Last time we ended by defining symplectic homology as a direct limit. Let us explore what this means:

Suppose $(I, \leq)$ is a directed set, i.e.

- $i \leq i$ for all $i \in I$
- If $i \leq j$ and $j \leq k$ then $i \leq k$
- for all $i, j \in I$ there is a $k \in I$ s.t. $i \leq k$ and $j \leq k$

Fix a ground ring R and let

$$C : (I, \leq) \longrightarrow R\text{-dgMod}$$

be a functor from I to the category of dg R-modules (i.e. chain complexes over R). This means we have

* chain complexes $C(i)$ for all $i \in I$ and
* a chain map $c_{ij} : C(i) \to C(j)$ whenever $i \leq j$.

Then the direct limit

$$C := \varinjlim_{I} C(i)$$

is a chain complex together with maps $\alpha(i) : C(i) \to C$ satisfying the following universal property:

If A is a chain complex and we are given chain maps $\varphi(i) : C(i) \to A$ s.t. $\varphi(i) = \varphi(j) \circ c_{ij}$ whenever $i \leq j$, then there is a unique chain map $\psi : C \to A$ making the following diagram commute for <u>all</u> $i, j \in I$

$$\begin{tikzcd}
\vdots \arrow[d] & & \\
C(i) \arrow[d, "c_{ij}"'] \arrow[dr, "\alpha(i)"] \arrow[drr, bend left, "\varphi(i)"] & & \\
C(j) \arrow[d] \arrow[r, "\alpha"'] \arrow[rr, bend right, "\varphi(j)"'] & C \arrow[r, dashed, "\exists! \psi"'] & A \\
\vdots & &
\end{tikzcd}$$

In particular, $\varphi(i) = \psi \circ \alpha(i)$ for all $i \in I$.

Facts: (1) Taking homology commutes with taking direct limits, i.e.

$$H_*\left(\varinjlim_{I} C(i)\right) = \varinjlim_{I} H_*(C(i))$$

(2) An explicit construction of $\varinjlim_{I} C(i)$ is given by

$$\left(\bigoplus_{i\in I} C(i)\right)\Big/ K \,,$$

where K is generated by all elements of the form $x - c_{ij}(x)$ for $x \in C(i)$, $i,j \in I$.

(3) Any element $c \in \varinjlim_{I} C(i)$ is of the form $c = \kappa(i)(x)$ for some $i \in I$ and some $x \in C(i)$.

For more background on this, see any book on homological algebra, e.g. C. Weibel "An introduction to homological algebra".

So as part of the package, symplectic homology $SH_*(W,\lambda)$ comes with morphisms

$$HF_*(H,J) \longrightarrow SH_*(W,\lambda)$$

for all regular Floer data (H,J) with an asymptotically linear Hamiltonian H.
Recall that the Floer homology $HF_*(H,J)$ depends only on the slope at infinity $0 < b(H)$.

Lemma: If $b(H)$ is smaller than the period of the shortest closed Reeb orbit of $(V = \partial W, \alpha = \lambda_{|V})$, then

$$HF_*(H,J) \cong H_{n+*}(W,\partial W) \cong H^{n-*}(W).$$

Pf: We use the same argument as in the computation of Floer homology for closed manifolds: We are free to choose the Hamiltonian, so we choose

$$H : W \longrightarrow \mathbb{R}$$

t-independent and C^2-small with $H|_V \equiv 0$, and extend it to a Hamiltonian on $\hat{W}$ with the prescribed slope at ∞.
In this case, the 1-periodic orbits of the Hamiltonian system are exactly the critical points of H, and Floer cylinders correspond to solutions of the equation

$$\partial_s u = -\nabla H .$$

But in our current setup we have "inputs" at $s=\infty$ and "outputs" at $s=-\infty$, so

$$HF_*(H,J) \cong HM_*(-H, g_J)$$

We can arrange that $V = \partial W$ is a regular level set of H with gradient pointing outwards, so that $-H$ has a local minimum along ∂W. So

$$HM_*(-H, g_J) \cong H_*(W, \partial W) \cong H^{2n-*}(W) .$$

$\square$

We conclude that $SH_*(W,\lambda)$ comes with a map

$$H^{2n-*}(W) \longrightarrow SH_*(W,\lambda) .$$

Note that if this is not an isomorphism, then (V,α) must have closed Reeb orbits (since all 1-periodic orbits of our Hamiltonians in the cylindrical end arise from these).
So in many cases computations of symplectic homology prove instances of the Weinstein conjecture.

To compute a direct limit, it is enough to pick a cofinal subset $I' \subseteq I$ in the index set and consider just the induced subsystem.
For symplectic homology this means that we pick a sequence of regular Floer data (H_k, J_k) with slopes $b(H_k) \to \infty$.

<u>Ex:</u> Consider the Hamiltonians

$$H_b : \mathbb{C} \to \mathbb{R} \quad , \quad H_b(z) = b|z|^2 - b$$

from problem sheet 4.
One can use $b_k = (2k+1)\pi$ to get a sequence as described above. Then the computations of that exercise imply:

<u>Thm:</u> $SH_*(\mathbb{C}, d\lambda_{can}) = 0$.

More generally, we have $SH_*(\mathbb{C}^n, d\lambda_{can}) = 0$ for all $n \geq 1$.

In principle, we do not care what the Hamiltonian function H looks like on the compact part W, as long as it has nondegenerate 1-periodic orbits there.
However, recall that the action of a periodic orbit on the cylindrical end (before the time-dependent perturbation) is

$$\mathcal{A}_H(\gamma) = \int_{S^1} \gamma^* \hat{\lambda} - \int_0^1 H(\gamma(t))\, dt$$

$$= r \cdot h'(r) - h(r)$$

where $\gamma(t) = (r, y(h'(r) \cdot t))$ for some Reeb orbit $y : [0, h'(r)] \to V$ of period $h'(r)$.

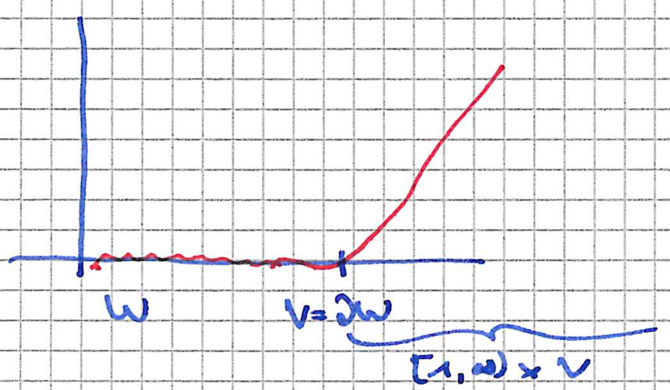
We saw that if $h''(r) \geq 0$ on the cylindrical end $[1,\infty) \times V$, then this action is an increasing function of r.
If we assume $h(1) = 0$ and $h'(1) > 0$, then all periodic orbits on the cylindrical end have action $\geq h'(1)$.

~~So~~ If we choose H on W to be C^2-small, then the only 1-periodic orbits in W are constant at the critical points of H, with action

$$\mathcal{A}_H(\gamma) = -H(\gamma(0)),$$

which will be close to 0.

In particular, we can arrange for the actions of all orbits in W to be <u>smaller</u> than the actions of the orbits on the cylindrical end. As the Floer boundary operator <u>decreases</u> action, we see that for such Hamiltonians there are no Floer cylinders from an orbit in W to an orbit in the cylindrical end. It follows that the part of the complex generated by periodic orbits in W is in fact a subcomplex of the Floer complex for Hamiltonians of this shape.



<u>Rem:</u> It is true in general that for any $b \in \mathbb{R}$ the orbits of action $\leq b$ generate a subcomplex of the Floer complex which we denote by $CF^{\leq b}(H, J)$.
For $a < b$ we get quotient complexes

$$CF^{[a,b]}(H, J) := CF^{\leq b}(H, J) \,/\, CF^{\leq a}(H, J).$$

The above discussion shows that the map $H^{2n-*}(W) \to SH_*(W)$ in the case of the special Hamiltonians described above is given as the inclusion-induced map

$$SH_*^{<a}(W, \lambda) := \varinjlim HF_*^{<a}(H, J) \to SH_*(W, \lambda)$$

for $a > 0$ small.

It is usually quite difficult to compute symplectic homology of a Liouville domain from the definitions, so we would like to have some tools to facilitate computations in some families of examples.

Here is a first result:

Thm (Oancea '04) (Künneth formula for symplectic homology)

Let $(W, d\lambda)$ and $(W', d\lambda')$ be Liouville domains with vanishing first Chern class. Then for any coefficient ring R there is a short exact sequence

$$0 \to \bigoplus_{r+s=k} SH_r(W,\lambda) \otimes SH_s(W',\lambda') \to SH_k(W \times W', \lambda + \lambda')$$

$$\to \bigoplus_{r+s=k-1} \mathrm{Tor}_1^R \left(SH_r(W,\lambda), SH_s(W',\lambda') \right) \to 0 .$$

This sequence splits (but not canonically).

In particular, if R is a field, then

$$SH_k(W \times W', \lambda + \lambda') \cong \bigoplus_{r+s=k} SH_r(W,\lambda) \otimes SH_s(W',\lambda') .$$

For the proof, see the paper:

A. Oancea: The Künneth formula in Floer homology for manifolds with restricted contact type boundary, Math. Annalen 334 (2006), 65-89

The main point of the proof is that while the sum $H + H'$ of linear Hamiltonians on $\hat{W}$ and $\hat{W}'$ is not linear on $\hat{W} \times \hat{W}'$, one can still define Floer homology for these Hamiltonians and compare the resulting theory to standard symplectic homology.

Cor: If $(\hat{W}, \lambda) \cong (\hat{W}', \lambda') \times (\mathbb{C}, \lambda_{can})$, then

$$SH_*(\hat{W}, \lambda) = 0 .$$

I have mentioned before that Weinstein domains $(W, d\lambda)$ have the homotopy type of a CW complex of at most half their dimension.
In fact, more is true: Every Weinstein domain can be obtained by starting from the ball $(B^{2n}, d\lambda_{can})$ and successively attaching "handles" $D^k \times D^{2n-k}$ along embeddings $(\partial D^k) \times D^{2n-k} \to \partial$ (previous part). Here $k \leq n$ and the handles come with a standard symplectic form for which D^k is isotropic.

Such a Weinstein handle is called <u>subcritical</u> if $k<n$, and a Weinstein domain is called subcritical if it can be built by only using subcritical handles.

<u>Thm</u> (Cieliebak '02)

Attaching a subcritical handle to a Liouville domain does not change ~~its~~ symplectic homology.

<u>Cor:</u> All subcritical Weinstein domains have vanishing symplectic homology.

For a class of nontrivial examples, we have

<u>Thm</u> (Viterbo '98)

For any closed manifold Q we have

$$SH_*(T^*Q, d\lambda_{can}; \mathbb{Z}_2) \cong H_{*+n}(LQ; \mathbb{Z}_2).$$

<u>Remarks:</u> ① If one wants to upgrade the statement to integer coefficients, one has to twist one of the two sides by a local coefficient system which is nontrivial whenever $w_2(TQ) \neq 0$.

② Besides Viterbo's original paper, there are several other proofs of this result in the literature:

- Abbondandolo-Schwarz (CPAM '06)

- Salamon-Weber (GAFA '06)
- Abouzaid (in: Loop spaces in geometry and topology, 2015)

Abouzaid's proof treats the most general situation ($\mathbb{Z}$-coefficients, Q not necessarily orientable) and also discusses relations between algebraic structures on the two sides.