Symplectic homology

Until now, we have mostly dealt with closed manifolds. To work on noncompact manifolds, one needs to put in place some additional restrictions to get a meaningful theory.

This is true already in Morse theory:

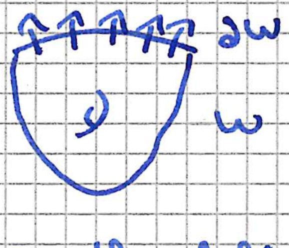
Ex: On $\mathbb{R}$, the function $f(x) = x$ has no critical points and so trivial Morse homology, whereas the function $h(x) = x^2$ has exactly one critical point of index 0, and so $HM_*(h, g_0) \cong H_*(\mathbb{R})$.

The manifolds for which we want to define a meaningful Floer theory are the following:

Def: A Liouville domain is a compact symplectic manifold with boundary $(W, d\lambda)$ with globally exact symplectic form such that the vector field Y satisfying

$$i_Y d\lambda = \lambda$$

is transverse to the boundary pointing outwards.



Rem: It follows from the definition that

(a) $\alpha = \lambda|_{\partial W}$ is a contact form on $\partial W =: V$

(b) the ~~negative~~ flow of Y is defined for all negative times, and one can use it to construct an embedding

$$((-\infty, 0] \times V, d(e^s \alpha)) \to (W, d\lambda)$$

as a "collar" of the boundary.

The completion of a Liouville domain is the symplectic manifold obtained from gluing on the positive half of the symplectization of (V, α):

$$(\hat{W}, d\lambda) = (W, d\lambda) \sqcup_V ([0, \infty) \times V, d(e^s \alpha)).$$

Aside on contact manifolds

Def: A contact manifold is an odd-dimensional manifold V with a maximally non-integrable hyperplane field $\xi \subseteq TV$.
Locally, one can always write $\xi = \ker\alpha$, and the maximal non-integrability means that $\alpha \wedge (d\alpha)^{top}$ is a local volume form.

Standard example: $J^1\mathbb{R}^n = T^*\mathbb{R}^n \times \mathbb{R}$
coord (x,y) z
$\alpha = dz - \sum_j y_j dx_j$ is a contact form

We are mainly interested in contact structures ξ which are co-orientable, meaning that the defining form α can be choosen globally.
Note that any two defining forms α and β differ by multiplication with a nonzero function.
Given a contact form α defining ξ, its Reeb vector field is the unique vector field R satisfying

$$\iota_R d\alpha = 0 \quad \text{and}$$
$$\alpha(R) = 1.$$

The Weinstein conjecture asserts that on a closed coorientable contact manifold, the Reeb vector field of every defining contact form will have at least one periodic orbit.

For more information on contact manifolds, see

H. Geiges — Introduction to contact topology
J. Etnyre — Introductory lectures on contact geometry
arxiv: math/0111118

Examples:

1. Any bounded convex (in the usual sense) or more generally any starshaped (w.r.t. 0) domain $W \subseteq (\mathbb{R}^{2n}, \omega_0)$ is a Liouville domain w.r.t. the standard Liouville form
$$\lambda = \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$
The Liouville vector field is
$$Y = \frac{1}{2} \sum_{j=1}^{n} (x_j \partial_{x_j} + y_j \partial_{y_j})$$
The completions are all symplectomorphic to $(\mathbb{R}^{2n}, \omega_0)$.

2. Let Q be a smooth closed manifold and let $W \subseteq (T^*Q, \omega_{can})$ be fiberwise starshaped and bounded. Then $(W, d\lambda_{can})$ is a Liouville domain with
$$Y = \sum_{j=1}^{n} p_j \partial_{p_j}$$
The completion are all symplectomorphic to (T^*Q, ω_{can}).

3. Let (X, J) be a Stein manifold, meaning that there exists an exhausting function
$$h : X \to \mathbb{R}$$
for which $-d(dh \circ J) =: -dd^c h$ is a Kähler form. By a theorem of Grauert, this is equivalent to the fact that X admits a proper holomorphic embedding as a complex submanifold in some $\mathbb{C}^N$.
For any regular value $a \in \mathbb{R}$ of h,
$$W_a := h^{-1}((-\infty, a]) \quad \text{is a Liouville domain}$$
w.r. to the Liouville form
$$\lambda = -d^c h = -dh \circ J.$$
One checks that the Liouville vector field Y is the gradient of h with respect to the metric
$$g_J = d\lambda(\cdot, J\cdot).$$
Specific examples are obtained from regular level sets of complex polynomials.

(i) Consider $p(z_1, \dots, z_n) = \sum_{j=1}^{n} z_j^2$

The only critical point of p is $0 \in \mathbb{C}^n$, and so for any $c \neq 0$ the level set $p^{-1}(c) =: H_c \subseteq \mathbb{C}^n$ is a smooth complex submanifold.

(ii) More generally, polynomials

$$p_\alpha(z_1, \dots, z_n) = \sum_{j=1}^{n} z_j^{\alpha_j}$$

for multiindices $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j \geq 2$ have the same property.

For submanifolds of $\mathbb{C}^n$, we may take $h(z) = \frac{1}{4}|z|^2$, so

$$dh = \frac{1}{4} \sum_{j=1}^{n} (z_j d\bar{z}_j + \bar{z}_j dz_j)$$

Now

$$d\bar{z}_j \circ J_0 = -i\, d\bar{z}_j \quad \text{and} \quad dz_j \circ J_0 = i\, dz_j, \text{ so that}$$

$$-dh \circ J = \frac{i}{4} \sum_{j=1}^{n} (z_j d\bar{z}_j - \bar{z}_j dz_j)$$

$$= \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$

So λ on $\mathbb{C}^n$ is the standard Liouville form, and it restricts as a Liouville form to any of the hypersurfaces $H(c) = p^{-1}(c)$ as above.

(4) Slightly more generally, a <u>Weinstein domain</u> is a Liouville domain $(W, d\lambda)$ such that the Liouville vector field Y is gradient-like for a Morse function

$$h : W \to \mathbb{R}$$

for which ∂W is (part of) a regular level set.

<u>Exercise:</u> Let $(W, d\lambda, h)$ be a Weinstein domain, and let $p \in W$ be a critical point of h (so a zero of Y). Prove that the restriction of λ to

$$W^s(p) = \{ x \in W : \lim_{t \to \infty} \varphi_t(x) \text{ exists and equals } p \}$$

vanishes, i.e. all the stable manifolds are isotropic for $\omega = d\lambda$.

It follows from this observation that every Weinstein domain of dimension $2n$ has the homotopy type of a CW complex of dimension at most n.

A general Liouville domain need not have this property, in fact there are examples of Liouville domains which have the homotopy type of a CW complex of codimension 1

We want to do (Hamiltonian) Floer theory on Liouville manifolds (or sometimes, when one is interested in geometric as opposed to topological questions, on Liouville domains).

As already mentioned, we need to work harder now to get an invariant. The basic first step is to restrict the class of functions we allow as Hamiltonian functions.

To simplify the notation, it is useful to introduce new coordinates on the cylindrical end by using the symplectomorphism

$$\begin{aligned} ((-\infty,\infty)\times V,\ d(e^s\alpha)) &\longrightarrow ((0,\infty)\times V,\ d(r\alpha)) \\ (s,x) &\longmapsto (e^s,x). \end{aligned}$$

Using the coordinates (r,x) we identify the cylindrical end of a Liouville manifold with $([1,\infty)\times V,\ d(r\alpha))$

<u>Exercise:</u> Suppose $(\hat{W}, d\lambda)$ is a Liouville manifold and

$$H: \hat{W} \to \mathbb{R} \quad \text{satisfies}$$

$$H_{|[1,\infty)\times V}(r,x) = h(r) \text{ for some function}$$

$$h: [1,\infty) \to \mathbb{R}.$$

Then the Hamiltonian vector field X_H has the form

$$X_H(r,x) = h'(r)\cdot R(x)$$

on the cylindrical end, where R is the Reeb vector field of the contact form α on V.

It follows that nonconstant 1-periodic orbits of X_H on the cylindrical end will have the form

$$\gamma(t) = (r, y(h'(r)\cdot t)),$$

where $y: [0, h'(r)] \to V$ is a periodic orbit of R with period $h'(r)$.

Since our symplectic form on $\hat{W}$ is globally exact, we can write the action functional

$$\mathcal{A}_H: \mathcal{L}\hat{W} \to \mathbb{R} \quad \text{as}$$

$$\mathcal{A}_H(\gamma) = \int_0^{\wedge} \gamma^*\lambda - \int_0^{\wedge} H(\gamma(t))\,dt$$

$$= r\cdot h'(r) - h(r).$$

Note in particular that if $h''(r) \geq 0$, for all r, then the action of periodic orbits with larger value of r will be bigger, since

$$\frac{d}{dr}(r\cdot h'(r) - h(r)) = r h''(r) \geq 0 \quad \text{in this case.}$$

<u>Def:</u> A Hamiltonian $H: \hat{W} \to \mathbb{R}$ is <u>(asymptotically) linear</u> of slope $b > 0$ if

$$H_{|[r,\infty)\times V}(r,x) = h(r) \quad \text{and}$$

$$h(r) = br + c \quad \text{for some } c \in \mathbb{R} \text{ and all sufficiently large } r \in [1,\infty).$$

<u>Remarks:</u>

1. The action spectrum of the contact manifold (V, α) is the set of periods of closed Reeb orbits. It is a closed, nowhere dense subset of $\mathbb{R}_+$, which is bounded away from 0.
   If the slope $b \in \mathbb{R}$ is choosen in the complement of the action spectrum, then all 1-periodic orbits of H will be contained in the compact subset
   $$W \cup_V [1, r_0] \times V$$
   (where $r_0 \in \mathbb{R}_+$ is choosen such that $h'(r) = b$ for all $r \geq r_0$).

2. If b is smaller than the period of the shortest closed Reeb orbit, then all 1-periodic orbits of H will

lie in W.

③ Time-independent Hamiltonians as introduced above are problematic for Floer theory because the periodic orbits in the cylindrical end come in S^1-families, and hence cannot be nondegenerate.
In practice, there are two solutions to this:

ⓐ Treat this as a Morse-Bott situation

ⓑ Use small time-dependent perturbations to make the 1-periodic orbits nondegenerate.
This has the effect that every closed Reeb orbit of period $< b$ gives rise to two 1-periodic orbits of X_H, whose indices will differ by 1.

Let $(\hat{W}, d\lambda)$ be a Liouville manifold.
Working with asymptotically linear Hamiltonians, we know that all 1-periodic orbits are contained in a fixed compact subset $W_0 \subseteq \hat{W}$.
To also control the behaviour of Floer cylinder, we need to restrict the type of compatible almost complex structure we allow.

<u>Def:</u> A compatible almost complex structure on the symplectization $((0,\infty)\times V, d(r\cdot\alpha))$ of a contact manifold $(V, \ker\alpha)$ is <u>convex</u> if

$$dr \circ J = -r\alpha \quad .$$

<u>Remarks:</u>

ⓐ The conditions can also be rewritten as follows:

(i) J leaves $\xi = \ker\alpha$ invariant and is compatible with $d\alpha$ there

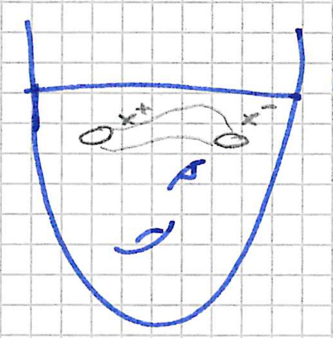
(ii) $J(R) = -r\partial_r$ (and therefore $J(\partial_r) = \frac{1}{r}R$)

ⓑ When $J|_{\ker\alpha}$ is independent of r, then J is invariant under the dilations

$$(r,x) \longmapsto (cr, x) \quad \text{with } c>0.$$

Compactness Lemma:

Let $\hat{W}$ be a Liouville manifold, $H:\hat{W}\to\mathbb{R}$ a Hamiltonian which is asymptotically linear and J an almost complex structure which is convex near some hypersurface $\{r_0\}\times V$ in the end.
If $x^\pm$ are 1-periodic orbits of H contained in the region "below" $\{r_0\}\times V$, then every Floer cylinder connecting them ~~does~~ does not enter $[r_0,\infty)\times V$.



Pf: Assume by contradiction that there is a Floer cylinder u connecting x^+ and x^- whose image rises above $\{r_0\}\times V$.
Fix a regular value r_1 near r_0 of $\pi_{\mathbb{R}}\circ u$ and let $\Sigma\subseteq \mathbb{R}\times S^1$ denote the preimage

$$u^{-1}([r_1,\infty)\times V)\subseteq \mathbb{R}\times S^1$$

and set $v:=u|_\Sigma$.

We know that

$$0\le E(v)=\frac{1}{2}\int_\Sigma\left(|\partial_s v|^2+|\partial_t v-X_H|^2\right)ds\wedge dt$$

$$=\int_\Sigma \omega(\partial_s v, J\partial_s v)\, ds\wedge dt$$

$$=\int_\Sigma\left(\omega(\partial_s v,\partial_t v)-\omega(\partial_s v, X_H\circ v)\right)ds\wedge dt$$

$H=br+c$ on the image of v $\longrightarrow$

$$=\int_\Sigma v^*\omega - b\, d(r\circ v\, dt)$$

$$=\int_{\partial\Sigma} v^*\lambda - b\,(r\circ v)\,dt$$

Now
$$
\begin{aligned}
v^*\lambda = v^*(r\alpha) &= -v^* dr\circ J \\
&= -dr\circ J\circ dv \\
&= -dr\circ dv\circ j + \underbrace{dr(X_H\circ v)\,ds}_{=0} - \underbrace{dr(J X_H\circ v)\,dt}_{=\,b(r\circ v)dt}
\end{aligned}
$$
Floer equation

So
$$\int_{\partial\Sigma} v^*\lambda - b(r\circ v)\,dt = -\int_{\partial\Sigma} d(r\circ v)\circ j .$$

Now the boundary orientation on $\partial\Sigma$ is such that if Y is tangent to $\partial\Sigma$, then $j(Y)$ points inward.

But by construction $r\circ v$ reaches its minimum on $\partial\Sigma$, so we conclude that $d(r\circ v)\circ j$ is a positive form on $\partial\Sigma$, implying that
$$\int_{\partial\Sigma} v^*\lambda - b(r\circ v)\,dt < 0$$

This contradicts (*), proving that the original cylinder
$$u : \mathbb{R}\times S^1 \to \widehat{W}$$
entering $[r_0,\infty)\times V$ does not exist. $\square$