Hamiltonian Floer theory

We now want to discuss our first version of Floer homology.
Standard references:
- D. Salamon "Lectures on Floer homology"
- M. Audin/M. Damian "Morse theory and Floer homology"
- + Floer's original papers

Recall the setup from lecture 1:

(M, ω) symplectic

Def: A diffeomorphism $\varphi: M \to M$ is called a Hamiltonian diffeomorphism if there exists a function $H: \mathbb{R}/_{\mathbb{Z}} \times M \to \mathbb{R}$ such that φ is the time 1 map of the associated "flow"

$$(*) \begin{cases} \dot{\varphi}_t(x) = X_{H_t}(\varphi_t(x)) \\ \varphi_0(x) = x \end{cases}$$

Here the functions $H_t = H(t, \cdot)$ determine the Hamiltonian vector fields X_t via

$$\omega(X_{H_t}, \cdot) = -dH_t .$$

Remarks:
- Any Hamiltonian diffeomorphism is symplectic, i.e. $\varphi^* \omega = \omega$, and so also volume preserving.
- The Hamiltonian function $\{H_t\}_{t \in \mathbb{R}}$ is not uniquely determined, there are many paths connecting a given Hamiltonian diffeomorphism to the identity
- Hamiltonian diffeomorphisms form a subgroup

$$\mathrm{Ham}(M, \omega) \subseteq \underbrace{\mathrm{Symp}_0}_{\text{component of } \mathrm{id}_M}(M, \omega)$$

We are interested in fixpoints of φ, or equivalently 1-periodic orbits of the system $(*)$. They are zeroes of the 1-form $\alpha_H \in \Omega^1(\mathcal{L}M)$

$$\alpha_H(\gamma, \xi) = \int \omega(\dot{\gamma}(t) - X_{H_t}(\gamma(t)), \xi(t))\, dt$$

We will for the moment restrict ourselves to 1-periodic solutions which form contractible loops in M.

We denote the component of contractible loops in the free loops space of M by $\mathcal{L}_0 M \subseteq \mathcal{L}M$, and its universal cover by $\widetilde{\mathcal{L}M_0} \xrightarrow{\pi} \mathcal{L}_0 M$.
Elements of $\widetilde{\mathcal{L}M_0}$ can be interpreted as pairs $(\gamma, [u])$ with $\gamma : S^1 \to M$ contractible and $[u]$ a homotopy class (rel boundary) of maps $u : D^2 \to M$ with $u_{|S^1} = \gamma$.

Lemma: The pullback $\pi^* \alpha_H$ of the 1-form α_H to $\widetilde{\mathcal{L}_0 M}$ is exact, indeed we have

$$\pi^* \alpha_H = - d \widetilde{\mathcal{A}}_H , \quad \text{where}$$

$$\widetilde{\mathcal{A}}_H : \widetilde{\mathcal{L}_0 M} \to \mathbb{R} \quad \text{is given by}$$

$$\widetilde{\mathcal{A}}_H (\gamma, [u]) = \int_{D^2} u^* \omega - \int_0^1 H_t(\gamma(t))\, dt$$

Pf: Consider a variation

$$\gamma_\varepsilon = \exp_\gamma (\varepsilon \cdot \xi) \quad \text{for some } \xi \in \gamma^* TM.$$

Then

$$\frac{d}{d\varepsilon} \Big(\int_{D^2} u_\varepsilon^* \omega \Big)_{|\varepsilon = 0} = \int_0^1 \omega(\xi(t), \dot\gamma(t))\, dt$$

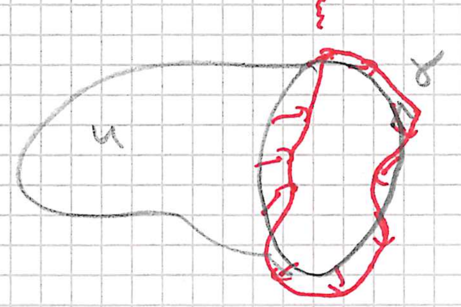
$$= - \int_0^1 \omega(\dot\gamma(t), \xi(t))\, dt$$

and

$$\frac{d}{d\varepsilon} \Big(\int_0^1 H_t(\gamma_\varepsilon(t))\, dt \Big)_{|\varepsilon=0} = \int_0^1 dH_t(\xi(t))\, dt$$

$$= - \int_0^1 \omega(X_{H_t}(\gamma(t)), \xi(t))\, dt$$

$\square$



<u>Remark:</u> If the symplectic area of all 2-spheres in M vanishes, i.e. we have

$$\int_{S^2} v^*\omega = 0 \quad \text{for all } v: S^2 \to M \text{ smooth},$$

then any two homotopy classes of discs give the same contribution to $\tilde{\mathcal{A}}_H$, i.e. the action functional becomes well-defined on $\mathcal{L}_0 M$.

Examples: - surfaces of genus ≥ 1 and their products, so in particular all tori
- any cotangent bundle (T^*Q, ω_{can})

<u>Def:</u> A 1-period orbit $\gamma: \mathbb{R} \to M$ of the system (*) is called <u>nondegenerate</u> if

$$(D\varphi_1)_{\gamma(0)} : T_{\gamma(0)} M \to T_{\gamma(0)} M$$

does not have 1 as an eigenvalue.

<u>Facts:</u> (1) A nondegenerate orbit gives rise to nondegenerate critical points of $\tilde{\mathcal{A}}_H$.

(2) For a generic choice of 1-periodic $H: \mathbb{R} \times M \to \mathbb{R}$, all 1-periodic orbits are nondegenerate.

We would like to do (an analogue of) Morse theory for $\tilde{\mathcal{A}}_H$. However, several problems occur:

(A) $\tilde{\mathcal{A}}_H$ is unbounded from above and below:

$$\mathcal{A}_H(\gamma, [u]) = \underbrace{\int_{D^2} u^*\omega}_{\substack{\text{dominant}\\ \text{term, unbounded}}} - \underbrace{\int_0^1 H_t(\gamma(t))\, dt}_{\substack{\text{bounded}\\ \text{(at least if } M \text{ is}\\ \text{compact)}}}$$

(B) Morse indices and coindices are infinite

(C) There are essential difficulties in defining a gradient flow.

A metric on $\mathcal{L}_0 M$ can be obtained from any metric on M. In the symplectic setting, we get Riemannian metrics nicely interacting with the symplectic geometry from certain almost complex structures on M.

Def: An almost complex structure on M is an endomorphism $J: TM \to TM$ with $J^2 = -\mathrm{Id}$.
It is compatible with ω if $\omega(\cdot, J\cdot)$ is a Riemannian metric on M, i.e.
symmetric: $\omega(v, Jw) = \omega(w, Jv)$ for all $w, v \in TM$
($\Leftrightarrow$ $\omega(v,w) = \omega(Jv, Jw)$ — " —)
positive definite: $\omega(v, Jv) > 0$ whenever $v \neq 0$.

Compatible a.c. structures always exist and form a contractible space.

Now given J compatible with ω, we can define a metric on $\mathcal{L}M$ as follows:
Given $\xi, \eta \in T_\gamma \mathcal{L}M$, let

$$\langle \xi, \eta \rangle_J := \int_0^1 \omega(\xi(t), J\eta(t))\, dt$$
$$= - \int_0^1 \omega(J\xi(t), \eta(t))\, dt$$

Remark: It is useful to note that J can also be a (1-periodic) time-dependent family of a.c. structures here.

From the fact that $d\tilde{\mathcal{A}}_H = -\pi^*\alpha_H$ we deduce that the L^2 gradient of $\tilde{\mathcal{A}}_H$ is

$$(\operatorname{grad} \tilde{\mathcal{A}}_H)_\gamma = -J(\dot{\gamma}(t) - X_{H_t}(\gamma(t)))$$

Indeed,
$$\langle (\operatorname{grad} \tilde{\mathcal{A}}_H)_\gamma, \xi \rangle_J = - \int_0^1 \omega(J \operatorname{grad} \tilde{\mathcal{A}}_H, \xi(t))\, dt$$
$$= - \underbrace{\int_0^1 \omega(\dot{\gamma}(t) - X_{H_t}(\gamma(t)), \xi(t))\, dt}_{= \alpha_H(\xi)}$$

So the formal gradient flow equation for a curve in $\mathcal{L}M$, i.e. for $\underset{s\ \ t}{u: \mathbb{R}\times S^1} \to M$

is

$$\boxed{\partial_s u + J\left(\partial_t u - X_{H_t}(u)\right) = 0} \quad (**)$$

As an ODE on (appropriate Banach space versions of) $\mathcal{L}M$ this is problematic, but Floer's seminal insight was that one can and should instead look at this as a PDE for maps into M.

What we need for the Morse-type complex is <u>not</u> a general notion of gradient flow line, but only a notion of <u>flow line connecting two critical points</u>.

<u>Assumptions</u> (for now):

- (M,ω) closed symplectic
- $H: \mathbb{R}\times M \to \mathbb{R}$ 1-periodic such that all 1-periodic orbits are nondegenerate

We denote the set of 1-periodic orbits by $\mathcal{P}(H)$. In the non-degenerate case, this set is finite.

<u>Def:</u> For $u: \mathbb{R}\times S^1 \to M$, we define its <u>energy</u> as

$$E(u) = \frac{1}{2}\int_{S^1}\int_{\mathbb{R}} |\partial_s u|^2 + |\partial_t u - X_{H_t}(u)|^2 \, ds\, dt$$

The first key observation is

<u>Proposition 1:</u> (Prop. 1.21 in Salamon) If $u: \mathbb{R}\times S^1 \to M$ solves $(**)$, then the following are equivalent:

(a) $E(u) < \infty$

(b) There are periodic orbits $x^\pm \in \mathcal{P}(H)$ such that

$$\lim_{s\to\pm\infty} u(s,t) = x^\pm(t) \quad \text{and} \quad \lim_{s\to\pm\infty} \partial_s u(s,t) = 0$$

uniformly in $t \in S^1$

(c) There are constants $\delta > 0$ and $c > 0$ such that

$$\| \partial_s u(s,t) \| \leq c \cdot e^{-\delta |s|}$$

for all $s, t \in \mathbb{R}$.

Remark: A solution $u: \mathbb{R} \times S^1 \to M$ of (**) with finite energy is called a Floer cylinder.

Exercise: For each Floer cylinder $u: \mathbb{R} \times S^1 \to M$ one has

$$E(u) = \widetilde{\mathcal{A}}_H(x^+, [u^+]) - \widetilde{\mathcal{A}}_H(x^-, [u^-]),$$

where u^+ is obtained from u^- by concatenation with u.

To set up the Floer complex in analogy with the Morse complex, we want to use the moduli spaces

$$\widetilde{\mathcal{M}}(x^+, x^-; H, J) := \{ u: \mathbb{R} \times S^1 \to M,\ \overline{\partial}_{H,J} u = 0,\ E(u) < \infty,\ \lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t) \}$$

and their $\mathbb{R}$-quotients to define a boundary operator on a complex generated by the periodic orbits $x \in \mathcal{P}(H)$, or more generally critical points $(x, [u])$ of $\widetilde{\mathcal{A}}_H$.

We now want to discuss the relevant notion of an index, which will play the role of the Morse index in Floer theory.

So let $x: S^1 \to M$ be a contractible 1-periodic solution of

$$\dot{\gamma} = X_{H_t} \circ \gamma$$

and let $u: D^2 \to M$ be a map with $u|_{S^1} = x$.

Since D^2 is contractible, there is a unique homotopy class of symplectic trivializations of $u^* TM$, and hence also of $x^* TM$. Choose such a trivialization, and denote by

$$\Psi : [0,1] \to Sp(2n)$$

the path of symplectic matrices representing the differentials

$$D\varphi_t : T_{x(0)} M \to T_{x(t)} M$$

in this trivialization. Clearly $\Psi(0) = \mathrm{Id}$, and nondegeneracy of x means that $\Psi(1)$ does not have 1 as an eigenvalue.

We set

$$SP(n) := \{ \Psi : [0,1] \to Sp(2n) \mid \Psi(0) = \mathrm{Id}, \; \Psi(1) \in Sp^*(2n) \}$$

where $Sp^*(2n) = \{ A \in Sp(2n) \mid \det(A - \mathrm{Id}) \neq 0 \}$

Fact 1: $Sp^*(2n)$ consists of two connected components, one containing $W^+ = -\mathrm{Id}$ and the other containing $W^- = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \oplus -\mathrm{Id}_{2n-2}$.

Fact 2: If $\phi : S^1 \to Sp^*(2n)$ is a loop, then ϕ is contractible ~~in~~ in $Sp(2n)$.

Fact 3: Every matrix $A \in Sp(2n)$ has a unique decomposition

$$A = U \cdot P$$

with $U \in U(n)$ and P positive definite, symmetric and symplectic

(In fact $P = (A^T A)^{\frac{1}{2}}$ and $U = A (A^T A)^{-\frac{1}{2}}$)

It follows that the inclusion $U(n) \hookrightarrow Sp(2n)$ is a homotopy equivalence

Fact 4: $\det_{\mathbb{C}} : U(n) \to S^1$ induces an isomorphism on fundamental groups

(In fact $U(n) \underset{\text{homeo}}{\cong} S^1 \times SU(n)$)

Putting these observations together, we can define a map
$$\mu_{CZ} : SP(n) \to \mathbb{Z}$$
as follows:

Given $\Psi : [0,1] \to Sp(2n)$ in $SP(n)$, we extend it by a path connecting $\Psi(1)$ to either $W^{\pm}$ in $Sp^*(2n)$, which is possible by Fact 1.

Denote this extension by $\tilde{\Psi} : [0,2] \to Sp(2n)$.

The map $\rho : Sp(2n) \to S^1$ sending $A = U \cdot P$ to $\det_{\mathbb{C}}(U)$ satisfies
$$\rho(W^{\pm}) \in \{\pm 1\}.$$

It follows that
$$R := (\rho \circ \tilde{\Psi})^2 : [0,2] \to S^1$$
is a loop, and we define
$$\mu_{CZ}(\Psi) := \deg(R).$$

By Facts 2 and 4, this is independent of the choice of extension $\tilde{\Psi}$.

The Conley-Zehnder index has a number of useful properties:

(A) (naturality) For any path $\Phi : [0,1] \to Sp(2n)$ we have
$$\mu_{CZ}(\Phi \Psi \Phi^{-1}) = \mu_{CZ}(\Psi)$$

(B) (homotopy) μ_{CZ} is constant on the connected components of $SP(n)$

(C) (zero) If $\Psi(t)$ does not have eigenvalues in S^1 for $t > 0$, then $\mu_{CZ}(\Psi) = 0$

(D) (product) If $\Psi = \Psi_1 \oplus \Psi_2$ for $\Psi_i \in SP(n_i)$ then
$$\mu_{CZ}(\Psi) = \mu_{CZ}(\Psi_1) + \mu_{CZ}(\Psi_2)$$

S1

E (loop) If $\Phi : [0,1] \to Sp(2n)$ is any loop based at $\mathbb{1} \in Sp(2n)$, then

$$\mu_{CZ}(\Phi\Psi) = \mu_{CZ}(\Psi) + 2\mu(\Phi)$$

where $\mu(\Phi) = \deg(\rho \circ \Phi)$ is the Maslov index of Φ

F (signature) If $S = S^T$ is a symmetric matrix which is nondegenerate with all eigenvalues of absolute value $< 2\pi$, then

$$\mu_{CZ}(\exp(J_0 S t)) = \frac{1}{2}\operatorname{sign}(S)$$
$$= \frac{1}{2}(\#\text{pos. EV} - \#\text{neg. EV})$$

G (determinant) $$(-1)^{n-\mu_{CZ}(\Psi)} = \operatorname{sgn}\det(\mathbb{1} - \Psi(1))$$

H (inverse) $$\mu_{CZ}(\Psi^{-1}) = \mu_{CZ}(\Psi^T) = -\mu_{CZ}(\Psi).$$

In fact, μ_{CZ} is characterized by B, E and F.

For details, see

Salamon

Audin-Damian

Conley-Zehnder — Birkhoff-Lewis fixed point theorem...

Robbin-Salamon — Maslov index for paths

Gutt — The Conley-Zehnder index for a paths of symplectic matrices