longrightarrow x_0 \in M.$$

Now Arzela-Ascoli gives us a further subsequence s.t.

$$v_k \xrightarrow[C^0_{loc}]{} v : \mathbb{R}^2 \to M,$$

where the limiting map v has the following properties:

a) $v(0) = x_0$

b) $|\partial_s v| \leq 1$ and $|\partial_s v(0)| = 1$, so v is not constant

c) v satisfies the limiting equation

$$\boxed{\partial_s v + J(v)\,\partial_t v = 0}$$

Elliptic regularity now implies that $v_k \to v$ in C^∞_{loc}.

Also, we know that

$$\int_{B(0,\varepsilon_k M_k)} \|\partial_s v_k\|^2 \, ds\,dt = \int_{B(z_k,\varepsilon_k)} \|\partial_s u_k\|^2 \, ds\,dt$$

$$\leq E(u) = \mathcal{A}_H(x^+) - \mathcal{A}_H(x^-)$$

So by Fatou's lemma

$$0 < E(v) = \int_{\mathbb{R}^2} \|\partial_s v\|^2 \, ds\,dt \leq E(u) < \infty.$$

Note that this energy also equals

$$E(v) = \int_{\mathbb{R}^2} \|\partial_s v\|^2 \, ds\, dt = \int_{\mathbb{R}^2} \omega_0(\partial_s u, J_0 \partial_s u)\, ds\, dt$$

$$= \int_{\mathbb{R}^2} \omega(\partial_s u, \partial_t u)\, ds\, dt$$

$$= \int_{\mathbb{R}^2} v^* \omega .$$

Now we derive the desired contradiction using the following

<u>Lemma:</u> In this situation, there is a sequence $r_k \to \infty$ such that the lengths of the image curves $v_k(\partial B(0, r_k))$ tend to 0 as $k \to \infty$.

<u>Remark:</u> In fact, more is true: the limiting map extends to a continuous and hence J-holomorphic map $\tilde{v} : S^2 = \mathbb{R} \cup \{\infty\} \to M$.

Let us first use the lemma to finish the proof of the proposition. As a consequence of the lemma, for $k \gg 1$ sufficiently large the images $v(\partial B(0, r_k))$ are contained in Darboux balls, so they bound disks D_k there. Gluing these disks to $v_k(B(0, r_k))$ get spheres in M, so since $\omega|_{\pi_2(M)} = 0$ we know that

$$0 = \int_{S^2} \omega = \int_{D_k} \omega + \int_{B(0,r_k)} v_k^* \omega .$$

Now we have

$$\left| \int_{D_k} \omega \right| = \left| \int_{D_k} d\lambda \right| = \left| \int_{\partial D_k} \lambda \right| \leq \underbrace{l(\partial D_k) \cdot \|\lambda\|}_{\substack{\to 0 \\ \text{as } k \to \infty}}$$

local primitive

But $\int_{B(0,r_k)} v_k^* \omega \longrightarrow \int_{\mathbb{R}^2} v^* \omega > 0$ and together these give us the desired contradiction.

It now remains to give the

Proof of the lemma:

Since v is J-holomorphic, in polar coordinates on $\mathbb{R}^2$ we have

$$v^*\omega = f(\rho,\theta)\,\rho\, d\rho \wedge d\theta$$

for some function $f : \mathbb{R}^2 \to [0,\infty)$, and the length $\ell(r)$ of $v(\partial B(0,r))$ is given by

$$\ell(r) = \int_0^{2\pi} \|\partial_\theta v\|_{g_J}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{v^*\omega(\partial_\theta, \underbrace{J_0\partial_\theta}_{-\rho\partial_\rho})}\, d\theta$$

$$= \int_0^{2\pi} r \sqrt{f(r,\theta)}\, d\theta$$

We now consider the area

$$A(r) := \int_{B(0,r)} v^*\omega = \int_0^{2\pi}\int_0^r \rho\, f(\rho,\theta)\, d\rho\, d\theta$$

and note that

$$A'(r) = \int_0^{2\pi} r\, f(r,\theta)\, d\theta$$

So using the Cauchy-Schwarz inequality we get

$$\ell(r) = r\int_0^{2\pi}\sqrt{f(r,\theta)}\, d\theta \le r\sqrt{\int_0^{2\pi} d\theta \int_0^{2\pi} f(r,\theta)\, d\theta}$$

$$= r\sqrt{2\pi\, \frac{A'(r)}{r}}$$

This can be rewritten as

$$\ell(r)^2 \le 2\pi\, r\, A'(r).$$

Now A is bounded, so clearly

$$\lim_{m\to\infty} \frac{A(m^2) - A(m)}{\ln m} = 0.$$

But

$$\frac{A(m^2)-A(m)}{\ln m} = \frac{A(m^2)-A(m)}{\ln m^2 - \ln m}$$

$$= \frac{A'(r_m)}{\frac{1}{r_m}} = A'(r_m)\cdot r_m$$

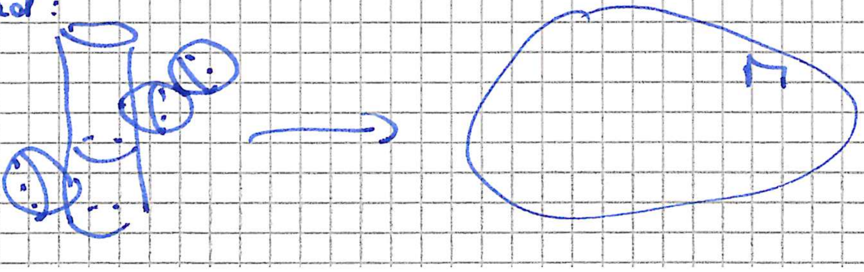
for some $r_m \in [m, m^2]$ by the mean value theorem.

Putting our estimates together, this proves the lemma, and so completes the proof of the proposition.

□

<u>Remarks:</u>

(1) We clearly see the use of the vanishing of symplectic area on $\pi_2(M)$ to exclude the existence of bubbles. In general, they will exist. In that case, one needs to do a more careful argument to find <u>all</u> (in a suitable sense). The idea is to repeat the bubbling argument, with the new starting domain $\mathbb{R}\times S^1 \setminus \{z_n\}$. One chooses a complete metric in the conformal class determined by the complex structure, and again either gets uniformly bounded derivatives or new points $z_k^1 \in \mathbb{R}\times S^1 \setminus \{z_n\}$ s.t. the norm of the derivative of u_k blows up there w.r.t. the <u>new</u> metric. Now the supremum of $\|du_k\|$ might not be finite, and an extra argument ("Hofer's lemma") is needed to select suitable points for the blow-up argument. Eventually, one finds a limit object which is a map on a cylinder with bubble trees of spheres attached:



The main (cylinder) component satisfies Floer's equation, the bubbles are J-holomorphic.

2) As in Morse theory, the asymptotic orbits associated to the cylinder components we find in this way need not be $x^\pm$. If that happens, we shift using regular values of the action to find the missing components just as in Morse theory. They, too, may have J-holomorphic bubbles attached in the general case.

In the symplectically aspherical case, we end up with the following result:

Prop: (M,ω) sympl. aspherical, closed
$H: \mathbb{R}\times M \to \mathbb{R}$ 1-periodic with all periodic orbits nondegenerate
J ω-compatible, $x^\pm \in \mathcal{P}(H)$ given.

Then any sequence of Floer cylinders $(u_k)_{k\geq 1}$ in $\widetilde{\mathcal{M}}(x^+,x^-;H,J)$ admits a subsequence which converges to a broken Floer cylinder, i.e. a finite chain of Floer cylinders $v^i \in \widetilde{\mathcal{M}}(x_{i-1},x_i)$ with $x_0 = x^+$ and $x_m = x^-$.
As in Morse theory, this means that there are times s_k^i, $1\leq i\leq m$ such that the shifted cylinders

$$u_k^i(s,t) := u_k(s+s_k^i,t)$$

converge to $v^i(s,t)$ in $C^\infty_{loc}(\mathbb{R}\times S^1, M)$.

We have

$$\mu_{CZ}(x^0) > \mu_{CZ}(x^1) > \ldots > \mu_{CZ}(x^m)$$

if (H,J) is generic.

Next one needs to prove a gluing theorem in much the same way that we did for Morse theory.

Then one arrives at

Corollary: ⓐ If $\mu_{CZ}(x^+) - \mu_{CZ}(x^-) = 1$, then

$$\mathcal{M}(x^+, x^-) := \widetilde{\mathcal{M}}(x^+, x^-)/\mathbb{R} \quad \text{is 0-dimensional}$$

and compact, i.e. finite.

ⓑ If $\mu_{CZ}(x^+) - \mu_{CZ}(x^-) = 2$, then $\mathcal{M}(x^+, x^-)$ has a natural compactification whose boundary corresponds bijectively to

$$\coprod_{\mu_{CZ}(x^+) > \mu_{CZ}(x) > \mu_{CZ}(x^-)} \mathcal{M}(x^+, x) \times \mathcal{M}(x, x^-).$$

This allows us to define Floer homology of a generic pair (H, J):

For each $x \in \mathcal{P}(H)$, we define its Floer degree by

$$|x| := n + \mu_{CZ}(x).$$

Now we set

$$CF_k(H, J) := \bigoplus_{|x| = k} \mathbb{Z}_2 \langle x \rangle$$

and define the boundary operator

$$\partial : CF_k(H, J) \longrightarrow CF_{k-1}(H, J)$$

as

$$\partial x^+ := \sum_{|x^-| = |x^+| - 1} \#_2 \mathcal{M}(x^+, x^-) \cdot x^-.$$

It is well-defined and satisfies $\partial^2 = 0$ by the above discussion.

Independence of Floer homology from the chosen data (H, J) is now proven in exactly the same way as in Morse theory by setting up continuation maps and chain homotopies between them.

Finally, it remains to compute $HF_*(H, J)$ for a particular choice of Floer data.